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On the Tensor Product of m-Partition Algebras

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Abstract. We study the tensor product algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$, where $P_k(x)$ is the partition algebra defined by Jones and Martin. We discuss the centralizer of this algebra and corresponding Schur–Weyl dualities and also index the inequivalent irreducible representations of the algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ and compute their dimensions in the semisimple case. In addition, we describe the Bratteli diagrams and branching rules. Along with that, we have also constructed the RS correspondence for the tensor product of m-partition algebras which gives the bijection between the set of tensor product of m-partition diagram of $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ and the pairs of m-vacillating tableaux of shape $[\lambda] \in \Gamma_k^m$, $\Gamma_k^m = \{[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_m)|\lambda_i \in \Gamma_k, i \in \{1, 2, ..., m\}\}$ where $\Gamma_k = \{\lambda_i \vdash t|0 \leq t \leq k\}$. Also, we provide proof of the identity $(n_1 n_2 \cdots n_m)^k = \sum_{[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}} f^{[\lambda]} m_k^{[\lambda]}$ where $m_k^{[\lambda]}$ is the multiplicity of the irreducible representation of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ module indexed by $[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}$, where $f^{[\lambda]}$ is the degree of the corresponding representation indexed by $[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}$ and $[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m} = \{[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_m)|\lambda_i \in \Lambda^k_{n_i}, i \in \{1, 2, ..., m\}\}$ where $\Lambda^k_{n_i} = \{\mu = (\mu_1, \mu_2, ..., \mu_t) \vdash n_i|n_i - \mu_1 \leq k\}$.

1. Introduction

The partition algebras $P_k(x)$ have been defined by Martin [7] and by Jones [5] independently. The algebra was studied as Potts model in statistical mechanics and generalization of the Temperley–Lieb algebras. In [7, 8] the algebra appears implicity and in [9] it appears explicitly. Jones considered the algebra $P_k(n)$, as the symmetric group's centralizer algebra on $V^{\otimes k}$ (see [5]).

The G-vertex colored partition algebras $P_k(x, G)$ has been recently introduced in [11]. The algebra $P_k(n, G)$ realized as the centralizer algebras of the direct product

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group $G \times S_n$ which is a subgroup of $G \wr S_n$ on $W^{\otimes k}$, where $W = \mathbb{C}^{n|G|}$. In [12], they also studied the inequivalent irreducible representations and their dimensions.

The class partition algebra $P_k(x, y)$ have been studied recently by Kennedy [6] and further studied by Martin and Elgamal [10] . The algebra $P_k(n, m)$ realized as the centralizer algebra of $S_m \wr S_n$ act on $W^{\otimes k}$, where $W = \mathbb{C}^{nm}$ and W is permutation module for S_{nm}.

The RS correspondence for the partition algebra by Halverson and Lewandowski [4] provides the bijection between the set partitions and the pairs of vacillating tableaux.

In this paper, we demonstrate that the algebra $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ is the centralizer algebra of the direct product $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ on $W^{\otimes k}$, where $W = \mathbb{C}^{n_1 n_2 \cdots n_m}$. We use centralizer theory to study the semisimplicity of $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ and by using the representation theory of $P_k(x)$ (from, [1, 3, 5]) the index of the inequivalent irreducible representations of $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ is studied and their dimensions in the semisimple case is computed. In addition, the Bratteli diagrams and branching rules for the towers $P_{k-1}(x_1) \otimes P_{k-1}(x_2) \otimes \cdots \otimes P_{k-1}(x_m) \subseteq P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ are described.

The RS correspondence for the partition algebra by Halverson and Lewandowski [4] influenced us to construct the RS correspondence for the tensor product of m-partition algebras which provides the bijection between the set of tensor product of m-partition diagrams and the pairs of m-vacillating tableaux. The proof of the identity $(n_1 n_2 \cdots n_m)^k = \sum_{[\lambda] \in \Lambda^k_{n_1, n_2, \ldots, n_m}} f^{[\lambda]} m_k^{[\lambda]}$ where $m_k^{[\lambda]}$ is the multiplicity of the irreducible representation of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ module indexed by $[\lambda] \in \Lambda^k_{n_1, n_2, \ldots, n_m}$, where $f^{[\lambda]}$ is the degree of the corresponding representation indexed by $[\lambda] \in \Lambda^k_{n_1, n_2, \ldots, n_m}$ and $[\lambda] \in \Lambda^k_{n_1, n_2, \ldots, n_m} = \{[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_m) | \lambda_i \in \Lambda^k_{n_i}, i \in \{1, 2, ..., m\}\}$ where $\Lambda^k_{n_i} = \{\mu = (\mu_1, \mu_2, ..., \mu_t) \vdash n_i | n_i - \mu_1 \leq k\}$ is discussed by constructing a bijection between the sequence $((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)$ of m-tuples of numbers where $1 \leq a_i \leq n_1, 1 \leq b_i \leq n_2, ..., 1 \leq l_i \leq n_m$ and the pair $(T_{[\lambda]}, P_{[\lambda]})$ where $T_{[\lambda]}$ is standard m-tableau of shape $[\lambda]$ and $P_{[\lambda]}$ is m-vacillating tableau of shape $[\lambda]$.

2. Preliminaries

In this section, some basic definitions and results are discussed herewith.

Definition 2.1.([13, §2.1]) A partition of non-negative integers n is a sequence of non-negative integers $\beta = (\beta_1, \beta_2, ..., \beta_i)$ such that $\beta_1 \geq \beta_2 \geq ... \geq \beta_i \geq 0$ and $|\beta| = \beta_1 + \beta_2 + ... + \beta_i = n$. It is denoted by $\beta \vdash n$.

A Young diagram is a diagrammatic representation of a partition β as an array of n boxes with β_1 boxes in the first row, β_2 boxes in the second row and so on.

Definition 2.2. A *m-partition* of size n is an ordered m-tuple of partitions $[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_m)$ where $\lambda_1 \vdash n_1, \lambda_2 \vdash n_2, ..., \lambda_m \vdash n_m$ with $n_1 + n_2 + ... + n_m = n$. We denote it $[\lambda] \vdash_m n$ where m is number of partition of n and λ_i is the i^{th} component of $[\lambda]$.

Remark 2.3. $\lambda \vdash n$ denotes a single partition of n and $[\lambda] \vdash_m n$ denotes a m-partition of n.

A young diagram of a 3-partition of size 9 is as follows:

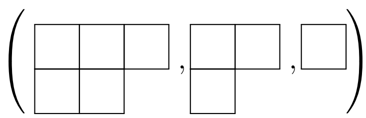


Figure 1: $(\lambda_1, \lambda_2, \lambda_3) = ([3, 2], [2, 1], [1])$

Definition 2.4.([13, 2.1.3]) Suppose $\lambda \vdash n$. A tableau of shape λ is an array t obtained by filling the boxes of the Young diagram of λ with the numbers 1, 2, . . . , n bijectively.

A tableau t is standard if the entries in the tableau increase along the rows from left to right and along the columns from top to bottom. Let $t_{i,j}$ stand for the entry of t in the position (i, j).

Definition 2.5. Suppose $[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_m) \vdash_m n$. A m-tableau of shape $[\lambda]$ is an m-tuple of array $[t] = (t_1, t_2, ..., t_m)$ obtained by filling the boxes of the each Young diagram of λ_i with the numbers 1, 2, . . . , n_i bijectively.

Definition 2.6. A m-tableau $[t]$ of shape $[\lambda]$ is standard if each t_i is standard tableau of shape λ_i.

Notation 2.7. Let $ST^m([\lambda]) = \{[t] \mid [t]$ is standard tableau of shape $[\lambda]\}$.

2.1. The partition algebra $P_k(x)$

A k-partition diagram is a simple graph of one above the other of two lines of k-vertices. The $2k$ vertices partitioned into l subsets, $1 \leq l \leq 2k$ by the connected components of a k-partition diagram. We state that two diagrams are equivalent when they determine the same partitions of $2k$ vertices.

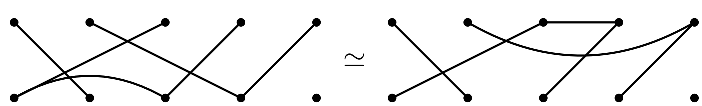


Figure 2: Two equivalent diagrams

When we are discussing about diagrams, we are really concerned about the associated equivalence classes. Define an equivalence classes of k-partition diagrams by stating that two classes are equivalent if they have same elements in any order.

Number the vertices $1, 2, ..., k$ in the upper line from left to right and $k+1, k+2, ..., 2k$ in the lower line from left to right in a k-partition diagram.

The field F will always represent a field of characteristic which is arbitrary throughout the paper and x represents a field element of the field F. The following is known as the product of two diagrams d and d' (see Figure 3):

1. Set d at the top and d' below it so that the lower line of d coincides with the upper line of d'.

2. Now, we have a diagram with upper line, middle line and lower line of vertices. This diagram is named as attachment of d and d'. Let the number of components that lie completely in the middle line is λ.

3. Make a new diagram d'' by deleting the vertices in the middle line but keeping the lower line and upper line and maintaining the connections between them. Replacing every "component" contained in the middle line with the variable x. That is, $d'd = x^{\lambda}d''$.

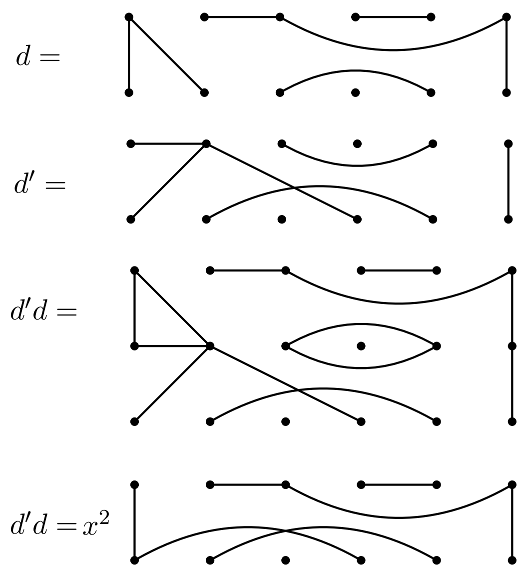


Figure 3: Product of two k-partition diagrams d and d'

This product is associative and well defined up to equivalence. Linearly extending this product makes the algebra $P_k(x)$ an associative algebra with identity.

The *partition algebra* $P_k(x)$ is the F-span of all k-partition diagrams for every x in the field F and a natural number k. The identity element is given by the partition diagram with every vertex in the upper line connected only to the vertex below it in the lower line. The dimension of the partition algebra $P_k(x)$ is the Bell number $B(2k)$, where

$$B(2k) = \sum_{l=1}^{l=2k} S(2k, l) \tag{2.1}$$

and where the number of equivalence relations with exactly l parts for a set of $2k$ elements is Stirling number $S(2k, l)$(see [14]). By convention, $P_0(x) = F$. Replacing the variable x by complex number ξ, we obtain a F-algebra $P_k(\xi)$.

