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On $*$-Conformal Ricci Solitons on a Class of Almost Kenmotsu Manifolds

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Abstract. The goal of this paper is to characterize a class of almost Kenmotsu manifolds admitting $*$-conformal Ricci solitons. It is shown that if a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold M admits $*$-conformal Ricci soliton, then the manifold M is $*$-Ricci flat and locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. The result is also verified by an example.

1. Introduction

In 1959, Tachibana [17] introduced the notion of $*$-Ricci tensors on almost Hermitian manifolds. Later in [13], Hamada defined $*$-Ricci tensors of real hypersurfaces in non-flat complex spaces by

$$S^*(X,Y) = g(Q^*X,Y) = \frac{1}{2}(trace\{\phi\circ R(X,\phi Y)\}) \tag{1.1}$$

for any vector fields X, Y on M, where Q^* is the $(1,1)$ $*$-Ricci operator. The $*$-scalar curvature is denoted by r^* and is defined by $r^* = trace(Q^*)$. An almost contact metric manifold M is called $*$-Ricci flat if the $*$-Ricci tensor S^* vanishes identically.

The concept of conformal Ricci flow was develpoed by Fischer [12] as a variation of the classical Ricci flow equation. The conformal Ricci flow on a smooth closed connected oriented n-manifold M is defined by the equation

$$\frac{\partial g}{\partial t} + 2(S+\frac{g}{n}) = -pg \text{ and } r=-1,$$

where p is a time dependent non-dynamical scalar field, S denotes the Ricci tensor

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and r is the scalar curvature of the manifold.

The concept of a conformal Ricci soliton was introduced by Basu and Bhattacharyya [1] on a $(2n+1)$-dimensional Kenmotsu manifold as

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g,$$

where λ is a constant and $\pounds_V g$ denotes the Lie derivative of g along the vector field V. This notion was studied by Dutta et al. [11], Nagaraja and Venu [15], Dey and Majhi [9] and many others.

Over the last decade, geometers and mathematical physicists have developed several notions related to the $*$-Ricci tensor. In 2014, the notion of a $*$-Ricci soliton ([14]) was introduced. Later in 2019, the notion of a $*$-critical point equation [7] was introduced and further studied in [8]. In this paper, we study the notion of $*$-conformal Ricci soliton defined as follows in [6].

Definition 1.1. An almost contact metric manifold (M, g) of dimension $(2n+1) \geq 3$ is said to admit $*$-conformal Ricci soliton (g, V, λ) if

$$\pounds_V g + 2S^* = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.2}$$

where λ is a constant. The $*$-conformal Ricci soliton is expanding, steady or shrinking according as λ is negative, zero or positive.

In [9], the authors proved that if the metric of a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold M admits a conformal Ricci soliton, then M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Thus a natural question is the following.

Question. Is the above result is true for a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold admitting $*$-conformal Ricci soliton?

We will answer this question affirmatively. Also, we get some additional results associated with the $*$-Ricci tensor and the vector field V.

The paper is organized as follows: In Section 2, we give some basic properties of $(k,\mu)'$-almost Kenmotsu manifolds. Section 3 deals with $(k,\mu)'$-almost Kenmotsu manifolds admitting $*$-conformal Ricci soliton. In the final section, the result is verified by an example.

2. $(k,\mu)'$-almost Kenmotsu Manifolds

An odd dimensional differentiable manifold M is said to have an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily. If a manifold M with an almost contact structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ of an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any X, Y on M. Almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are called almost Kenmotsu manifolds ([10], [16]).

Let us denote by $\mathcal{D}$ the distribution orthogonal to ξ. It is defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

Let M be a $(2n + 1)$-dimensional almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $l = R(\cdot, \xi)\xi$ on M. The tensor fields l and h are symmetric operators and satisfy the following relations [16]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X\xi = X - \eta(X)\xi - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.3}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y, \tag{2.5}$$

for any vector fields X, Y. The $(1, 1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([10], [18])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2). \tag{2.6}$$

In [10], Dileo and Pastore introduced the notion of $(k, \mu)'$-nullity distribution, on an almost $(2n + 1)$-dimensional Kenmotsu manifold (M, ϕ, ξ, η, g), which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \mu)' = \{Z \in T_p(M) : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+\mu[g(Y, Z)h'X - g(X, Z)h'Y]\}. \end{aligned} \tag{2.7}$$

The $(k, \mu)'$-nullity distribution is called generalized $(k, \mu)'$-nullity distribution when one allows k, μ to be smooth functions.

Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue α. Then from (2.6) it is clear that $\alpha^2 = -(k + 1)$, a constant. Therefore $k \leq -1$ and $\alpha = \pm\sqrt{-k - 1}$. We denote by $[\alpha]'$ and $[-\alpha]'$ the corresponding eigenspaces related to the non-zero eigenvalue α and $-\alpha$ of h', respectively. In [10], it is proved that in a $(2n+1)$ dimensional $(k, \mu)'$-almost Kenmotsu manifold M with $h' \neq 0$, $k < -1$, $\mu =$

-2 and Spec$(h') = \{0, \alpha, -\alpha\}$, with 0 as simple eigenvalue and $\alpha = \sqrt{-k-1}$. From (2.7), we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.8}$$

where $k, \mu \in \mathbb{R}$. Also we get from (2.8)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{2.9}$$

Using (2.3), we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y). \tag{2.10}$$

For further details on $(k,\mu)'$-almost Kenmotsu manifolds, we refer the reader to the references ([5], [10], [16]).

3. $*$-Conformal Ricci Soliton

In this section, we study the notion of $*$-conformal Ricci solitons in the framework of $(k,\mu)'$-almost Kenmotsu manifolds. To prove the main theorem, we need the following lemmas:

Lemma 3.1. ([4]) *On a $(k,\mu)'$-almost Kenmotsu manifold with $k < -1$, the $*$-Ricci tensor is given by*

$$S^*(X,Y) = -(k+2)(g(X,Y) - \eta(X)\eta(Y)) \tag{3.1}$$

for any vector fields X, Y.

Lemma 3.2. ([9]) *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, $(\pounds_X h')Y = 0$ for any X, $Y \in [\alpha]'$ or X, $Y \in [-\alpha]'$, where Spec$(h') = \{0, \alpha, -\alpha\}$.*

Lemma 3.3. *On a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold M, the $*$-Ricci tensor S^* satisfies the following relation:*

$$\begin{aligned}&(\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(X,Z)\\ &= -2(k+2)\eta(Z)[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\end{aligned}$$

for any vector fields X, Y and Z on M.

Proof. Differentiating (3.1) covariantly along any vector field Z, we have

$$\nabla_Z S^*(X,Y) = -(k+2)[\nabla_Z g(X,Y) - (\nabla_Z \eta(X))\eta(Y) - (\nabla_Z \eta(Y))\eta(X)]. \tag{3.2}$$

Now,

$$(\nabla_Z S^*)(X,Y) = \nabla_Z S^*(X,Y) - S^*(\nabla_Z X, Y) - S^*(X, \nabla_Z Y).$$

Using (3.1) and (3.2) in the foregoing equation, we obtain

$$\begin{aligned}(\nabla_Z S^*)(X,Y) &= -(k+2)[\nabla_Z g(X,Y) - (\nabla_Z\eta(X))\eta(Y) - (\nabla_Z\eta(Y))\eta(X)] \\ &\quad +(k+2)[g(\nabla_Z X,Y) - \eta(\nabla_Z X)\eta(Y)] \\ &\quad +(k+2)[g(X,\nabla_Z Y) - \eta(\nabla_Z Y)\eta(X)] \\ &= (k+2)[((\nabla_Z\eta)X)\eta(Y) + ((\nabla_Z\eta)Y)\eta(X)]. \qquad (3.3)\end{aligned}$$

Again, using (2.10) in (3.3), we infer that

$$\begin{aligned}(\nabla_Z S^*)(X,Y) &= (k+2)[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z) \\ &\quad +g(h'Z,X)\eta(Y) + g(h'Z,Y)\eta(X)]. \qquad (3.4)\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}(\nabla_X S^*)(Y,Z) &= (k+2)[g(Y,X)\eta(Z) + g(Z,X)\eta(Y) - 2\eta(Y)\eta(Z)\eta(X) \\ &\quad +g(h'X,Y)\eta(Z) + g(h'X,Z)\eta(Y)]. \qquad (3.5)\end{aligned}$$

$$\begin{aligned}(\nabla_Y S)^*(Z,X) &= (k+2)[g(Z,Y)\eta(X) + g(X,Y)\eta(Z) - 2\eta(Z)\eta(X)\eta(Y) \\ &\quad +g(h'Y,Z)\eta(X) + g(h'Y,X)\eta(Z)]. \qquad (3.6)\end{aligned}$$

Now, using (3.4)-(3.6), we compute

$$\begin{aligned}&(\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(X,Z) \\ &= -2(k+2)\eta(Z)[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)].\end{aligned}$$

This completes the proof. □

We are now ready to prove our main theorem which is stated below.

