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Integrability of the Metallic Structures on the Frame Bundle

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Abstract. Earlier investigators have made detailed studies of geometric properties such as integrability, partial integrability, and invariants, such as the fundamental 2-form, of some canonical f-structures, such as $f^3 \pm f = 0$, on the frame bundle FM. Our aim is to study metallic structures on the frame bundle: polynomial structures of degree 2 satisfying $F^2 = pF + qI$ where p, q are positive integers. We introduce a tensor field $F_\alpha, \alpha = 1, 2..., n$ on FM show that it is a metallic structure. Theorems on Nijenhuis tensor and integrability of metallic structure F_α on FM are also proved. Furthermore, the diagonal lifts g^D and the fundamental 2-form Ω_α of a metallic structure F_α on FM are established. Moreover, the integrability condition for horizontal lift F^H_α of a metallic structure F_α on FM is determined as an application. Finally, the golden structure that is a particular case of a metallic structure on FM is discussed as an example.

1. Introduction

The geometry of frame bundles is a powerful method in the geometry that permits to get rich results while studying the various structures such as an almost complex structure, f-structures, etc. on the base manifold admit lifts to the frame bundle. Cordero at el [4, 5] studied horizontal and diagonal lifts of connections and tensor fields of a different type; for example, tensor fields of type (1,1) and (0,2). They studied the integrability and the partial integrability of an f-structure F_α on the frame bundle. Kowalski and Sekizawa [17] investigated curvatures of diagonal lift from an affine manifold to the linear frame bundle.

On the other hand, Goldenberg et al [7, 8] introduced the polynomial structure $Q(J) = J^n + a_n J_{n-1} + ... + a_2 J + a_1 I$ where J is the tensor field of type (1,1) and I is an identity operator on differentiable manifold M. The polynomial structure of degree 2 satisfying $J^2 = pJ + qI$ is called a metallic structure on differentiable

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manifold M. The notion of the metallic mean family introduced by Spinadel [24, 25]. The golden mean, Silver mean, Bronze mean, Subtal mean, etc. are the members of the Metallic Mean Family [21, 22, 26]. The general quadratic equation $x^2 - px - q = 0, p$ and q some positive integers, has the positive solution denoted by

$$\theta_p^q = \frac{p + \sqrt{p^2 + 4q}}{2},$$

is called Metallic Mean Family.

The differential geometry of the metallic structure on a Riemannian manifold is an effective domain of differential geometry. Hretcamu and Crasmareanu [10] studied and analyzed the geometric properties of the metallic structure on the Riemannian manifold. The metallic structures on the tangent bundle of a Riemannian manifold by using complete and horizontal lifts studied by Kazmi [1]. The author [15] studied complete and horizontal lifts of the metallic structures and investigated integrability conditions for these structures. Turanli et. al. [27] constructed metallic Kähler and nearly metallic Kähler structures on Riemannian manifolds and studied curvature properties of such structures on Riemannian manifold. The geometry of metallic structure was studied in [9, 11, 12, 13, 16, 20].

The main contributions of the paper can be listed as follows:

- A tensor field $F_\alpha, \alpha = 1, 2..., n$ of type (1,1) is introduced and shows that it is a metallic structure on the frame bundle FM.
- Nijenhuis tensor N_{F_α} of a tensor field F_α and its integrability is calculated.
- The diagonal lift g^D of Riemannian metric g to the frame bundle FM is adopted to tensor field F_α.
- The fundamental 2-Form Ω_α of the tensor field F_α is determined.
- The horizontal lift F^H of metallic structure F_α i.e $F_\alpha^2 - pF_\alpha - qI = 0$ to the frame bundle FM is integrable with certain conditions.
- An example of the golden structure is constructed that is a particular case of metallic structure on FM.