Schur-Weyl Duality

We follow the notations, as given in [3]. Let $V = \mathbb{C}^n$, where V is the permutation module for S_n with standard basis $v_1, v_2, ..., v_n$. Then $\pi(v_i) = v_{\pi(i)}$, for $\pi \in S_n$ and $1 \leq i \leq n$. For every positive integer k, the tensor product space $V^{\otimes k}$ is a module for S_n with a standard basis given by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, where $1 \leq i_j \leq n$. The action of $\pi \in S_n$ on a basis vector is given by

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\pi(i_1)} \otimes v_{\pi(i_2)} \otimes \cdots \otimes v_{\pi(i_k)}. \tag{2.2}$$

For every diagram d and every integer sequence $i_1, i_2, ..., i_{2k}$ with $1 \leq i_s \leq n$, define

$$\psi(d)^{i_1, i_2, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} = \begin{cases} 1 & \text{if } i_r = i_s \text{ whenever vertices } s \text{ and } r \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Define the action of a diagram $d \in P_k(n)$ on $V^{\otimes k}$ by stating it on the standard basis as

$$d(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \leq i_{k+1}, \ldots, i_{2k} \leq n} \psi(d)^{i_1, i_2, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{2k}}. \tag{2.4}$$

Theorem 2.8.([5]). *$\mathbb{C}[S_n]$ and $P_k(n)$ generate full centralizers of each other in $End(V^{\otimes k})$. In particular, for $n \geq 2k$,*

(a) $P_k(n) \cong End_{S_n}(V^{\otimes k})$

(b) *S_n generates $End_{P_k(n)}(V^{\otimes k})$.*

2.2. The Irreducible Representations of $P_k(x)$

Double centralizer Theory

We follow the notations as in [1]. Let $\mathcal{A}$ be a *finite-dimensional* associative algebra over $\mathbb{C}$, the field of complex numbers. The algebra $\mathcal{A}$ is said to be *semi-simple* if

$$\mathcal{A} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathcal{M}_{d_\lambda}(\mathbb{C})$$

where $\mathcal{M}_{d_\lambda}(\mathbb{C})$ denotes full matrix algebras, $\widehat{\mathcal{A}}$ a finite index set and d_λ be any positive integer. Corresponding to every $\lambda \in \widehat{\mathcal{A}}$ there is a single irreducible $\mathcal{A}$-module, call it V^λ, which has dimension d_λ. If $\widehat{\mathcal{A}}$ is a singleton set then $\mathcal{A}$ is said to be simple. Maschke's Theorem (see [2]) says that if G is finite, $\mathbb{C}[G]$ is semisimple.

A finite dimensional $\mathcal{A}$-module M is *completely reducible* if it is the direct sum of irreducible $\mathcal{A}$-modules, i.e.,

$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} m_\lambda V^\lambda$$

where the non-negative integer m_λ is the multiplicity (dimension) of the irreducible $\mathcal{A}$-module V^λ in M (some of the m_λ may be zero). Wedderburn's Theorem (see [2]) discuss that for $\mathcal{A}$ being semi-simple every $\mathcal{A}$ is completely reducible.

The algebra $\mathrm{End}(M)$ comprises of all $\mathbb{C}$-linear transformations on M, where the composition of transformations is the algebra multiplication. If the representation $\rho : \mathcal{A} \to \mathrm{End}(M)$ is injective we say that M is $faithful$ $\mathcal{A}$-module. The *centralizer algebra* of $\mathcal{A}$ on M denoted $\mathrm{End}_{\mathcal{A}}(M)$, is the subalgebra of $\mathrm{End}(M)$ comprising of all operators that commute with the $\mathcal{A}$-action:

$$End_{\mathcal{A}}(M) = \{T \in End(M) \mid T\rho(a) \cdot m = \rho(a)T \cdot m, \forall a \in \mathcal{A}, m \in M\}.$$

If M is irreducible, then Schur's Lemma says that $\mathrm{End}_{\mathcal{A}}(M) \cong \mathbb{C}$. If G is a finite group and M is a G-module, then we often write $\mathrm{End}_G(M)$ in place of $\mathrm{End}_{\mathbb{C}[G]}(M)$.

Theorem 2.9. Double centralizer Theorem(see [2]).
Suppose that $\mathcal{A}$ and M decomposes as above. Then

(a)

$$End_{\mathcal{A}}(M) \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathcal{M}_{m_\lambda}(\mathbb{C}).$$

(b) *As an $End_{\mathcal{A}}(M)$-module,*

$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} d_\lambda U^\lambda,$$

where dim $U^\lambda = m_\lambda$ and U^λ is an irreducible module for $End_{\mathcal{A}}(M)$ when $m_\lambda > 0$.

(c) *As $\mathcal{A} \otimes End_{\mathcal{A}}(M)$-bimodule,*

$$M \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}} \text{ such that } m_\lambda \neq 0} V^\lambda \otimes U^\lambda.$$

(d) *$\mathcal{A}$ generates $End_{End_{\mathcal{A}}(M)}(M)$.*

This theorem tells us that if $\mathcal{A}$ is semisimple then so is $\mathrm{End}_{\mathcal{A}}(M)$. It also says that the set $\widehat{\mathcal{A}}_M = \{m_\lambda \in \widehat{\mathcal{A}} | m_\lambda > 0\}$ indexes all the irreducible representations of $\mathrm{End}_{\mathcal{A}}(M)$. Finally, we see from this theorem that the roles of multiplicity and dimension are interchanged when M is viewed as an $\mathrm{End}_{\mathcal{A}}(M)$- module as against the $\mathcal{A}$-module. When the hypothesis of the above theorem are satisfied, we say that $\mathcal{A}$ and $\mathrm{End}_{\mathcal{A}}(M)$ generate full centralizers of each other in M. This is often called *Schur-Weyl Duality* between $\mathcal{A}$ and $\mathrm{End}_{\mathcal{A}}(M)$.

Branching Rules

Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple algebras and where $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let M be a $\mathcal{A}$-module and M can be viewed as $\mathcal{B}$-module by restricting the action of $\mathcal{A}$ on M to $\mathcal{B}$. This $\mathcal{B}$-module is called the *restriction* of M from $\mathcal{A}$ to $\mathcal{B}$ and is denoted $M \downarrow^{\mathcal{A}}_{\mathcal{B}}$. On other side, let N be a $\mathcal{B}$-module. A $\mathcal{A}$-module produced from a $\mathcal{B}$-module is called *induction* of N from $\mathcal{B}$ to $\mathcal{A}$ and is denoted $N \uparrow^{\mathcal{A}}_{\mathcal{B}}$. Let $\{V^\lambda\}_{\lambda \in \widehat{\mathcal{A}}}$ denote the irreducible $\mathcal{A}$-modules and $\{\check{V}^\mu\}_{\mu \in \widehat{\mathcal{B}}}$ denote the irreducible $\mathcal{B}$-modules. The decomposition

$$V^\lambda \downarrow^{\mathcal{A}}_{\mathcal{B}} = \bigoplus_{\mu \in \widehat{\mathcal{B}}} g_{\lambda\mu} \check{V}^\mu,$$

where the $g_{\lambda\mu}$ are non-negative integers are called the (restriction) *branching rule* for $\mathcal{B} \subseteq \mathcal{A}$. Frobenius reciprocity (see [2]) tells us that

$$\check{V}^\mu \uparrow^{\mathcal{A}}_{\mathcal{B}} = \bigoplus_{\lambda \in \widehat{\mathcal{A}}} g_{\lambda\mu} V^\lambda.$$

Proposition 2.10. (Branching rule for $End_G(M^{\otimes(k-1)}) \subseteq End_G(M^{\otimes k})$).
Let G be a finite group and $\rho : \mathbb{C}[G] \to End(M)$ be a representation of G. Let $M^{\otimes k}$ denote the k-fold tensor product of M and $\{V^\lambda\}_{\lambda \in \widehat{\mathcal{G}}_k}$ denote the irreducible G-modules that appear in $M^{\otimes k}$ where $\widehat{\mathcal{G}}_k$ indexes the irreducible G-modules that appear in $M^{\otimes k}$. $\{U^\lambda_k\}_{\lambda \in \widehat{\mathcal{G}}_k}$ denote the irreducible $End_G(M^{\otimes k})$-modules that appear in $M^{\otimes k}$. View the algebra $End_G(M^{\otimes(k-1)})$ as a subalgebra of $End_G(M^{\otimes k})$ by identifying it with the subalgebra $End_G(M^{\otimes(k-1)}) \otimes id$, with $id \in End_G(M)$, the identity transformation. For V^μ a summand of $M^{\otimes(k-1)}$ consider that as a G-module

$$V^\mu \otimes M = \bigoplus_{\lambda \in \widehat{\mathcal{G}}_k} g_{\mu\lambda} V^\lambda.$$

Suppose further that

$$U^\lambda_k \downarrow^{End_G(M^{\otimes k})}_{End_G(M^{\otimes(k-1)})} = \bigoplus_{\mu \in \widehat{\mathcal{G}}_k - 1} g'_{\lambda\mu} U^\mu_{k-1}.$$

Then $g_{\mu\lambda} = g'_{\lambda\mu}$ for all μ and λ.

Theorem 2.11. (see [5]). *Let S^λ be an irreducible S_n-module and let V denote the permutation representation of S_n. Then*

$$S^\lambda \otimes V \cong (S^\lambda \downarrow^{S_n}_{S_{n-1}}) \uparrow^{S_n}_{S_{n-1}} \cong \bigoplus_{\mu=(\lambda^-)^+} S^\mu,$$

where $(\lambda^-)^+$ denotes a partition of n obtained by removing a box from λ and then adding a box.

Definition 2.12. The *Bratteli diagram* is a graph which contains a rows of vertices with the rows labeled by $0, \frac{1}{2}, 1, 1\frac{1}{2}, ..., k$ where vertices in a row i and $i+\frac{1}{2}$ are from the index sets Λ^i_n and Λ^i_{n-1} respectively. There is a edge between two vertices when they are in consecutive rows and they differing by one box.

Proposition 2.13. (see [1]). (Branching rule for $P_{k-1}(n) \subseteq P_k(n)$).
The lines in the $(S_n, P_k(n))$-Bratteli diagram when read upward from row k to $k-1$, provides the restriction branching rule, the lines downward gives the induction branching rule $P_{k-1}(n) \subseteq P_k(n)$. In particular, for $n \geq 2k$,

$$P^\lambda \downarrow^{P_k(n)}_{P_{k-1}(n)} = \bigoplus_{\mu=(\lambda^-)^+, n-\lambda_1 \leq k-1} P^\mu,$$

and

$$P^\mu \uparrow^{P_k(n)}_{P_{k-1}(n)} = \bigoplus_{\lambda=(\mu^-)^+} P^\lambda.$$

3. The Tensor Product of m-Partition Algebras

3.1. The tensor product partition algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$

In this subsection, the structure of the tensor product partition algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$, where $x_1, x_2, ..., x_m \in F$ are discussed.
Consider the tensor product partition algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$. Note that the standard basis for this algebra is

$$\mathcal{T}_k := \{(d_1 \otimes d_2 \otimes \cdots \otimes d_m) |\ d_1, d_2, ..., d_m \text{ are } k\text{-partition diagrams } \}$$

and the dimension is $[B(2k)]^m$.

Let $(d'_1 \otimes d'_2 \otimes \cdots \otimes d'_m), (d''_1 \otimes d''_2 \otimes \cdots \otimes d''_m) \in \mathcal{T}_k$, then $(d''_1 \otimes d''_2 \otimes \cdots \otimes d''_m)(d'_1 \otimes d'_2 \otimes \cdots \otimes d'_m) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$, where $d''_1 d'_1 = x_1^{\lambda_1} d_1$ in $P_k(x_1), d''_2 d'_2 = x_2^{\lambda_2} d_2$ in $P_k(x_2), ..., d''_m d'_m = x_m^{\lambda_m} d_m$ in $P_k(x_m)$. Thus the product of any two element in $\mathcal{T}_k$ is a scalar product of some element in $\mathcal{T}_k$. Hence, the extension of partition algebras are defined to be the F-span of the tensor product of m-partition diagrams with identity.