Theorem 3.4. *Let M be a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting $*$-conformal Ricci soliton (g,V,λ). Then, the manifold M is $*$-Ricci flat and locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$, provided that $\lambda \neq \frac{p}{2} + \frac{1}{2n+1}$.*

Proof. From (1.2), we have

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \qquad (3.7)$$

Differentiating the above equation covariantly along any vector field Z, we get

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z S^*)(X,Y). \qquad (3.8)$$

It is well known that ([19], p-23)

$$(\pounds_V\nabla_X g - \nabla_X\pounds_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y).$$

Since g is parallel with respect to the Levi-Civita connection ∇, then the above relation becomes

$$(3.9)\qquad (\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y).$$

Since $\pounds_V \nabla$ is symmetric, then it follows from (3.9) that

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ (3.10)\qquad &\quad -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned}$$

Using (3.8) in (3.10) we have

$$(3.11)\qquad g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(X,Z).$$

Now using Lemma 3.3 in (3.11) we have

$$g((\pounds_V \nabla)(X,Y),Z) = -2(k+2)[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\eta(Z),$$

which implies

$$(3.12)\quad (\pounds_V \nabla)(X,Y) = -2(k+2)[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\xi.$$

Substituting $Y = \xi$ in (3.12) we get $(\pounds_V \nabla)(X,\xi) = 0$. From which we obtain $\nabla_Y(\pounds_V \nabla)(X,\xi) = 0$. This gives

$$(3.13)\qquad (\nabla_Y \pounds_V \nabla)(X,\xi) + (\pounds_V \nabla)(\nabla_Y X,\xi) + (\pounds_V \nabla)(X,\nabla_Y \xi) = 0.$$

Using $(\pounds_V \nabla)(X,\xi) = 0$, (3.12) and (2.3) in (3.13), we infer that

$$\begin{aligned} (\nabla_Y \pounds_V \nabla)(X,\xi) &= 2(k+2)[g(X,Y) - \eta(X)\eta(Y) + g(X,h'Y) + g(h'X,Y) \\ (3.14)\qquad &\quad + g(h'^2 X,Y)]\xi. \end{aligned}$$

It is known that ([19], p.23)

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

Using the equation (3.14) in the above formula, we obtain

$$(3.15)\qquad (\pounds_V R)(X,\xi)\xi = (\nabla_X \pounds_V \nabla)(\xi,\xi) - (\nabla_\xi \pounds_V \nabla)(X,\xi) = 0.$$

Now, substituting $Y = \xi$ in (3.7) and applying (3.1), we have

$$(3.16)\qquad (\pounds_V g)(X,\xi) = [2\lambda - (p + \frac{2}{2n+1})]\eta(X),$$

which implies

$$(3.17)\qquad (\pounds_V \eta)X - g(X,\pounds_V \xi) - [2\lambda - (p + \frac{2}{2n+1})]\eta(X) = 0.$$

From (3.17), after putting $X=\xi$ we can easily obtain that

$$\eta(\pounds_V\xi) = -[\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{3.18}$$

From (2.8), we have

$$R(X,\xi)\xi = k(X-\eta(X)\xi) - 2h'X. \tag{3.19}$$

Now, using (3.17)-(3.19) and (2.8)-(2.9) we obtain

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi &= \pounds_V R(X,\xi)\xi - R(\pounds_V X,\xi)\xi - R(X,\pounds_V\xi)\xi - R(X,\xi)\pounds_V\xi \\ &= k[2\lambda - (p+\frac{2}{2n+1})](X-\eta(X)\xi) - 2(\pounds_V h')X \\ &\quad -2[2\lambda - (p+\frac{2}{2n+1})]h'X - 2\eta(X)h'(\pounds_V\xi) \\ &\quad -2g(h'X,\pounds_V\xi)\xi. \end{aligned} \tag{3.20}$$

Equating (3.15) and (3.20) and then taking inner product with Y yields

$$\begin{aligned}& k[2\lambda - (p+\frac{2}{2n+1})](g(X,Y)-\eta(X)\eta(Y)) \\ & -2g((\pounds_V h')X,Y) - 2[2\lambda - (p+\frac{2}{2n+1})]g(h'X,Y) \\ & -2\eta(X)g(h'(\pounds_V\xi),Y) - 2g(h'X,\pounds_V\xi)\eta(Y) = 0.\end{aligned}$$

Replacing X by ϕX and Y by ϕY in the above equation, we infer that

$$\begin{aligned}& k[2\lambda - (p+\frac{2}{2n+1})]g(\phi X,\phi Y) - 2g((\pounds_V h')\phi X,\phi Y) \\ & -2[2\lambda - (p+\frac{2}{2n+1})]g(h'\phi X,\phi Y) = 0.\end{aligned} \tag{3.21}$$

Let $X \in [-\alpha]'$ and $V \in [\alpha]'$, then $\phi X \in [\alpha]'$. Then from (3.21), we have

$$(k-2\alpha)[2\lambda - (p+\frac{2}{2n+1})]g(\phi X,Y) - 2g((\pounds_V h')\phi X,Y) = 0. \tag{3.22}$$