The structure of the paper is as follows: Section 2 presents a brief account of frame bundle, metallic structure and Nijenhuis tensor. In Section 3, a tensor field $F_\alpha, \alpha = 1, 2..., n$ of type (1,1) is defined and showed that it is a metallic structure on the frame bundle FM. Integrability, diagonal lift g^D of a Riemannian metric g and the fundamental 2-Form of a metallic structure F_α on the frame bundle FM are also obtained. In Section 4, an application of the horizontal lift F_α^H of a metallic structure F_α on the frame bundle FM is investigated. Finally, in Section 5, an example of the golden structure is constructed.

2. Preliminaries

Let M be an n-dimensional differentiable manifold of class C^∞ and FM its frame bundle over the manifold M. Suppose the base space M is covered by a system of coordinate neighborhoods (U, x^i) such that $F(U) = \pi^{-1}(U)$ where (x^i) is a system of local coordinates defined in the neighborhood U and $\pi : FM \to M$ the projection map. The local components of the vector X_α of the frame $p_x \in U$ are given by $X_\alpha = X^i_\alpha \left(\frac{\partial}{\partial x^i}\right)_x$. Thus $\{FU, (x^i, X^i_\alpha)\}$ is a coordinate system in FM [4, 18, 19] .

Let ∇ be a linear connection and X a vector field on M with local components Γ^h_{ij} and X^i, respectively. Let vector fields X^H and $X^\alpha, \alpha = 1, 2, ...n.$ be the horizontal lift and the α^{th}-vertical lift of X on FM and defined by

$$X^H = X^i \frac{\partial}{\partial x^i} - X^i \Gamma^h_{ik} X^k_\alpha \frac{\partial}{\partial x^h}, \tag{2.1}$$

$$X^{(\alpha)} = X^i \frac{\partial}{\partial X^i_\alpha}. \tag{2.2}$$

Let f be a differentiable function on M, we write f^V for function i.e. vertical lift in FM and $f^H = 0$ its horizontal lift [3].

If F is a tensor field on M of type (1,1) with components F^h_j in U, then

$$F^H = F^h_j \frac{\partial}{\partial X^h} \otimes dx^j + X^k_\alpha (\Gamma^i_{jk} F^h_i - \Gamma^h_{ik} F^i_j) \frac{\partial}{\partial X^h_\alpha} \otimes dx^j + \delta^\beta_\alpha F^h_j \frac{\partial}{\partial X^h_\alpha} \otimes dX^j_\beta \tag{2.3}$$

is local components of F^H in FU.

Let τ be a 1-form on M with local components τ_i in U, then

$$\begin{aligned} \tau^V &= \tau_i dx^i, \\ \tau^{H_\alpha} &= X^j_\alpha \Gamma^h_{ij} \tau_h dx^i + \tau_i dX^i_\alpha, \\ X^H &= \sum_{\alpha=1}^{m} (X^j_\alpha \Gamma^h_{ij} \tau_h dx^i + \tau_i dX^i_\alpha) \end{aligned} \tag{2.4}$$

are local components of τ^V, τ^{H_α} and X^H in FU.

The following formulas of horizontal and vertical lifts are given by

$$\begin{aligned} X^H(f^V) &= (X(f))^V, \\ X^{(\alpha)}(f^V) &= 0, \\ F^H(X^{(\alpha)}) &= (F(X))^\alpha, \\ F^H(X^H) &= (F(X))^H, \\ F^H(\lambda A) &= F^C(\lambda A) = \lambda(F^\circ A), \\ \tau^V(X^H) &= (F(X))^V, \\ \tau^V(X^{(\alpha)}) &= 0, \\ \tau^{H_\alpha}(X^H) &= 0, \\ \tau^{H_\alpha}(X^{(\beta)}) &= \delta^\beta_\alpha(\tau(X))^V, \end{aligned} \tag{2.5}$$

for all vector fields X, Y on M and λA is fundamental vector field associated to A where $A \in gl(n, \Re), gl(n, \Re)$ is general linear group and $\Re$ is Euclidean space [4, 15].