3.2. Two bases for $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$

In this subsection, two bases for $End_{S_{n_1} \times .... \times S_{n_m}}(W^{\otimes k})$, where $W = \mathbb{C}^{n_1 n_2 \cdots n_m}$ are discussed.

$$\text{Let } W = \text{Span}_{\mathbb{C}}\{v_{(i,j,...,s)} |\ 1 \leq i \leq n_1, 1 \leq j \leq n_2, ..., 1 \leq s \leq n_m\}. \tag{3.1}$$

The action of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ on W is defined as

$$(\pi_1, \pi_2, ..., \pi_m)(v_{(i,j,...,s)}) = v_{(\pi_1(i), \pi_2(j), ..., \pi_m(s))}. \tag{3.2}$$

Note that when $n_i = 1$, for all $i \in \{2, 3, ..., m\}$, $S_{n_1} \times S_{n_2} \times .... \times S_{n_m} \cong S_{n_1}$; in this case W specializes to V_1, the permutation representation of S_{n_1}.

$$\text{Let } \mathbb{S} := \{1, 2, ..., n_1\} \times \{1, 2, ..., n_2\} \times .... \times \{1, 2, ..., n_m\} \tag{3.3}$$

be an index set for the basis of W and $I = ((i_1, j_1, ..., s_1), (i_2, j_2, ..., s_2), ..., (i_k, j_k, ..., s_k))$, $J = ((i_{k+1}, j_{k+1}, ..., s_{k+1}), (i_{k+2}, j_{k+2}, ..., s_{k+2}), ..., (i_{2k}, j_{2k}, ..., s_{2k}))$ in $\mathbb{S}^k$. The action of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ on $\mathbb{S}$ by $(\pi_1, \pi_2, ..., \pi_m)(i, j, ..., s) = (\pi_1(i), \pi_2(j), ..., \pi_m(s))$ can be extended to an action on $\mathbb{S}^{2k}$ by $(\pi_1, \pi_2, ..., \pi_m)(I, J) = ((\pi_1, \pi_2, ..., \pi_m)(I), (\pi_1, \pi_2, ..., \pi_m)(J))$.

Diagonally extend the action of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ on W to an action of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ on $W^{\otimes k}$ as follows:

$$\begin{aligned}(\pi_1, \pi_2, ..., \pi_m)&(v_{(i_1, j_1, ..., s_1)} \otimes \cdots \otimes v_{(i_k, j_k, ..., s_k)}) \\ &= v_{(\pi_1(i_1), ..., \pi_m(s_1))} \otimes \cdots \otimes v_{(\pi_1(i_k), ..., \pi_m(s_k))}\end{aligned} \tag{3.4}$$

We will write the above as $(\pi_1, \pi_2, ..., \pi_m)(v_I) = v_{(\pi_1, \pi_2, ..., \pi_m)(I)}$.
Let $A \in \text{End}(W^{\otimes k})$. Define $A(v_J) = \sum_I A_I^J(v_I)$, where $I, J \in \mathbb{S}^k$ and $A_I^J \in \mathbb{C}$ is the $(I, J)^{th}$ entry of A and v_I is a basis element of $W^{\otimes k}$.

The following is our analog of Jones's result.

Lemma 3.1. $A \in End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{(\pi_1, \pi_2, ..., \pi_m)(I)}^{(\pi_1, \pi_2, ..., \pi_m)(J)}$, $\forall$ $(\pi_1, \pi_2, ..., \pi_m) \in S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$.

Proof. We have $A \in End_{S_{n_1} \times S_{n_2} \times \dots \times S_{n_m}}(W^{\otimes k})$

$$\begin{aligned}
&\Leftrightarrow (\pi_1, \pi_2, ..., \pi_m)A = A(\pi_1, \pi_2, ..., \pi_m), \ \forall \ (\pi_1, \pi_2, ..., \pi_m) \in S_{n_1} \times S_{n_2} \times ... \times S_{n_m} \\
&\Leftrightarrow (\pi_1, \pi_2, ..., \pi_m)A(v_J) = A(\pi_1, \pi_2, ..., \pi_m)(v_J), \ \forall \ v_J \\
&\Leftrightarrow (\pi_1, \pi_2, ..., \pi_m) \sum_I A_I^J(v_I) = A(v_{(\pi_1, \pi_2, ..., \pi_m)(J)}) \\
&\Leftrightarrow \sum_I A_I^J(\pi_1, \pi_2, ..., \pi_m)(v_I) = \sum_I A_I^{(\pi_1, \pi_2, ..., \pi_m)(J)}(v_I) \\
&\Leftrightarrow \sum_I A_I^J(v_{(\pi_1, \pi_2, ..., \pi_m)(I)}) = \sum_I A_{(\pi_1, \pi_2, ..., \pi_m)(I)}^{(\pi_1, \pi_2, ..., \pi_m)(J)}(v_{(\pi_1, \pi_2, ..., \pi_m)(I)})
\end{aligned}$$

since the action of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ is by the permutation representation. The result follows from equating the scalars and linearly independence. □

Lemma 3.2.

$$dim \ End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k}) = \sum_{i=1, j=1, ..., s=1}^{i=n_1, j=n_2, ..., s=n_m} S(2k, i)S(2k, j) \cdots S(2k, s).$$

$$when \ n_1, n_2, ..., n_m \geq 2k, \quad dim \ End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k}) = [B(2k)]^m.$$

Proof. By lemma 3.1. A commutes with the $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-action on $W^{\otimes k}$ if and only if the matrix entries of A are equal on $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbits. Thus, dim $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$ is the number of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbits on $\mathbb{S}^{2k}$. Fix a tuple of indices $(I, J) = ((i_1, j_1, ..., s_1), (i_2, j_2, ..., s_2), ..., (i_{2k}, j_{2k}, ..., s_{2k})) \in \mathbb{S}^{2k}$ which determine the partitions $d_1 := \bar{d}(i_1, i_2, ..., i_{2k}), d_2 := \bar{d}(j_1, j_2, ..., j_{2k}), ...,$ $d_m := \bar{d}(s_1, s_2, ..., s_{2k})$ of $\{1, ..., 2k\}$ (into at most $n_1, n_2, ..., n_m$ subsets respectively) according to those that have an equal value. Let $[(I, J)]$ be the orbit of $(I, J) \in \mathbb{S}^{2k}$. Then $(I', J') \in [(I, J)]$

$\Leftrightarrow$ $(I', J') = (\pi_1, \pi_2, ..., \pi_m)(I, J)$, for some $(\pi_1, \pi_2, ..., \pi_m) \in S_{n_1} \times S_{n_2} \times ... \times S_{n_m}$

$\Leftrightarrow$ $(i'_r, j'_r, ..., s'_r) = (\pi_1, \pi_2, ..., \pi_m)(i_r, j_r, ..., s_r)$, $\forall$ r such that $1 \leq r \leq 2k$, where $(i'_r, j'_r, ..., s'_r)$ and $(i_r, j_r, ..., s_r)$ are the r^{th} component of (I', J') and (I, J) respectively.

$\Leftrightarrow$ $(i'_r, j'_r, ..., s'_r) = (\pi_1(i_r), \pi_2(j_r), ..., \pi_m(s_r))$

$\Leftrightarrow$ $i'_r = \pi_1(i_r), j'_r = \pi_2(j_r), ..., s'_r = \pi_m(s_r)$

(3.5)

$\Leftrightarrow$ $[i_p = i_q \text{ iff } i'_p = i'_q], [j_p = j_q \text{ iff } j'_p = j'_q], ..., [s_p = s_q \text{ iff } s'_p = s'_q], (1 \leq p, q \leq 2k)$

$\Leftrightarrow$ $\bar{d}(i_1, i_2, ..., i_{2k}) = \bar{d}(i'_1, i'_2, ..., i'_{2k}), \bar{d}(j_1, j_2, ..., j_{2k}) = \bar{d}(j'_1, j'_2, ..., j'_{2k}), ...,$ $\bar{d}(s_1, s_2, ..., s_{2k}) = \bar{d}(s'_1, s'_2, ..., s'_{2k}).$

Thus, every $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbits determine the partitions $d_1, d_2, ..., d_m$ of the set of $2k$ elements and vice-versa. Hence, the result. $\square$

For a fixed tuple of indices $(I, J) \in \mathbb{S}^{2k}$, define the matrix $E_J^I \in End(W^{\otimes k})$ to be the $(n_1 n_2 \cdots n_m)^k \times (n_1 n_2 \cdots n_m)^k$ matrix with a 1 in the (I, J)-position and zero elsewhere. For every $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbit $[(I, J)]$, we define a matrix $T_J^I \in End(W^{\otimes k})$ by

$$T_J^I = \sum_{(I', J') \in [(I,J)]} E_{J'}^{I'},$$

In fact, $T_J^I \in End(W^{\otimes k})$, since such a matrix satisfies the Lemma 3.1. condition: The entries of the matrix are equal on $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbits. By using the equation (3.5), we obtained

$$(3.6) \quad T^{(i_1,j_1,...,s_1),...,(i_k,j_k,...,s_k)}_{(i_{k+1},j_{k+1},...,s_{k+1}),...,(i_{2k},j_{2k},...,s_{2k})} = \sum E^{(i'_1,j'_1,...,s'_1),...,(i'_k,j'_k,...,s'_k)}_{(i'_{k+1},j'_{k+1},...,s'_{k+1}),...,(i'_{2k},j'_{2k},...,s'_{2k})},$$

where the sum is over $i_p = i_q \Leftrightarrow i'_p = i'_q, j_p = j_q \Leftrightarrow j'_p = j'_q, ..., s_p = s_q \Leftrightarrow s'_p = s'_q, (1 \leq p, q \leq 2k)$.

Since every matrix T_J^I is the sum of different matrix units, the set $\{T_J^I \mid [(I, J)]$is an $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbit$\}$ is linearly independent set.

For $A \in End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$, we obtain $A = \sum_{[(I,J)]} A_J^I T_J^I$ by using the lemma 3.1. Thus, the matrices T_J^I span $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$ and so they are a basis for $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$.

Definition 3.3. Let $\bar{d}$ and $\bar{d}'$ be partitions of $[2k]$. We say that $\bar{d}'$ is coarser than $\bar{d}$ if any class in $\bar{d}$ is contained in some class in $\bar{d}'$. In this case we write $\bar{d}' \leq \bar{d}$.

Now, we state another basis for $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$ as follows: Define for every $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbit $[(I, J)] = [(i_1, j_1, ..., s_1), ..., (i_{2k}, j_{2k}, ..., s_{2k})]$, the matrix

$$L_J^I = \sum T_{J'}^{I'},$$

where the sum is over $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbit $[(I', J')] = [(i'_1, j'_1, ..., s'_1), ..., (i'_{2k}, j'_{2k}, ..., s'_{2k})]$ such that $\bar{d}(i_1, i_2, ..., i_{2k}) \geq \bar{d}(i'_1, i'_2, ..., i'_{2k}), ..., \bar{d}(s_1, s_2, ..., s_{2k}) \geq \bar{d}(s'_1, s'_2, ..., s'_{2k})$. The matrix T_J^I can be expressed in terms of the matrix L_J^I by using Möbius inversion (see [14]). So they also span $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$. By using the equation (3.6), we obtain

$$(3.7) \quad L^{(i_1,j_1,...,s_1),...,(i_k,j_k,...,s_k)}_{(i_{k+1},j_{k+1},...,s_{k+1}),...,(i_{2k},j_{2k},...,s_{2k})} = \sum E^{(i'_1,j'_1,...,s'_1),...,(i'_k,j'_k,...,s'_k)}_{(i'_{k+1},j'_{k+1},...,s'_{k+1}),...,(i'_{2k},j'_{2k},...,s'_{2k})},$$

where the sum is over $i_p = i_q \Rightarrow i'_p = i'_q, j_p = j_q \Rightarrow j'_p = j'_q, ..., s_p = s_q \Rightarrow s'_p = s'_q, (1 \leq p, q \leq 2k)$. The matrices T_J^I and L_J^I form two different basis for $End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$.