Since, V, $\phi X \in [\alpha]'$, using Lemma 3.2 we have $(\pounds_V h')\phi X = 0$. Therefore, equation (3.22) reduces to

$$(k-2\alpha)[2\lambda - (p+\frac{2}{2n+1})]g(\phi X,Y) = 0,$$

which implies $k=2\alpha$, since by hypothesis $\lambda \neq (\frac{p}{2}+\frac{1}{2n+1})$.
If $k=2\alpha$, then from $\alpha^2 = -(k+1)$ we get $\alpha=-1$, and hence $k=-2$. Therefore,

from Lemma 3.1, we have $S^* = 0$. Thus the manifold is $*$-Ricci flat.
Again from Proposition 4.2 of [10], we have

$$R(X_\alpha, Y_\alpha)Z_\alpha = 0$$

and

$$R(X_{-\alpha}, Y_{-\alpha})Z_{-\alpha} = -4[g(Y_{-\alpha}, Z_{-\alpha})X_{-\alpha} - g(X_{-\alpha}, Z_{-\alpha})Y_{-\alpha}],$$

for any $X_\alpha, Y_\alpha, Z_\alpha \in [\alpha]'$ and $X_{-\alpha}, Y_{-\alpha}, Z_{-\alpha} \in [-\alpha]'$. Also noticing $\mu = -2$ it follows from Proposition 4.3 of [10] that $K(X, \xi) = -4$ for any $X \in [-\alpha]'$ and $K(X, \xi) = 0$ for any $X \in [\alpha]'$. Again from Proposition 4.3 of [10] we see that $K(X, Y) = -4$ for any $X, Y \in [-\alpha]'$ and $K(X, Y) = 0$ for any $X, Y \in [\alpha]'$. As is shown in [10] that the distribution $[\xi] \oplus [-\alpha]'$ is integrable with totally geodesic leaves and the distribution $[\alpha]'$ is integrable with totally umbilical leaves by $H = -(1+\alpha)\xi$, where H is the mean curvature tensor field for the leaves of $[\alpha]'$ immersed in M^{2n+1}. Here $\alpha = -1$, then the two orthogonal distributions $[\xi] \oplus [-\alpha]'$ and $[\alpha]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. □

Remark 3.5. If $\lambda = (\frac{p}{2} + \frac{1}{2n+1})$, then from (1.2), we can say that the $*$-conformal Ricci soliton reduces to a steady $*$-Ricci soliton. To discuss this situation we need the following well known definition.

Definition 3.6. On an almost contact metric manifold M, a vector field V is said to be Killing if $\pounds_V g = 0$ and an infinitesimal contact transformation if $\pounds_V \eta = f\eta$ for some smooth function f on M. In particular, if $f = 0$, then V is said to be strict infinitesimal contact transformation.

We consider the following two cases.
Case 1: If $k \neq -2$ and $\lambda = (\frac{p}{2} + \frac{1}{2n+1})$, then from (3.17), we have $(\pounds_V \eta)X = g(X, \pounds_V \xi)$. From this we can easily say that V will be an infinitesimal contact transformation if $\pounds_V \xi$ is parallel to ξ, that is, there is a smooth function f on M such that $\pounds_V \xi = f\xi$. But in view of (3.18), we have $\eta(\pounds_V \xi) = 0$, that is, $g(\pounds_V \xi, \xi) = 0$, which implies $\pounds_V \xi$ and ξ are orthogonal. Hence $\pounds_V \xi \neq f\xi$, for any smooth function f on M, unless $f = 0$ identically. Then V is an strict infinitesimal contact transformation if $\pounds_V \xi = 0$.
Case 2: If $k = -2$ and $\lambda = (\frac{p}{2} + \frac{1}{2n+1})$, then from (1.2), we have $\pounds_V g = 0$. Hence V is a Killing vector field.

4. Example

In [10], Dileo and Pastore give an example of a $(2n + 1)$-dimensional $(k, \mu)'$-almost Kenmotsu manifold which is connected but not compact. In [9], the authors obtained the following expressions for 5-dimensional case, when $k = -2$:

$$(\pounds_\xi g)(\xi, \xi) = (\pounds_\xi g)(e_4, e_4) = (\pounds_\xi g)(e_5, e_5) = 0$$

$$(\pounds_\xi g)(e_2, e_2) = (\pounds_\xi g)(e_3, e_3) = 4.$$

Also $k = -2$ implies that the manifold is locally isometric to $\mathbb{H}^3(-4) \times \mathbb{R}^2$ and $S^* = 0$, that is, the manifold is $*$-Ricci flat.
Considering $V = \xi$ and tracing (1.2), we obtain $\lambda = \frac{p+2}{2}$. Hence $(g, \xi, \frac{p+2}{2})$ is a $*$-conformal Ricci soliton on M. This verifies our theorem 3.4.

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