The brackets of vertical and horizontal lifts are expressed by the following formulas

$$\begin{aligned} [X^{(\alpha)}, Y^{(\beta)}] &= 0, \\ [X^H, Y^{(\alpha)}] &= (\nabla_X Y)^{(\alpha)}, \\ [X^H, Y^H] &= [X^H, Y^H] - \gamma R(X, Y), \end{aligned} \tag{2.6}$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Definition 2.1. Let F be a tensor field of type (1,1) on a differentiable manifold M and satisfies the equation

$$F^2 - pF - qI = 0, \tag{2.7}$$

where p, q are positive integers and I is an identity operator. Then the tensor field F is called a metallic structure on M and (M, F) is called a metallic manifold [1].

The Nijenhuis tensor N of a metallic structure F is given by

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]. \tag{2.8}$$

where X and Y are vector fields on a differentiable manifold M. The metallic structure F is called integrable if $N(X, Y) = 0$ [28, 29].

3. Integrability of Metallic Structure on the Frame Bundle

In this section, a tensor field $F_\alpha, \alpha = 1, 2, ..., m$ of type (1,1) is introduced and proved that it is a metallic structure on the frame bundle FM. The integrability of a metallic structure F_α on FM has been studied. Furthermore, the fundamental 2-form $\Omega_\alpha, \alpha = 1, 2..., n$ of F_α is determined.

3.1. Metallic structures on the frame bundle

Let (M, g) be an n-dimensional Riemannian manifold and FM its frame bundle. Let X^H and $X^{(\alpha)}, \alpha = 1, 2, ..., n$, be horizontal and vertical lifts of a vector field X on FM with respect to the Levi-Civita connection ∇ of a Riemannian metric g.

In [4], Cordero et al defined a tensor field $F_\alpha, \alpha = 1, 2, ..., n$ of type (1,1) on FM as

$$F_\alpha X^H = -X^{(\alpha)}, \quad F_\alpha X^{(\beta)} = \delta_\alpha^\beta X^H, \tag{3.1}$$

where X^H and $X^{(\alpha)}$ are 'the horizontal lift' and '$\alpha-$th vertical lift' of a vector field X on M. It is proved that $F_\alpha^3 + F_\alpha = 0$ i.e. F-structure.

Also, Gezer and Kamran [23] defined a tensor field $\tilde{J}$ of type (1,1) in the tangent bundle TM of M by

$$\begin{aligned}\tilde{J}X^H &= \frac{1}{2}(\alpha X^H + (2\sigma_\alpha^\beta - \alpha)(X \otimes \tilde{E}^V),\\ \tilde{J}X^V &= \frac{1}{2}(\alpha(X \otimes \tilde{E}^V + (2\sigma_\alpha^\beta - \alpha)X^H),\\ \tilde{J}A^V &= \sigma_\alpha^\beta A^V,\end{aligned}$$

for any vector field X, tensor field A of type (1,1), $\tilde{E} = g \circ E$ and g a Riemannian metric on M. It is proved that $\tilde{J}$ is a metallic structure on TM.

From Cordero et al [4] and Gezer and Kamran [23], a tensor field $F_\alpha, \alpha = 1, 2, ..., n$, of type (1,1) in FM is introduced as

$$\begin{aligned}F_\alpha X^H &= \frac{1}{2}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\},\\ F_\alpha X^{(\beta)} &= \frac{1}{2}\delta_\alpha^\beta\{pX^{(\beta)} + (2\theta_p^q - p)X^H\},\end{aligned} \tag{3.2}$$

where $\theta_p^q = \frac{p+\sqrt{p^2+4q}}{2}$.

Theorem 3.1. *Let FM be the frame bundle of M. Then a tensor field F_α, defined by equation (3.2), is a metallic structure on FM.*

Proof. In order to prove F_α is a metallic structure, it suffices to show that $F_\alpha^2 - pF_\alpha - qI = 0$.