Note: For a given tuple $(i_1, i_2, ..., i_{2k}) \in \{1, 2, ..., n\}^{\times 2k}$ collect the numbers $i_1, i_2, ..., i_{2k}$ into (at most n) subsets then i_p and i_q are in the same subset if and only if $i_p = i_q$. This determines the relation $\sim$ on $\{1, 2, ..., 2k\}$, i.e., $p \sim q$ if and only if i_p and i_q are in the same subset. Naturally this relation in turn determines a partition $d = d(i_1, i_2, ..., i_{2k})$ of $\{1, 2, ..., 2k\}$ into subsets.

3.3. Schur-Weyl Duality

An action of $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ on $W^{\otimes k}$ is defined as follows: Define a map $\bar{\phi} : P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) \longrightarrow End(W^{\otimes k})$ by defining it on a basis element $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ as follows:

$$\begin{aligned}\bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m) &= \left(\bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)^{(i_1, j_1, \ldots, s_1), \ldots, (i_k, j_k, \ldots, s_k)}_{(i_{k+1}, j_{k+1}, \ldots, s_{k+1}), \ldots, (i_{2k}, j_{2k}, \ldots, s_{2k})}\right) \\ &= \left(\psi(d_1)^{i_1, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} \psi(d_2)^{j_1, \ldots, j_k}_{j_{k+1}, \ldots, j_{2k}} \cdots \psi(d_m)^{s_1, \ldots, s_k}_{s_{k+1}, \ldots, s_{2k}}\right),\end{aligned}$$

where ψ is defined as in equation (2.3). Alternatively, in terms of matrix units we have

$$\bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m) = \sum_{\substack{p \sim q \ in \ d_1 \Rightarrow i_p = i_q \\ p \sim q \ in \ d_2 \Rightarrow j_p = j_q \\ \vdots \\ p \sim q \ in \ d_m \Rightarrow s_p = s_q}} E^{(i_1, j_1, \ldots, s_1), \ldots, (i_k, j_k, \ldots, s_k)}_{(i_{k+1}, j_{k+1}, \ldots, s_{k+1}), \ldots, (i_{2k}, j_{2k}, \ldots, s_{2k})} \tag{3.8}$$

where $1 \leq i_1, i_2, ..., i_{2k} \leq n_1, 1 \leq j_1, j_2, ..., j_{2k} \leq n_2, ..., 1 \leq s_1, s_2, ..., s_{2k} \leq n_m$. Then, we have an action of $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ on $W^{\otimes k}$ defined by

$$(d_1 \otimes d_2 \otimes \cdots \otimes d_m)(v_J) = \bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)(v_J), \qquad \text{for all } J \in \mathbb{S}^k.$$

when $n_1 = n$ and $n_i = 1$, for all $i \in \{2, 3, ..., m\}$, this action restricted to the partition algebra coincides with the action defined by Jones [5] on tensors.

Thus, we have an action of a basis element $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ on $W^{\otimes k}$ by defining it on the standard basis element by

$$(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \cdot (v_{(i_1, j_1, \ldots, s_1)} \otimes \cdots \otimes v_{(i_k, j_k, \ldots, s_k)})$$

$$= \sum_{\substack{1 \leq i_{k+1}, \ldots, i_{2k} \leq n_1 \\ 1 \leq j_{k+1}, \ldots, j_{2k} \leq n_2 \\ \vdots \\ 1 \leq s_{k+1}, \ldots, s_{2k} \leq n_m}} \psi(d_1)^{i_1, i_2, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} \cdots \psi(d_m)^{s_1, s_2, \ldots, s_k}_{s_{k+1}, \ldots, s_{2k}} (v_{(i_{k+1}, \ldots, s_{k+1})} \otimes \cdots \otimes v_{(i_{2k}, \ldots, s_{2k})}).$$

Lemma 3.4. *The map $\bar{\phi} : P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) \longrightarrow End_{S_{n_1} \times \ldots \times S_{n_m}}(W^{\otimes k})$ is an algebra homomorphism.*

Proof. From (3.8) we have,

$$\bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m) = \sum_{\substack{\bar{d}(i_1,i_2,...,i_{2k}) \leq d_1 \\ \bar{d}(j_1,j_2,...,j_{2k}) \leq d_2 \\ \vdots \\ \bar{d}(s_1,s_2,...,s_{2k}) \leq d_m}} E^{(i_1,j_1,...,s_1),...,(i_k,j_k,...,s_k)}_{(i_{k+1},j_{k+1},...,s_{k+1}),...,(i_{2k},j_{2k},...,s_{2k})}, \tag{3.9}$$

where $1 \leq i_1, i_2, ..., i_{2k} \leq n_1, 1 \leq j_1, j_2, ..., j_{2k} \leq n_2, ..., 1 \leq s_1, s_2, ..., s_{2k} \leq n_m$.

$$= \sum_{\substack{\bar{d}(i_1,i_2,...,i_{2k}) \leq d_1 \\ \bar{d}(j_1,j_2,...,j_{2k}) \leq d_2 \\ \vdots \\ \bar{d}(s_1,s_2,...,s_{2k}) \leq d_m}} T^{(i_1,j_1,...,s_1),...,(i_k,j_k,...,s_k)}_{(i_{k+1},j_{k+1},...,s_{k+1}),...,(i_{2k},j_{2k},...,s_{2k})},$$

where the sum over one representative $(i_1, j_1, ..., s_1), (i_2, j_2, ..., s_2), ..., (i_{2k}, j_{2k}, ..., s_{2k})$ for one $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-orbit. Thus, $\bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in End_{S_{n_1} \times S_{n_2} \times .... \times S_{n_m}}(W^{\otimes k})$.

Claim: The map $\bar{\phi}$ is an algebra homomorphism.

Let $(d_1' \otimes d_2' \otimes \cdots \otimes d_m'), (d_1'' \otimes d_2'' \otimes \cdots \otimes d_m'') \in P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ and $(d_1'' \otimes d_2'' \otimes \cdots \otimes d_m'')(d_1' \otimes d_2' \otimes \cdots \otimes d_m') = n_1^{\lambda_1} n_2^{\lambda_2} \cdots n_m^{\lambda_m}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$, where $d_1'' d_1' = n_1^{\lambda_1} d_1$ in $P_k(n_1), d_2'' d_2' = n_2^{\lambda_2} d_2$ in $P_k(n_2), ..., d_m'' d_m' = n_m^{\lambda_m} d_m$ in $P_k(n_m)$. From (3.9), we have

$\bar{\phi}(d_1'' \otimes d_2'' \otimes \cdots \otimes d_m'')\bar{\phi}(d_1' \otimes d_2' \otimes \cdots \otimes d_m')$

$$= \sum_{\substack{\bar{d}(i_1'',...,i_{2k}'') \leq d_1'' \\ \vdots \\ \bar{d}(s_1'',...,s_{2k}'') \leq d_m''}} E^{(i_1'',j_1'',...,s_1''),...,(i_k'',j_k'',...,s_k'')}_{(i_{k+1}'',...,s_{k+1}''),...,(i_{2k}'',...,s_{2k}'')} \sum_{\substack{\bar{d}(i_1',...,i_{2k}') \leq d_1' \\ \vdots \\ \bar{d}(s_1',...,s_{2k}') \leq d_m'}} E^{(i_1',j_1',...,s_1'),...,(i_k',j_k',...,s_k')}_{(i_{k+1}',...,s_{k+1}'),...,(i_{2k}',...,s_{2k}')}$$

where $1 \leq i_z'', i_z' \leq n_1, 1 \leq j_z'', j_z' \leq n_2, ..., 1 \leq s_z'', s_z' \leq n_m$ and $1 \leq z \leq 2k$.

$$= \sum_{\substack{\bar{d}(i_1'',...,i_{2k}'') \leq d_1'',....,\bar{d}(s_1'',...,s_{2k}'') \leq d_m'' \\ \bar{d}(i_1',...,i_{2k}') \leq d_1',....,\bar{d}(s_1',...,s_{2k}') \leq d_m'}} \delta^{(i_1'',j_1'',...,s_1''),...,(i_k'',j_k'',...,s_k'')}_{(i_{k+1}',...,s_{k+1}'),...,(i_{2k}',...,s_{2k}')} E^{(i_1',j_1',...,s_1'),...,(i_k',j_k',...,s_k')}_{(i_{k+1}'',...,s_{k+1}''),...,(i_{2k}'',...,s_{2k}'')}$$

since $E_p^q E_r^s = \delta_{qr} E_p^s$, where δ_{qr} is the Kronecker delta.

$$= \sum_{\substack{\bar{d}(i_1'',...,i_{2k}'') \leq d_1'',....,\bar{d}(s_1'',...,s_{2k}'') \leq d_m'' \\ \bar{d}(i_1',...,i_{2k}') \leq d_1',....,\bar{d}(s_1',...,s_{2k}') \leq d_m'}} \delta^{i_1'',i_2'',...,i_k''}_{i_{k+1}',...,i_{2k}'} \cdots \delta^{s_1'',s_2'',...,s_k''}_{s_{k+1}',...,s_{2k}'} E^{(i_1',j_1',...,s_1'),...,(i_k',j_k',...,s_k')}_{(i_{k+1}'',...,s_{k+1}''),...,(i_{2k}'',...,s_{2k}'')}$$

$$= n_1^{\lambda_1} n_2^{\lambda_2} \cdots n_m^{\lambda_m} \sum_{\substack{\bar{d}(i_1,\ldots,i_{2k}) \le d_1 \\ \bar{d}(j_1,\ldots,j_{2k}) \le d_2 \\ \vdots \\ \bar{d}(s_1,\ldots,s_{2k}) \le d_m}} E^{(i_1,j_1,\ldots,s_1),\ldots,(i_k,j_k,\ldots,s_k)}_{(i_{k+1},\ldots,s_{k+1}),\ldots,(i_{2k},\ldots,s_{2k})}, \text{as in the partition case.}$$

$$= n_1^{\lambda_1} n_2^{\lambda_2} \cdots n_m^{\lambda_m} \bar{\phi}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$$
$$= \bar{\phi}((d_1'' \otimes d_2'' \otimes \cdots \otimes d_m'')(d_1' \otimes d_2' \otimes \cdots \otimes d_m')).$$

□

Theorem 3.5. $\mathbb{C}[S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}]$ *and* $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ *generate full centralizers of each other in* $End(W^{\otimes k})$. *In particular, for* $n_1, n_2, \ldots, n_m \ge 2k$,

(a) $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) \cong End_{S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}}(W^{\otimes k})$,

(b) $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$ *generates* $End_{P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)}(W^{\otimes k})$.

Proof. Since $n_1, n_2, \ldots, n_m \ge 2k$, $\dim P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) = \dim End_{S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}}(W^{\otimes k})$. Therefore, (a) follows from Lemma 3.1 and (b) follows from (a) and Double Centralizer Theorem. □

As the centralizer of the semisimple group algebra $\mathbb{C}[S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}]$, the $\mathbb{C}$-algebra $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ is semisimple for $n_1, n_2, \ldots, n_m \ge 2k$.

4. The Irreducible Representations of $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$

In this section, the inequivalent irreducible representations of the tensor product partition algebra $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ by using the representation theory of the partition algebra $P_k(x)$ (from [1, 3, 5]) and the centralizer theory is being indexed. Also, their dimensions are computed. When $n_1, n_2, \ldots, n_m \ge 2k$, the Bratteli diagrams and the branching rules for the tower $P_{k-1}(n_1) \otimes P_{k-1}(n_2) \otimes \cdots \otimes P_{k-1}(n_m) \subseteq P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ are described.