In the view of equation (3.2), then

$$\begin{aligned}(F_\alpha^2 - pF_\alpha - qI)X^H &= F_\alpha(F_\alpha X^H) - pF_\alpha X^H - qX^H,\\ &= F_\alpha \frac{1}{2}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\}\\ &- \frac{p}{2}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\} - qX^H,\\ &= \frac{p}{4}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\}\\ &+ \frac{(2\theta_p^q - p)}{4}\{pX^{(\beta)} + (2\theta_p^q - p)X^H\}\\ &- \frac{p}{2}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\} - qX^H,\\ &= 0.\end{aligned}$$

Similarly, it is easily proved that $(F_\alpha^2 - pF_\alpha - qI)X^{(\alpha)} = 0$ which imply that $F_\alpha^2 - pF_\alpha - qI = 0$.

Hence, F_α is a metallic structure on FM. □

The rank of F_α is constant and equal to $2n$ and rank of metallic structure $F_\alpha^2 - pF_\alpha - qI = 0$ is equal to $2n$ on FM.

Define the projection operators by [14]

$$l_\alpha = \frac{F_\alpha^2 - pF_\alpha}{q}, \tag{3.3}$$

$$m_\alpha = I - \frac{F_\alpha^2 - pF_\alpha}{q}, \tag{3.4}$$

then there exists on FM the complementary distributions L_α and M_α corresponding to l_α and m_α, respectively. When the rank of F_α is $2n$. Then dim of $L_\alpha = 2n$ and dim of $M_\alpha = n^2 - n$.

Theorem 3.2. *Let a tensor field F_α be a metallic structure and l_α and m_α be projection operators on FM. Then*

$$l_\alpha + m_\alpha = I, \quad l_\alpha^2 = l_\alpha, \quad m_\alpha^2 = m_\alpha, \quad l_\alpha m_\alpha = m_\alpha l_\alpha = 0, \tag{3.5}$$

$$F_\alpha l_\alpha = l_\alpha F_\alpha = F_\alpha, \quad F_\alpha m_\alpha = m_\alpha F_\alpha = 0. \tag{3.6}$$

Proof. Using equations (3.3) and (3.4), it can be easily proved. □

3.2. Integrability

To study of integrability and partial integrability of metallic structure F_α on FM, first state the following propositions for later use [3]:

Proposition 3.3.([4]) *Let H be a tensor field of type (1,1) on M. Then*

$$(i)\ F_\alpha(\gamma H) = \sigma_\alpha H, \tag{3.7}$$

$$(ii)\ F_\alpha(\sigma_\alpha H) = -\gamma^\alpha H, \tag{3.8}$$

where σH and γH are horizontal and vertical lifts on FM, respectively.

Theorem 3.4. *Let X^H and $X^{(\alpha)}, \alpha = 1, 2, ..., n$, be horizontal and vertical lifts of a vector field X on FM with respect to the Levi-Civita connection of g. If*

$$l_\alpha(X^H) = X^H, \quad l_\alpha(X^{(\beta)}) = \delta_\alpha^\beta X^{(\beta)}), \tag{3.9}$$

$$m_\alpha(X^H) = 0, \quad m_\alpha(X^{(\beta)}) = q(I - \delta_\alpha^\beta)X^{(\beta)}. \tag{3.10}$$

Then $\{X^H, X^{(\alpha)}\}$ span L_α and $\{X^{(\beta)}; \beta \neq 0\}$ span M_α

Proof. Operating X^H and $X^{(\alpha)}$ on equation (3.3) and using equation (3.2), then

$$\begin{aligned} l_\alpha(X^H) &= \frac{1}{q}F_\alpha^2(X^H) - \frac{p}{q}F_\alpha(X^H), \\ &= \frac{1}{2q}F_\alpha[pX^H + (2\theta_p^q - p)X^{(\alpha)}] - \frac{p}{2q}[pX^H + (2\theta_p^q - p)X^{(\alpha)}], \\ &= \frac{p}{2q}F_\alpha(X^H) + (2\theta_p^q - p)F_\alpha X^{(\alpha)}] - \frac{p^2}{2q}X^H - (2\theta_p^q - p)\frac{p}{2q}X^{(\alpha)}], \\ &= X^H. \end{aligned}$$

Similarly, other identities can be easily obtained.

Thus $\{X^H, X^{(\alpha)}\}$ span L_α and $\{X^{(\beta)}; \beta \neq 0\}$ span M_α □

In the view of Proposition 3.3, the complementary distribution M_α is always completely integrable. Integrability of the complementary distribution L_α is obtained by the following Theorem:

Theorem 3.5. *The complementary distribution L_α is completely integrable if and only if (M, g) is locally Euclidean.*

Theorem 3.6. *Let X^H and $X^{(\alpha)}, \alpha = 1, 2, ..., n$, be horizontal and vertical lifts of a vector field X on FM. The Nijenhuis tensor N_{F_α} of F_α is given by*

$$\begin{aligned}
(i)\ \ N_{F_\alpha}(X^H, Y^H) &= -\frac{p^2}{2}\gamma^\alpha R(X,Y) - q\gamma^\alpha R(X,Y) + q[X,Y]^H,\\
(ii)\ \ N_{F_\alpha}(X^H, Y^{(\alpha)}) &= -\frac{p(2\theta_p^q - p)}{4}\gamma^\alpha R(X,Y) + \frac{(2\theta_p^q - p)}{2}\sigma_\alpha R(X,Y),\\
(iii)\ \ N_{F_\alpha}(X^{(\beta)}, Y^{(\mu)}) &= -\delta_\alpha^\beta \delta_\alpha^\mu \frac{p(2\theta_p^q - p)^2}{4}\gamma R(X,Y).
\end{aligned}$$

for all vector fields X, Y, Z on M, and $1 \leq \beta \leq n$.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be vector fields on the frame bundle FM and N_{F_α} be Nijenhuis tensor of a tensor field F_α of type (1,1) is given by

$$\tilde{N}_{F_\alpha}(\tilde{X}, \tilde{Y}) = [F_\alpha \tilde{X}, F_\alpha \tilde{Y}] - F_\alpha[F_\alpha \tilde{X}, \tilde{Y}] - F_\alpha[\tilde{X}, F_\alpha \tilde{Y}] + F_\alpha^2[\tilde{X}, \tilde{Y}]. \tag{3.11}$$

(i) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ in equation (3.11) and using equation (3.2), then

$$\begin{aligned}
\tilde{N}_{F_\alpha}(X^H, Y^H) &= [F_\alpha X^H, F_\alpha Y^H] - F_\alpha[F_\alpha X^H, Y^H]\\
&\quad -F_\alpha[X^H, F_\alpha Y^H] + F_\alpha^2[X^H, Y^H],\\
&= [\frac{1}{2}\{pX^H + (2\theta_p^q - p)\delta_\alpha^\beta X^{(\beta)}\}, \frac{1}{2}\{pY^H + (2\theta_p^q - p)\delta_\alpha^\beta Y^{(\beta)}\}]\\
&- F_\alpha[\frac{1}{2}\{pX^H + (2\theta_p^q - p)\delta_\alpha^\beta X^{(\beta)}\}, Y^H],\\
&- [X^H, \frac{1}{2}\{pY^H + (2\theta_p^q - p)\delta_\alpha^\beta Y^{(\beta)}\}] + F_\alpha^2\{[X,Y]^H - \gamma R(X,Y)\},\\
&= -\frac{p^2}{2}\gamma^\alpha R(X,Y) - q\gamma^\alpha R(X,Y) + q[X,Y]^H.
\end{aligned}$$