The $\mathbb{C}$-vector space $V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes \cdots \otimes V_m^{\otimes k}$ is a $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$-module under the action is given by

$(\pi_1, \pi_2, \ldots, \pi_m)((v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) \otimes (v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}) \otimes \cdots \otimes (v_{s_1} \otimes v_{s_2} \otimes \cdots \otimes v_{s_k})) = (v_{\pi_1(i_1)} \otimes v_{\pi_2(i_2)} \otimes \cdots \otimes v_{\pi_m(i_k)}) \otimes (v_{\pi_1(j_1)} \otimes v_{\pi_2(j_2)} \otimes \cdots \otimes v_{\pi_m(j_k)}) \otimes \cdots \otimes (v_{\pi_1(s_1)} \otimes v_{\pi_2(s_2)} \otimes \cdots \otimes v_{\pi_m(s_k)})$.

Lemma 4.1. *The index set of the irreducible* $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$*-modules appearing as summands in* $V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes \cdots \otimes V_m^{\otimes k}$ *is* $\widehat{P_k(n_1)} \times \widehat{P_k(n_2)} \times \ldots. \times \widehat{P_k(n_m)}$, *where* $\widehat{P_k(n_i)}$ *is the index set of the irreducible* S_{n_i}*-modules.*

Proof. The representation $V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes \cdots \otimes V_m^{\otimes k}$ of $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$ is the product representation of $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$ afforded by $V_1^{\otimes k}$ of $S_{n_1}, V_2^{\otimes k}$ of

$S_{n_2}, ..., V_m^{\otimes k}$ of S_{n_m}, where the representation $V_i^{\otimes k}$ of S_{n_i}, $i \in \{1, 2, ..., m\}$ is the tensor product permutation representation which is decomposed as (see § 2.2)

$$V_i^{\otimes k} \cong \bigoplus_{\lambda_i \in \widehat{P_k(n_i)}} m_{\lambda_i} S^{\lambda_i}$$

(where m_{λ_i} is the multiplicity of the irreducible S_{n_i}-module appearing as summands in $V_i^{\otimes k}$).
Hence, as $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-module

$$V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes \cdots \otimes V_m^{\otimes k} \cong \bigoplus_{\substack{\lambda_i \in \widehat{P_k(n_i)} \\ i \in \{1,2,...,m\}}} m_{\lambda_1} m_{\lambda_2} \cdots m_{\lambda_m} S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m},$$

where $S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m}$ is the irreducible $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-module induced by the irreducible S_{n_1}-module S^{λ_1}, S_{n_2}-module S^{λ_2}, ..., S_{n_m}-module S^{λ_m}. □

Theorem 4.2.

(a) *As a* $S_{n_1} \times S_{n_2} \times .... \times_{n_m}$*-module*

$$W^{\otimes k} \cong \bigoplus_{\substack{\lambda_i \in \widehat{P_k(n_i)} \\ i \in \{1,2,...,m\}}} m_{\lambda_1} m_{\lambda_2} \cdots m_{\lambda_m} S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m}.$$

(b) *For* $n_1, n_2, ..., n_m \geq 2k$,

$$P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) \cong \bigoplus_{\substack{\lambda_i \in \widehat{P_k(n_i)} \\ i \in \{1,2,...,m\}}} \mathcal{M}_{[\lambda_1, \lambda_2, ..., \lambda_m]}(\mathbb{C}),$$

where $[\lambda_1, \lambda_2, ..., \lambda_m] = m_{\lambda_1} m_{\lambda_2} \cdots m_{\lambda_m}$.

(c) *For* $n_1, n_2, ..., n_m \geq 2k,$ *as* $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$*-module*

$$W^{\otimes k} \cong \bigoplus_{\substack{\lambda_i \in \widehat{P_k(n_i)} \\ i \in \{1,2,...,m\}}} d^{\lambda_1} d^{\lambda_2} \cdots d^{\lambda_m} P^{\lambda_1, \lambda_2, ..., \lambda_m},$$

where d^{λ_i} *is the dimension of* S^{λ_i} *and* $P^{\lambda_1, \lambda_2, ..., \lambda_m}$ *is the irreducible* $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$*-module indexed by* $\lambda_1 \in \widehat{P_k(n_1)}, \lambda_2 \in \widehat{P_k(n_2)}, ..., \lambda_m \in \widehat{P_k(n_m)}$ *with dimension* $[\lambda_1, \lambda_2, ..., \lambda_m]$.

(d) *For $n_1, n_2, ..., n_m \geq 2k$, as a $\mathbb{C}[S_{n_1} \times S_{n_2} \times .... \times S_{n_m}] \otimes (P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m))$-bimodule,*

$$W^{\otimes k} \cong \bigoplus_{\substack{\lambda_i \in \widehat{P_k(n_i)} \\ i \in \{1,2,...,m\}}} (S^{\lambda_1,\lambda_2,...,\lambda_m} \otimes P^{\lambda_1,\lambda_2,...,\lambda_m}),$$

where $S^{\lambda_1,\lambda_2,...,\lambda_m} = S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m}$.

Proof. Since $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ acts on the suffix of $v_{(i,j,...,s)}$, we have the permutation representation V_i of S_{n_i} with respect to S_{n_i-1} for $i \in \{1, 2, ..., m\}$. Hence,

$$W \cong V_1 \otimes V_2 \otimes \cdots \otimes V_m.$$

Moreover,

$$W^{\otimes k} \cong (V_1 \otimes V_2 \otimes \cdots \otimes V_m)^{\otimes k} \cong V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes \cdots \otimes V_m^{\otimes k}.$$

(a) follows from lemma 4.1. $(b), (c)$ and (d) follows from theorems 2.9. and 3.5. □

Corollary 4.3. *Let $S^{\lambda_1,\lambda_2,...,\lambda_m}$ be an irreducible $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$-module and let W be the permutation representation of $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$. Then*

$$\begin{aligned} S^{\lambda_1,\lambda_2,...,\lambda_m} \otimes W &\cong (S^{\lambda_1,\lambda_2,...,\lambda_m} \downarrow^{S_{n_1} \times ... \times S_{n_m}}_{S_{n_1-1} \times ... \times S_{n_m-1}}) \uparrow^{S_{n_1} \times ... \times S_{n_m}}_{S_{n_1-1} \times ... \times S_{n_m-1}} \\ &\cong \bigoplus_{\substack{\mu_i = (\lambda_i^-)^+ \\ i \in \{1,2,...,m\}}} S^{\mu_1,\mu_2,...,\mu_m} \end{aligned}$$

where $(\lambda_i^-)^+$ denotes a partition of n_i obtained by removing a box from λ_i and then adding a new box.

Proof. This follows from theorems 4.2. and 2.11. □

From Corollary 4.3. the Bratteli diagram for $(S_{n_1} \times S_{n_2} \times .... \times S_{n_m}, P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m))$ as they act on $W^{\otimes k}$ is the tensor product of the Bratteli diagram for $(S_{n_1}, P_k(n_1))$ as they act on $V_1^{\otimes k}, (S_{n_2}, P_k(n_2))$ as they act on $V_2^{\otimes k}, ..., (S_{n_m}, P_k(n_m))$ as they act on $V_m^{\otimes k}$. Note that if $n_i = 1$ for $i \in \{1, 2, ..., m\}$ except one n_i then the Bratteli diagram for $(S_{n_1} \times S_{n_2} \times .... \times S_{n_m}, P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m))$ as they act on $W^{\otimes k}$ is the Bratteli diagram for $(S_{n_i}, P_k(n_i))$ as they act on $V_i^{\otimes k}$.

Now, we may write the Bratteli diagram for $S_{n_1} \times S_{n_2} \times .... \times S_{n_m}$ and $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ as they act on $W^{\otimes k}$ when $m = 2, n_1 = 4, n_2 = 4$ (see Figure 4).

For $k = 2$ and $m = 2, n_1 = 4, n_2 = 4$, from Figure 4: the dimensions of the irreducible $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$-modules $P^{\lambda_1,\lambda_2,...,\lambda_m}$ are

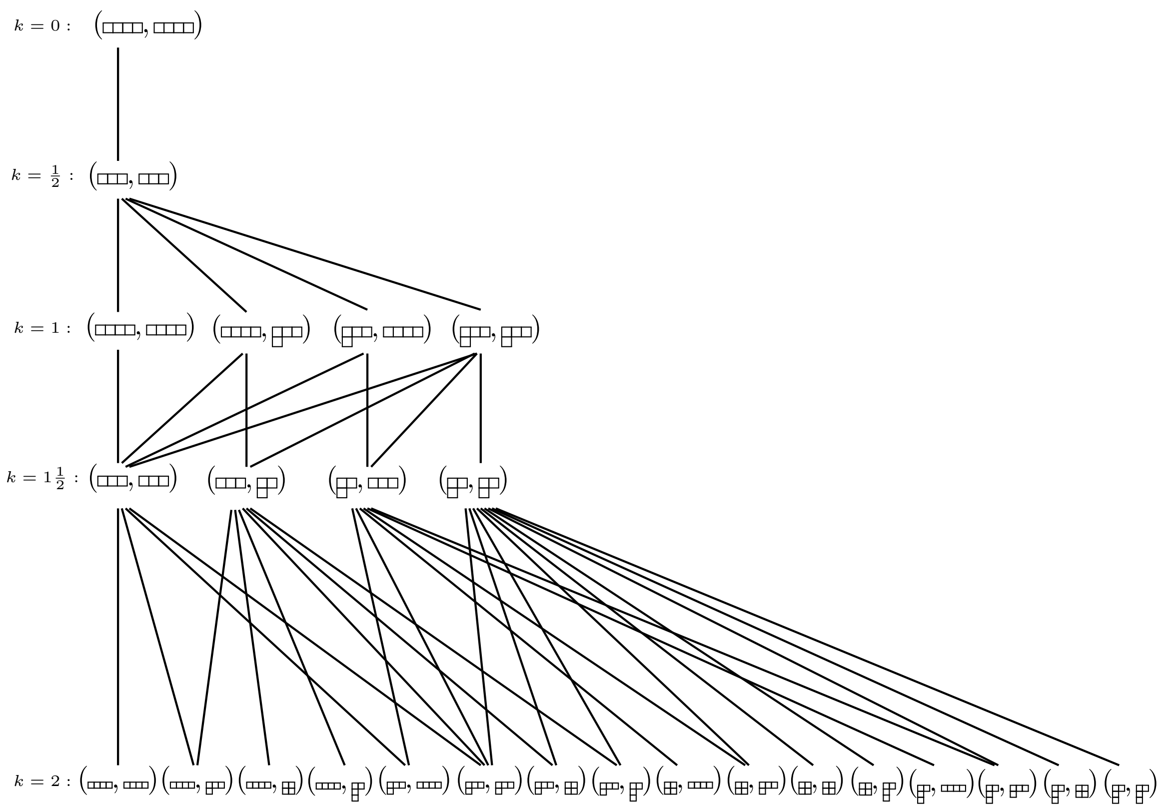


Figure 4: Bratteli diagram for $\mathbb{C}\{P_k(4) \otimes P_k(4)\}$

$4,6,2,2,6,9,3,3,2,3,1,1,2,3,1,1$ (which are multiplicity of the irreducible $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$-module $S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m}$ reading from left to right) and $4^2+6^2+2^2+2^2+6^2+9^2+3^2+3^2+2^2+3^2+1^2+1^2+2^2+3^2+1^2+1^2 = 225 = dim(P_2(4) \otimes P_2(4))$. The multiplicity of $P^{\lambda_1,\lambda_2,\ldots,\lambda_m}$ are $1,3,2,3,3,9,6,9,2,6,4,6,3,9,6,9$ (which are the dimensions of $S^{\lambda_1} \otimes S^{\lambda_2} \otimes \cdots \otimes S^{\lambda_m}$ respectively). Hence, the dimension of $W^{\otimes 2} = 16^2 = 256 = (1\times 4)+(3\times 6)+(2\times 2)+(3\times 2)+(3\times 6)+(9\times 9)+(6\times 3)+(9\times 3)+(2\times 2)+(6\times 3)+(4\times 1)+(6\times 1)+(3\times 2)+(9\times 3)+(6\times 1)+(9\times 1)$.

Proposition 4.4. (*Branching rule for* $P_{k-1}(n_1) \otimes P_{k-1}(n_2) \otimes \cdots \otimes P_{k-1}(n_m) \subseteq P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$).
The lines in the $(S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}, P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m))$*-bratteli diagram when read upward from row* k *to* $k-1$ *leads to the restriction branching rule, the lines downward leads to the induction branching rule for* $P_{k-1}(n_1) \otimes P_{k-1}(n_2) \otimes \cdots \otimes P_{k-1}(n_m) \subseteq P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$. *In particular, for* $n_1, n_2, \ldots, n_m \geq 2k$,

$$P^{\lambda_1,\lambda_2,\ldots,\lambda_m} \downarrow^{P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)}_{P_{k-1}(n_1)\otimes\cdots\otimes P_{k-1}(n_m)} = \bigoplus_{\substack{\mu_i=(\lambda_i^-)^+, n_i-\mu_i^1 \leq k-1\\ i\in\{1,2,\ldots,m\}}} P^{\mu_1,\mu_2,\ldots,\mu_m},$$

and

$$P^{\mu_1,\mu_2,\ldots,\mu_m} \uparrow^{P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)}_{P_{k-1}(n_1)\otimes\cdots\otimes P_{k-1}(n_m)} = \bigoplus_{\substack{\lambda_i=(\mu_i^-)^+\\ i\in\{1,2,\ldots,m\}}} P^{\lambda_1,\lambda_2,\ldots,\lambda_m}.$$

Proof. The proposition follows from proposition 2.10. and corollary 4.3. □

5. Vacillating Tableaux

Let

$$\Lambda^k_{n_1,n_2,\ldots,n_m} = \{[\lambda] = (\lambda_1,\lambda_2,\ldots,\lambda_m) | \lambda_i \in \Lambda^k_{n_i}, i \in \{1,2,\ldots,m\}\},$$
$$\Lambda^k_{n_1-1,n_2-1,\ldots,n_m-1} = \{[\lambda] = (\lambda_1,\lambda_2,\ldots,\lambda_m) | \lambda_i \in \Lambda^k_{n_i-1}, i \in \{1,2,\ldots,m\}\},$$
$$\Gamma^m_k = \{[\lambda] = (\lambda_1,\lambda_2,\ldots,\lambda_m) | \lambda_i \in \Gamma_k, i \in \{1,2,\ldots,m\}\}$$

where $\Lambda^k_{n_i} = \{\mu = (\mu_1,\mu_2,\ldots,\mu_t) \vdash n_i | n_i - \mu_1 \leq k\}$ and $\Gamma_k = \{\lambda_i \vdash t | 0 \leq t \leq k\}$.

Let $T_k^{[\lambda]}$ denote the irreducible $\mathbb{C}\{P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)\}$ representation indexed by $\Lambda^k_{n_1,n_2,\ldots,n_m}$. Since, the dimension of $T_k^{[\lambda]}$ equals the multiplicity of $V^{[\lambda]}$ in $V^{\otimes k}$.

Here, we discuss the vacillating tableau in the case of m-partitions following the procedure in [4] for partitions of n. The dimension of the irreducible $S_{n_1} \times S_{n_2} \times \ldots. \times S_{n_m}$ module $V^{[\lambda]}$ equals the number of standard m-tableaux of shape $[\lambda]$. We can identify a standard m-tableau $T_{[\lambda]}$ of shape $[\lambda]$ with a sequence $(\emptyset = [\lambda]^{(0)}, [\lambda]^{(1)}, \ldots, [\lambda]^{(n)} = [\lambda])$ of m-tableaux such that $|[\lambda]^{(i)}| = i, (i.e).|\lambda_l^{(i)}| = i$ for all

$l \in \{1, 2, ..., m\}, [\lambda]^{(i)} \subseteq [\lambda]^{(i+1)}$ and such that $[\lambda]^{(i)}/[\lambda]^{(i-1)}$ is the box containing i in $T_{[\lambda]}$. For example,

$$\left(\ytableaushort{134,25}\ ,\ \ytableaushort{125,34} \right) = \left((\emptyset, \emptyset), \left(\ydiagram{1}, \ydiagram{1}\right), \left(\ydiagram{1,1}, \ydiagram{2}\right), \left(\ydiagram{2,1}, \ydiagram{2,1}\right), \left(\ydiagram{3,1}, \ydiagram{2,2}\right), \left(\ydiagram{3,2}, \ydiagram{3,2}\right) \right).$$

Let $[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}$. A *$m$-vacillating tableaux* of shape $[\lambda]$ and length $2k$ is a sequence of m-partitions,

$$\left(((n_1), (n_2), ..., (n_m)) = [\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)} = [\lambda] \right),$$

satisfying for each i,

1. $[\lambda]^{(i)} \in \Lambda^i_{n_1,n_2,...,n_m}$ and $[\lambda]^{(i+\frac{1}{2})} \in \Lambda^i_{n_1-1,n_2-1,...,n_m-1}$,
2. $[\lambda]^{(i)} \supseteq [\lambda]^{(i+\frac{1}{2})}$ and $|[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}| = 1$,
3. $[\lambda]^{(i+\frac{1}{2})} \subseteq [\lambda]^{(i+1)}$ and $|[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}| = 1$.

The m-vacillating tableaux of shape $[\lambda]$ corresponds exactly with the paths from the top of the Bratteli diagram to $[\lambda]$. By the double centralizer theorem, we have $m_k^{[\lambda]} = \dim(T_k^{[\lambda]})$. Thus, if we let $VT_k^m([\lambda])$ denote the set of m-vacillating tableaux of shape $[\lambda]$ and length k then

$$m_k^{[\lambda]} = dim(T_k^{[\lambda]}) = |VT_k^m([\lambda])|$$

where $m_k^{[\lambda]}$ is the multiplicity of $V^{[\lambda]}$ in the decomposition of $V^{\otimes k}$ as a $S_{n_1} \times S_{n_2} \times ... \times S_{n_m}$ module.
Let $n_1, n_2, ..., n_m \geq 2k$. The sets $\Lambda^k_{n_1,n_2,...,n_m}$ and Γ^m_k are in bijection with one another using the maps,

$$\Lambda^k_{n_1,n_2,...,n_m} \to \Gamma^m_k \qquad\qquad \Gamma^m_k \to \Lambda^k_{n_1,n_2,...,n_m}.$$

via these bijections can be used either to Γ^m_k or $\Lambda^k_{n_1,n_2,...,n_m}$ so as to index the irreducible representations of $\mathbb{C}\{P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)\}$.

The following sequences represent the same m-vacillating tableau $P_{[\lambda]}$, the first one is obtained using the diagrams from $\Lambda^k_{n_1,n_2,...,n_m}$ and the second from Γ^m_k,

$$\begin{aligned} P_{[\lambda]} &= \left(\left(\ydiagram{4}, \ydiagram{4}\right), \left(\ydiagram{3}, \ydiagram{3}\right), \left(\ydiagram{3,1}, \ydiagram{3,1}\right), \left(\ydiagram{2,1}, \ydiagram{2,1}\right), \left(\ydiagram{2,2}, \ydiagram{2,1,1}\right) \right) \\ &= \left((\emptyset, \emptyset), (\emptyset, \emptyset), \left(\ydiagram{1}, \ydiagram{1}\right), \left(\ydiagram{1}, \ydiagram{1}\right), \left(\ydiagram{2}, \ydiagram{1,1}\right) \right) \end{aligned}$$

For our bijection, in section 6 we use $\Lambda^k_{n_1,n_2,...,n_m}$ and in section 7 we use Γ^m_k .

6. A Bijective Proof of $(n_1 n_2 \cdots n_m)^k = \sum_{[\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}} f^{[\lambda]} m_k^{[\lambda]}$

We follow the notations as given below:

1. $\tilde{n_i} = \{1, 2, ..., n_i\}$
2. $\tilde{k} = \{1, 2, ..., k\}$

To give a combinatorial proof of identity

$$(n_1 n_2 \cdots n_m)^k = \sum_{[\lambda]\in\Lambda^k_{n_1,n_2,\dots,n_m}} f^{[\lambda]} m_k^{[\lambda]}, \qquad \text{for } n_1, n_2, ..., n_m \geq 2k. \tag{6.1}$$

We need to find a bijection of the form

$$\{((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k))|a_q \in \tilde{n_1}, b_q \in \tilde{n_2}, ..., l_q \in \tilde{n_m}, q \in \tilde{k}\}$$
$$\longleftrightarrow \bigsqcup_{[\lambda]\in\Lambda^k_{n_1,n_2,\dots,n_m}} SYT^m([\lambda]) \times VT^m_k([\lambda]).$$

To do so, construct an invertible function that turns a sequence $((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k))$ of m-tuples of numbers in the range $1 \leq a_i \leq n_1, 1 \leq b_i \leq n_2, ..., 1 \leq l_i \leq n_m$ into a pair $(T_{[\lambda]}, P_{[\lambda]})$ consisting of a standard m-tableaux $T_{[\lambda]}$ of shape $[\lambda]$ and m-vacillating tableaux $P_{[\lambda]}$ of shape $[\lambda]$ and length $2k$ for some $[\lambda] \in \Lambda^k_{n_1,n_2,\dots,n_m}$.

Note: Here the RS insertion and reverse RS algorithm as in [4] is used. Also, we used the jeu de taquin in each component of the m-partition. If $T = (T_1, T_2, ..., T_m)$ is a standard m-tableau of shape $[\lambda] \vdash_m n$ and for $r \in \{1, 2, ..., m\}, T_r$ is a standard tableau of shape $\lambda_r \vdash n_r$ then jeu de taquin provides an algorithm for removing the box containing x_r from T_r and producing a standard tableau S_r of shape $\mu_r \vdash (n_r - 1)$ and entries from $\{1, 2, ..., n_r\}\backslash\{x_r\}$. Let $S = (S_1, S_2, ..., S_m)$ be the standard m-tableau and $S_r^{i,j}$ denotes the entry of S_r in row i and column j. We say that a box whose removal leaves the young diagram of a partition is *corner* of S_r. Thus, the corner of S_r are the boxes that are end of both the row and column. The following algorithm will delete x_r from T_r leaving a standard tableau S_r with x_r removed. We denote this process by $x_r \xleftarrow{\text{jdt}} T_r$.

1. Let $c = S_r^{i,j}$ be the box containing x_r.
2. While c is not a corner, do
   a. Let c' be the box containing $\min\{S_r^{i+1,j}, S_r^{i,j+1}\}$;
   b. Exchange the positions of c and c'.
3. Delete c.

If only one of $S_r^{i+1,j}, S_r^{i,j+1}$ exits at step $2.a$ then the minimum is taken to be the single value.

Let $S = (S_1, S_2, ..., S_m)$ be the standard m-tableau and S_r be a tableau of shape μ_r with $|\mu_r| < n_r$ and distinct entries from $\{1, 2, ..., n_r\}$. Let x_r be a positive integer that is not in S_r. The following algorithm insets x_r into S_r producing a standard tableau T_r of shape λ_r with $\mu_r \subseteq \lambda_r, |\lambda_r/\mu_r| = 1$ whose entries are the union of those from S and $\{x_r\}$. We denote this process by $x_r \xrightarrow{RS} S_r$.