(ii) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^{(\beta)}$ in equation (3.11) and using equation (3.2), then

$$\begin{aligned}
\tilde{N}_{F_\alpha}(X^H, Y^{(\beta)}) &= [F_\alpha X^H, F_\alpha Y^{(\beta)}] - F_\alpha[F_\alpha X^H, Y^{(\beta)}] \\
&\quad -F_\alpha[X^H, F_\alpha Y^{(\beta)}] + F_\alpha^2[X^H, Y^{(\beta)}], \\
&= [\frac{1}{2}\{pX^H + (2\theta_p^q - p)\delta_\alpha^\beta X^{(\beta)}\}, \frac{1}{2}\{p\delta_\alpha^\beta Y^{(\beta)} + (2\theta_p^q - p)Y^H\}] \\
&- F_\alpha[\frac{1}{2}\{pX^H + (2\theta_p^q - p)\delta_\alpha^\beta X^{(\beta)}\}, Y^{(\beta)}] \\
&- F_\alpha[X^H, \frac{1}{2}\{p\delta_\alpha^\beta Y^{(\beta)} + (2\theta_p^q - p)\delta_\alpha^\beta Y^H\}] \\
&\quad +F_\alpha^2\{[X,Y]^H - \gamma R(X,Y)\}, \\
&= -\frac{p(2\theta_p^q - p)}{4}\gamma^\alpha R(X,Y) + \frac{(2\theta_p^q - p)}{2}\sigma_\alpha R(X,Y).
\end{aligned}$$

(iii) Setting $\tilde{X} = X^{(\beta)}$ and $\tilde{Y} = Y^{(\mu)}$ in equation (3.11) and using equation (3.2), then

$$\begin{aligned}
\tilde{N}_{F_\alpha}(X^{(\beta)}, Y^{(\mu)}) &= [F_\alpha X^{(\beta)}, F_\alpha Y^{(\mu)}] - F_\alpha[F_\alpha X^{(\beta)}, Y^{(\mu)}] - F_\alpha[X^{(\beta)}, F_\alpha Y^{(\mu)}] \\
&+ F_\alpha^2[X^{(\beta)}, Y^{(\mu)}], \\
&= [\frac{1}{2}\{p\delta_\alpha^\beta X^{(\beta)} + (2\theta_p^q - p)X^H\}, \frac{1}{2}\{p\delta_\alpha^\mu Y^{(\mu)} + (2\theta_p^q - p)Y^H\}] \\
&- F_\alpha[\frac{1}{2}\{p\delta_\alpha^\beta X^{(\beta)} + (2\theta_p^q - p)X^H\}, Y^{(\mu)}] \\
&- F_\alpha[X^H, \frac{1}{2}\{p\delta_\alpha^\beta Y^{(\mu)} + (2\theta_p^q - p)Y^H\}], \quad as \quad [X^{(\beta)}, Y^{(\mu)}] = 0, \\
&= -\delta_\alpha^\beta \delta_\alpha^\mu \frac{p(2\theta_p^q - p)^2}{4}\gamma R(X,Y).
\end{aligned}$$

□

Theorem 3.7. *The following statements are equivalent:*
(i) (M, g) *is locally Euclidean;*
(ii) L_α *is completely integrable;*
(iii) F_α *is partially integrable;*
(iv) F_α *is integrable.*

Proof. Using Theorem 3.5 and Theorem 3.6, then Theorem 3.7 can be easily obtained. □

3.3. Diagonal metric and fundamental 2-Form on the frame bundle

Let (M, g) be a Riemannian manifold with the Levi-Civita connection ∇ and diagonal lift g^D of a Riemannian metric g to the frame bundle FM. The diagonal lift g^D is Riemannian metric on FM [4]. Then

$$\begin{aligned} g^D(X^H, Y^H) &= \{g(X,Y)\}^V, \\ g^D(X^H, Y^{(\alpha)}) &= 0, \\ g^D(X^{(\alpha)}, Y^{(\beta)}) &= \delta^{\alpha\beta}\{g(X,Y)\}^V, \end{aligned} \tag{3.12}$$

where X, Y are vector fields on $M, \alpha, \beta = 1, 2, ..., n$.

Definition 3.8. Let F_α be a tensor field of type (1,1) defined (3.2) and g^D be the diagonal lift of g on FM with Levi-Civita connection ∇. The fundamental 2-Form Ω_α of F_α is defined by

$$\Omega_\alpha(\tilde{X}, \tilde{Y}) = g^D(F_\alpha \tilde{X}, \tilde{Y}), \tag{3.13}$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on FM.