1. Let R be the first row of S_r.
2. While x_r is less than some element in R, do

   a. Let y_r be the smallest element of R greater than x_r;

   b. Replace $y_r \in R$ with x_r;

   c. Let $x_r := y_r$ and let R be the next row.
3. Place x_r at the end of R (which is possibly empty).

It is possible to invert the process of insertion using the R-S reverse algorithm.

Theorem 6.1. *The function* $((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]})$ *provides a bijection between sequence of m-tuples in* $\{((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)) | 1 \leq a_i \leq n_1, 1 \leq b_i \leq n_2, ..., 1 \leq l_i \leq n_m\}$ *and* $\bigsqcup_{[\lambda]\in\Lambda^k_{n_1,n_2,...,n_m}} SYT^m([\lambda]) \times VT^m_k([\lambda])$ *and thus gives a combinatorial proof of* (6.1).

Proof. The proof is based on [4]. Given $(a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)$ with $1 \leq a_i \leq n_1, 1 \leq b_i \leq n_2, ..., 1 \leq l_i \leq n_m$, we will produce a pair $(T_{[\lambda]}, P_{[\lambda]}), [\lambda] \in \Lambda^k_{n_1,n_2,...,n_m}$, consisting of a standard m-tableau $T_{[\lambda]}$ and a m-vacillating tableau $P_{[\lambda]}$.

Let $T^{(j)} = (T^{(j)}_1, T^{(j)}_2, ..., T^{(j)}_m)$. First, we initialize the 0^{th} tableau to be the standard m-tableau of shape $(n_1), (n_2), ..., (n_m)$, namely,

$$T^{(0)} = (T^{(0)}_1, T^{(0)}_2, ..., T^{(0)}_m)$$

$$= \left(\boxed{1 \mid 2 \mid \cdots \mid n_1}\ ,\ \boxed{1 \mid 2 \mid \cdots \mid n_2}\ ,\ .\ .\ .\ .\ ,\ \boxed{1 \mid 2 \mid \cdots \mid n_m} \right)$$

Then recursively define standard m-tableau $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ for $0 \leq j \leq k-1$ by

$$T^{(j+\frac{1}{2})} = \left(T^{(j+\frac{1}{2})}_1 = a_{j+1} \xleftarrow{jdt} T^{(j)}_1, ..., T^{(j+\frac{1}{2})}_m = l_{j+1} \xleftarrow{jdt} T^{(j)}_m \right)$$

$$T^{(j+1)} = \left(T^{(j+1)}_1 = a_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}_1, ..., T^{(j+1)}_m = l_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}_m \right)$$

Let $[\lambda]^{(j)} \in \Lambda^j_{n_1,n_2,...,n_m}$ be the shape of $T^{(j)}$ and $[\lambda]^{(j+\frac{1}{2})} \in \Lambda^j_{n_1-1,n_2-1,...,n_m-1}$ be the shape of $T^{(j+\frac{1}{2})}$. Then let

$$P_{[\lambda]} = ([\lambda]^{(0)}, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k)}) \qquad \text{and} \qquad T_{[\lambda]} = T^{(k)}$$

so that $P_{[\lambda]}$ is a m-vacillating tableau of shape $[\lambda] = [\lambda]^{(k)} \in \Lambda^k_{n_1,n_2,...,n_m}$ and $T_{[\lambda]}$ is a standard m-tableau of the same shape $[\lambda]$. We denote this iterative process of deletion and insertion that associates the pair $(T_{[\lambda]}, P_{[\lambda]})$ to the sequence of m-tuples $(a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)$ by

$$((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]}).$$

Let $[\lambda]^{(j+\frac{1}{2})} \subseteq [\lambda]^{(j+1)}$ with $[\lambda]^{(j+1)} \in \Lambda^{j+1}_{n_1,n_2,...,n_m}, [\lambda]^{(j+\frac{1}{2})} \in \Lambda^{j}_{n_1-1,...,n_m-1}$ and $T^{(j+1)}$ be a standard m-tableau of shape $[\lambda]^{(j+1)}$. We can uniquely determine $a_{j+1}, b_{j+1}, ..., l_{j+1}$ and a m-tableau $T^{(j+\frac{1}{2})}$ of shape $[\lambda]^{(j+\frac{1}{2})}$ such that $T^{(j+1)} = \left(a_{j+1} \xrightarrow{RS} T_1^{(j+\frac{1}{2})}, b_{j+1} \xrightarrow{RS} T_2^{(j+\frac{1}{2})}, ..., l_{j+1} \xrightarrow{RS} T_m^{(j+\frac{1}{2})}\right)$. To do this, let $z_1, z_2, ..., z_m$ be the boxes in $\lambda_1^{(j+1)}/\lambda_1^{(j+\frac{1}{2})}, \lambda_2^{(j+1)}/\lambda_2^{(j+\frac{1}{2})} ..., \lambda_m^{(j+1)}/\lambda_m^{(j+\frac{1}{2})}$. We use reverse RS insertion to delete the numbers in the boxes $z_1, z_2, ..., z_m$ which gives a_{j+1} and $T_1^{(j+\frac{1}{2})}, b_{j+1}$ and $T_2^{(j+\frac{1}{2})}, ..., l_{j+1}$ and $T_m^{(j+\frac{1}{2})}$. Thus, $T^{(j+\frac{1}{2})} = \left[T_1^{(j+\frac{1}{2})}, T_2^{(j+\frac{1}{2})}, ..., T_m^{(j+\frac{1}{2})}\right]$.

Now, let $T^{(j+\frac{1}{2})} = [T_1^{(j+\frac{1}{2})}, T_2^{(j+\frac{1}{2})}, ..., T_m^{(j+\frac{1}{2})}]$ be a m-tableau of shape $[\lambda]^{(j+\frac{1}{2})} \in \Lambda^{j}_{n_1-1,n_2-1,...,n_m-1}$ with increasing rows and columns and entries $\{1, 2, ..., n_1\}\backslash\{a_{j+1}\}$, $\{1, 2, ..., n_2\}\backslash\{b_{j+1}\}, ..., \{1, 2, ..., n_m\}\backslash\{l_{j+1}\}$ respectively and let $[\lambda]^{(j)} \subseteq [\lambda]^{(j+\frac{1}{2})}$ with $[\lambda]^{(j)} \in \Lambda^{j}_{n_1,n_2,...,n_m}$. We can uniquely produce a standard m-tableau $T^{(j)}$ such that $T^{(j+\frac{1}{2})} = \left(a_{j+1} \xleftarrow{jdt} T_1^{(j)}, b_{j+1} \xleftarrow{jdt} T_2^{(j)}, ..., l_{j+1} \xleftarrow{jdt} T_m^{(j)}\right)$. To do this, let z_1 be the box in $\lambda_1^{(j)}/\lambda_1^{(j+\frac{1}{2})}$, put a_{j+1} in position of z_1 of $T_1^{(j+\frac{1}{2})}$ and perform the inverse of jeu de taquin to produce $T_1^{(j)}$, i.e., move a_{j+1} into a standard position by iteratively swapping it with larger of the numbers just above it or just left of it. Similarly, we can produce $T_2^{(j)}, ..., T_m^{(j)}$. Thus, $T^{(j)} = \left[T_1^{(j)}, T_2^{(j)}, ..., T_m^{(j)}\right]$.

Given $[\lambda] \in \Lambda^{k}_{n_1,n_2,...,n_m}$ and $(T_{[\lambda]}, P_{[\lambda]}) \in SYT^m([\lambda]) \times VT^m_k([\lambda])$ we apply the process above to $[\lambda]^{(k-\frac{1}{2})} \subseteq [\lambda]^{(k)}$, $T^{(k)} = T_{[\lambda]}$ producing $(a_k, b_k, ..., l_k)$ and $T^{(k-1)}$ respectively. Continuing this way, we can produce $((a_k, b_k, ..., l_k)$, $(a_{k-1}, b_{k-1}, ..., l_{k-1}), ..., (a_1, b_1, ..., l_1))$ and $T^{(k)}$, $T^{(k-1)}$, $..., T^{(1)}$ such that $((a_1, b_1, ..., l_1), (a_2, b_2, ..., l_2), ..., (a_k, b_k, ..., l_k)) \xrightarrow{Fd} (T_{[\lambda]}, P_{[\lambda]})$. $\square$

Example 6.2. For $((6, 2), (3, 5), (1, 4))$ the pair $(T_{[\lambda]}, P_{[\lambda]})$ is as follows.

j	(a_j, b_j)		$T^{(j)}$
0			$\left(\young(123456)\,,\young(123456)\right)$
$\frac{1}{2}$	$(6,2)$	$\xleftarrow{jdt}$	$\left(\young(12345)\,,\young(13456)\right)$
1	$(6,2)$	$\xrightarrow{RS}$	$\left(\young(123456)\,,\young(12456,3)\right)$
$1\frac{1}{2}$	$(3,5)$	$\xleftarrow{jdt}$	$\left(\young(12456)\,,\young(1246,3)\right)$
2	$(3,5)$	$\xrightarrow{RS}$	$\left(\young(12356,4)\,,\young(1245,36)\right)$
$2\frac{1}{2}$	$(1,4)$	$\xleftarrow{jdt}$	$\left(\young(2356,4)\,,\young(125,36)\right)$
3	$(1,4)$	$\xrightarrow{RS}$	$\left(\young(1356,2,4)\,,\young(124,35,6)\right)$

$$T_{[\lambda]} = \left(\young(1356,2,4)\,,\ \young(124,35,6)\right)$$

$$P_{[\lambda]} = \Big((\yng(6), \yng(6)), (\yng(5), \yng(5)), (\yng(6), \yng(5,1)), (\yng(5), \yng(4,1)), (\yng(5,1), \yng(4,2)), (\yng(4,1), \yng(3,2)), (\yng(4,1,1), \yng(3,2,1))\Big).$$

7. The RS Correspondence for the Tensor Product of Partition Algebras

To give a combinatorial proof of the identity

$$[B(2k)]^m = \sum_{[\lambda]\in\Gamma_k^m} (m_k^{[\lambda]})^2 \tag{7.1}$$

we need to find a bijection of the form

$$\mathcal{T}_k \longleftrightarrow \bigsqcup_{[\lambda]\in\Gamma_k^m} VT_k^m([\lambda]) \times VT_k^m([\lambda])$$

by constructing a function that takes a tensor product partition diagram $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathcal{T}_k$ and produce a pair $(P_{[\lambda]}, Q_{[\lambda]})$ of m-vacillating tableaux.

Represent $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathcal{T}_k$ as a m-tuple of k-partition diagrams and draw diagrams for every component (k-partition diagram) $d_t, t \in \{1, 2, ..., m\}$ of

m-tuple using a *standard representation* as single row with the vertices in order $1, 2, ..., 2k$ where the vertex j' is relabeled as $2k - j + 1$. We draw the edges of the standard representation of each component of a m-tuple of k-partition diagrams of $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ in a specific way: connect vertices i and j with $i \le j$ if and only if i and j are related in $d_t, t \in \{1, 2, ..., m\}$ and there does not exits k related to i and j with $i < k < j$. In this way, each vertex is connected only to its nearest neighbors in its block.