Theorem 3.7. *Let (FM, g^D) be the frame bundle of Riemannian manifold (M, g). The Riemannian metric g^D is adopted to a tensor field F_α on FM.*

Proof. Let $\tilde{X}$ be a vector field and g^D be a Riemannian metric on FM. Let L_α and M_α be complementary distributions corresponding to projection operators l_α and m_α. Since complementary distributions L_α and M_α are mutually orthogonal with respect to g^D then $g^D(\tilde{X}, F_\alpha \tilde{X}) = 0$.

This completes the proof. □

Theorem 3.8. *Let M be a manifold and FM its frame bundle admits a tensor field $F_\alpha, \alpha = 1, 2, ..., n$ of type (1,1) defined by (3.2). Then the fundamental 2-Form Ω_α is given by*

$$\begin{aligned} \Omega_\alpha(X^H, Y^H) &= \frac{1}{2}p(g(X,Y))^V, \\ \Omega_\alpha(X^H, Y^{(\beta)}) &= \frac{(2\theta^q_p - p)}{2}\delta^{\alpha\beta}(g(X,Y))^V, \\ \Omega_\alpha(X^{(\beta)}, Y^{(\mu)}) &= \frac{(2\theta^q_p - p)}{2}\delta^{\beta\mu}(g(X,Y))^V, \end{aligned}$$

for all vector fields X, Y on $M, 1 \leq \beta, \mu \leq n$.

Proof. (i) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ in equation (3.13) and using equation (3.2) and (3.12), then

$$\begin{aligned} \Omega_\alpha(X^H, Y^H) &= g^D(F_\alpha X^H, Y^H), \\ &= g^D(\frac{1}{2}\{pX^H + (2\theta^q_p - p)X^{(\alpha)}\}, Y^H), \\ &= \frac{1}{2}p(g(X,Y))^V, \quad as \;\; g^D(X^{(\alpha)}, Y^H) = 0. \end{aligned}$$

(ii) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^{(\beta)}$ in equation (3.13) and using equations (3.2) and (3.12), then

$$\begin{aligned}\Omega_\alpha(X^H, Y^{(\beta)}) &= g^D(F_\alpha X^H, Y^{(\beta)}),\\ &= g^D(\frac{1}{2}\{pX^H + (2\theta_p^q - p)X^{(\alpha)}\}, Y^{(\beta)}),\\ &= \frac{(2\theta_p^q - p)}{2}\delta^{\alpha\beta}(g(X,Y))^V, \quad as \;\; g^D(X^H, Y^{(\beta)}) = 0.\end{aligned}$$

(iii) Setting $\tilde{X} = X^{(\beta)}$ and $\tilde{Y} = Y^{(\mu)}$ in equation (3.13) and using equations (3.2) and (3.12), then

$$\begin{aligned}\Omega_\alpha(X^{(\beta)}, Y^{(\mu)}) &= g^D(F_\alpha X^{(\beta)}, Y^{(\mu)}),\\ &= g^D(\frac{1}{2}\delta_\alpha^\beta\{pX^{(\beta)} + (2\theta_p^q - p)X^H\}, Y^{(\mu)}),\\ &= \frac{p}{2}\delta^{\beta\mu}(g(X,Y))^V, \quad as \;\; g^D(X^H, Y^{(\mu)}) = 0.\end{aligned}$$

This completes the proof. □

4. Application

In this section, a study is done on the Nijenhuis tensor $N_{F_\alpha^H}$ of a tensor field F_α^H of type (1,1) on FM is integrable.

Let $\tilde{X}$ and $\tilde{Y}$ be vector fields on the frame bundle FM and $N_{\tilde{F}}$ be Nijenhuis tensor of a tensor field $\tilde{F}$ of type (1,1) on FM is given by

$$N_{\tilde{F}}(\tilde{X}, \tilde{Y}) = [\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\tilde{X}, \tilde{Y}] - \tilde{F}[\tilde{X}, \tilde{F}\tilde{Y}] + \tilde{F}^2[\tilde{X}, \tilde{Y}], \tag{4.1}$$

where Nijenhuis tensor $N_{\tilde{F}}$ is a tensor field of type (1,2).

By using equations (2.5) and (2.6), the following identities have been obtained.

$$\begin{aligned}N_{F_\alpha^H}(\lambda A, \lambda B) &= 0,\\ N_{F_\alpha^H}(\lambda A, X^H) &= \lambda(F_\alpha \nabla_X F_\alpha - \nabla_{F_\alpha X} F_\alpha)^\circ A,\\ N_{F_\alpha^H}(X^H, Y^H) &= \{N_{F_\alpha}(X,Y)\}^H - \gamma(R(F_\alpha X, F_\alpha Y) - F_\alpha R(F_\alpha X, Y)\\ &- F_\alpha R(X, F_\alpha Y) + F_\alpha^2 R(X,Y)),\end{aligned}$$

for all vector fields X, Y on M and λA is fundamental vector field associated to A where $A \in gl(n, \Re), gl(n, \Re)$ is general linear group and R is Euclidean space [4].

Theorem 4.1. *Let M be a differentiable manifold of C^∞ admitting metallic structure F_α and F_α^H its horizontal lift to FM with respect to ∇ is integrable that is $N_{F_\alpha^H} = 0$ if*
(i) $F_\alpha \nabla_X F_\alpha - \nabla_{F_\alpha X} F_\alpha = 0$ and
(ii) The curvature tensor R of ∇ satisfies

$$R(F_\alpha X, F_\alpha Y) + (pF_\alpha + qI)R(X,Y) = 0, \tag{4.2}$$

for all vector fields X, Y on M.

Proof. Let M be a differentiable manifold admitting a metallic structure F and given that

$$F_\alpha \nabla_X F_\alpha - \nabla_{F_\alpha X} F_\alpha = 0 \tag{4.3}$$

and the curvature tensor R of ∇ satisfies

$$R(F_\alpha X, F_\alpha Y) + (pF_\alpha + qI)R(X,Y) = 0. \tag{4.4}$$

Replace X by $F_\alpha X$ in equation (4.4) and using $F_\alpha^2 - pF_\alpha - qI = 0$, the obtained equation is

$$R(X, F_\alpha Y) + R(F_\alpha X, Y) = 0, \quad pF_\alpha + qI \neq 0. \tag{4.5}$$

Making use of equations (2.7), (4.3), (4.4), (4.5) in equation (4.1), we get

$$\begin{aligned} N_{F_\alpha^H}(\lambda A, \lambda B) &= 0, \\ N_{F_\alpha^H}(\lambda A, X^H) &= 0, \\ N_{F_\alpha^H}(X^H, Y^H) &= 0. \end{aligned}$$

Thus $N_{F_\alpha^H} = 0$,

so, F^H is integrable.

This completes the proof. □

5. Example

Setting $p = 1, q = 1$ in equation (2.7), then obtained equation is $F^2 - F - I = 0$. It is named as golden structure which is a particular case of metallic structure [2, 6].

Define a tensor field $F'_\alpha, \alpha = 1, 2, .., n$ of type (1,1) on FM such that ${F'}_\alpha^2 - F'_\alpha - I = 0$ and equation (3.2) becomes

$$\begin{aligned} F'_\alpha X^H &= \frac{1}{2}\{X^H + \sqrt{5}X^{(\alpha)}\}, \\ F'_\alpha X^{(\beta)} &= \frac{1}{2}\delta_\alpha^\beta\{X^{(\beta)} + \sqrt{5}X^H\}. \end{aligned} \tag{5.1}$$

The study of golden structure is omitted on FM due to the similarity with a metallic structure.

Conflicts of Interest. The author declares no conflict of interest.

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