Example 7.1. Consider the diagram $(d_1 \otimes d_2) \in \mathfrak{T}_4$

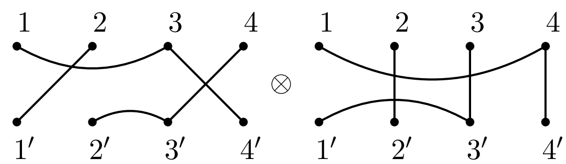


Figure 5:

The above diagram has a standard one line representation as follows:

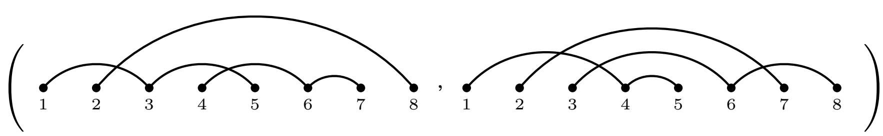


Figure 6:

We label each edge e_t of the diagram $d_t, t \in \{1, 2, ..., m\}$ with $2k+1-v$ where v is the right vertex of e_t. Define the *insertion sequence* of m-tuple of diagrams to be the sequence $E = (E_j) = (E_j^1, E_j^2, ..., E_j^m)$ indexed by the sequence $\frac{1}{2}, 1, 1\frac{1}{2}, ..., 2k - 1, 2k - \frac{1}{2}, 2k$.

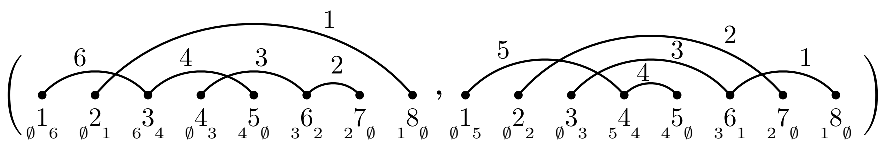
$$E_j = (E_j^1, E_j^2, ..., E_j^m),$$

$$\text{where } E_j^i = \begin{cases} e_i, & \text{if vertex } j \text{ is left end point of edge } e_i \text{ in } i^{th} \text{ component,} \\ & i \in \{1, 2, ..., m\} \\ \emptyset, & \text{if vertex } j \text{ is not left end point.} \end{cases}$$

$$E_{j-\frac{1}{2}} = \left(E^1_{j-\frac{1}{2}}, E^2_{j-\frac{1}{2}}, ..., E^m_{j-\frac{1}{2}}\right),$$

where $E^i_{j-\frac{1}{2}} = \begin{cases} e_i, & \text{if vertex } j \text{ is right end point of edge } e_i \text{ in } i^{th} \text{ component,} \\ & i \in \{1, 2, ..., m\} \\ \emptyset, & \text{if vertex } j \text{ is not right end point.} \end{cases}$

The edge labeling for Example 7.1 is as follows:



The insertion sequence of the above edge labeling diagram is

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
(E^1_j, E^2_j)	$(\emptyset, \emptyset)$	$(6, 5)$	$(\emptyset, \emptyset)$	$(1, 2)$	$(6, \emptyset)$	$(4, 3)$	$(\emptyset, 5)$	$(3, 4)$	$(4, 4)$	$(\emptyset, \emptyset)$	$(3, 3)$

j	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
(E^1_j, E^2_j)	$(2, 1)$	$(2, 2)$	$(\emptyset, \emptyset)$	$(1, 1)$	$(\emptyset, \emptyset)$

The insertion sequence of a m-tuple of standard diagram completely determines the edges and thus the connected components of the diagram and therefore the following proposition follows immediately.

Proposition 7.2. $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ *is completely determined by its insertion sequence.*

For $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ with insertion sequence $E_j = (E^1_j, E^2_j, ..., E^m_j)$ we generate a pair $(P_{[\lambda]}, Q_{[\lambda]})$ of m-vacillating tableaux. Begin with the empty tableaux,

$$T^{(0)} = (T^{(0)}_1, T^{(0)}_2, ..., T^{(0)}_m) = (\emptyset, \emptyset, ..., \emptyset)$$

Then recursively define standard m-tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ for $0 \leq j \leq 2k-1$ as follows: The m-tuple of numbers $E_{j+\frac{1}{2}}$ is removed from the m- tableau $T^{(j)}$ by the process of applying jeu de taquin on the components in which it appears

$$T^{(j+\frac{1}{2})} = \begin{cases} E_{j+\frac{1}{2}} \xleftarrow{jdt} T^{(j)}, & \text{if } E_{j+\frac{1}{2}} \neq \emptyset \text{ (as given below)} \\ T^{(j)}, & \text{if } E_{j+\frac{1}{2}} = \emptyset. \end{cases}$$

The process of insertion is as follows:

$$T^{(j+1)} = \begin{cases} E_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} \neq \emptyset \text{ (as given below)} \\ T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} = \emptyset. \end{cases}$$

Let $E_{j+1} \xrightarrow{RS} T^{(j+\frac{1}{2})}$ denotes the insertion of all $E^i_{j+1} \neq \emptyset$ of E_{j+1} into the i^{th} component $T_i^{(j+\frac{1}{2})}$ of $T^{(j+\frac{1}{2})}$ and other components remain unchanged.

If $E_{j+1} \neq \emptyset$, then

$$T^{(j+1)} = \left[T_1^{(j+1)}, T_2^{(j+1)}, ..., T_m^{(j+1)}\right] \quad \text{and} \quad E_{j+1} = \left[E^1_{j+1}, E^2_{j+1}, ..., E^m_{j+1}\right]$$

$$\text{where } T_i^{(j+1)} = \begin{cases} E^i_{j+1} \xrightarrow{RS} T_i^{(j+\frac{1}{2})}, & \text{if } E^i_{j+1} \neq \emptyset \text{ for } i \in \{1, 2, ..., m\} \\ T_i^{(j+\frac{1}{2})}, & \text{if } E^i_{j+1} = \emptyset. \end{cases}$$

if $E_{j+\frac{1}{2}} \neq \emptyset$, then

$$T^{(j+\frac{1}{2})} = \left[T_1^{(j+\frac{1}{2})}, T_2^{(j+\frac{1}{2})}, ..., T_m^{(j+\frac{1}{2})}\right] \quad \text{and} \quad E_{j+\frac{1}{2}} = \left[E^1_{j+\frac{1}{2}}, E^2_{j+\frac{1}{2}}, ..., E^m_{j+\frac{1}{2}}\right]$$

$$\text{where } T_i^{(j+\frac{1}{2})} = \begin{cases} E^i_{j+\frac{1}{2}} \xleftarrow{jdt} T_i^{(j)}, & \text{if } E^i_{j+\frac{1}{2}} \neq \emptyset \text{ for } i \in \{1, 2, ..., m\} \\ T_i^{(j)}, & \text{if } E^i_{j+\frac{1}{2}} = \emptyset. \end{cases}$$

Let $[\lambda]^{(i)}$ be the shape of $T^{(i)}$, $[\lambda]^{(i+\frac{1}{2})}$ be the shape of $T^{(i+\frac{1}{2})}$ and $[\lambda] = [\lambda]^{(k)}$. Define

$$Q_{[\lambda]} = \left(\emptyset, [\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)}\right) \in VT_k^m([\lambda]),$$

$$P_{[\lambda]} = \left([\lambda]^{(2k)}, [\lambda]^{(2k-\frac{1}{2})}, ..., [\lambda]^{(k+\frac{1}{2})}, [\lambda]^{(k)}\right) \in VT_k^m([\lambda]).$$

In this way, we associate a pair of m-vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ to a tensor product of m-partition diagrams $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathcal{T}_k$ which we denote by

$$(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \longrightarrow (P_{[\lambda]}, Q_{[\lambda]}).$$

For the insertion sequence in Example 7.1:

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
(E^1_j, E^2_j)	$(\emptyset,\emptyset)$	$(6,5)$	$(\emptyset,\emptyset)$	$(1,2)$	$(6,\emptyset)$	$(4,3)$	$(\emptyset,5)$	$(3,4)$	$(4,4)$	$(\emptyset,\emptyset)$	$(3,3)$

j	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
(E^1_j, E^2_j)	$(2,1)$	$(2,2)$	$(\emptyset,\emptyset)$	$(1,1)$	$(\emptyset,\emptyset)$

the pair of 2-vacillating tableaux is given by

j	(E_j^1, E_j^2)		$T^{(j)}$	j	(E_j^1, E_j^2)		$T^{(j)}$
0			$(\emptyset,\emptyset)$	8	$(\emptyset,\emptyset)$	$\xrightarrow{RS}$	$(\emptyset,\emptyset)$
$\frac{1}{2}$	$(\emptyset,\emptyset)$	$\xleftarrow{jdt}$	$(\emptyset,\emptyset)$	$7\frac{1}{2}$	$(1,1)$	$\xleftarrow{jdt}$	$(\emptyset,\emptyset)$
1	$(6,5)$	$\xrightarrow{RS}$	$\left(\ytableaushort{6}, \ytableaushort{5}\right)$	7	$(\emptyset,\emptyset)$	$\xrightarrow{RS}$	$\left(\ytableaushort{1}, \ytableaushort{1}\right)$
$1\frac{1}{2}$	$(\emptyset,\emptyset)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{6}, \ytableaushort{5}\right)$	$6\frac{1}{2}$	$(2,2)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{1}, \ytableaushort{1}\right)$
2	$(1,2)$	$\xrightarrow{RS}$	$\left(\ytableaushort{1,6}, \ytableaushort{2,5}\right)$	6	$(2,1)$	$\xrightarrow{RS}$	$\left(\ytableaushort{12}, \ytableaushort{1,2}\right)$
$2\frac{1}{2}$	$(6,\emptyset)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{1}, \ytableaushort{2,5}\right)$	$5\frac{1}{2}$	$(3,3)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{1}, \ytableaushort{2}\right)$
3	$(4,3)$	$\xrightarrow{RS}$	$\left(\ytableaushort{14}, \ytableaushort{23,5}\right)$	5	$(\emptyset,\emptyset)$	$\xrightarrow{RS}$	$\left(\ytableaushort{13}, \ytableaushort{23}\right)$
$3\frac{1}{2}$	$(\emptyset,5)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{14}, \ytableaushort{23}\right)$	$4\frac{1}{2}$	$(4,4)$	$\xleftarrow{jdt}$	$\left(\ytableaushort{13}, \ytableaushort{23}\right)$
4	$(3,4)$	$\xrightarrow{RS}$	$\left(\ytableaushort{13,4}, \ytableaushort{234}\right)$	4			$\left(\ytableaushort{13,4}, \ytableaushort{234}\right)$

$$Q_{[\lambda]} = \Big((\emptyset,\emptyset),(\emptyset,\emptyset),(\ydiagram{1},\ydiagram{1}),(\ydiagram{1},\ydiagram{1}),(\ydiagram{1,1},\ydiagram{1,1}),(\ydiagram{1},\ydiagram{1,1}),(\ydiagram{2},\ydiagram{2,1}),(\ydiagram{2},\ydiagram{2}),(\ydiagram{2,1},\ydiagram{3})\Big)$$
$$P_{[\lambda]} = \Big((\emptyset,\emptyset),(\emptyset,\emptyset),(\ydiagram{1},\ydiagram{1}),(\ydiagram{1},\ydiagram{1}),(\ydiagram{2},\ydiagram{1,1}),(\ydiagram{1},\ydiagram{1}),(\ydiagram{2},\ydiagram{2}),(\ydiagram{2},\ydiagram{2}),(\ydiagram{2,1},\ydiagram{3})\Big)$$

We have numbered the edges of each standard diagram of m-tuple of diagrams in increasing order from right to left so if $E^i_{j+\frac{1}{2}} \neq \emptyset, i \in \{1,2,...,m\}$ then $E^i_{j+\frac{1}{2}}$ is the largest element of $T_i^{(j)}$. Thus, in $T_i^{(j+\frac{1}{2})} = \left(E^i_{j+\frac{1}{2}} \xleftarrow{jdt} T_i^{(j)}\right)$ we know that $E^i_{j+\frac{1}{2}}$ is in a corner box and jeu de taquin simply deletes that box.

Theorem 7.3. *The function* $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ *provides a bijection between the set of tensor product of partition diagrams in* $\mathfrak{T}_k$ *and pair of* m*-vacillating tableaux in* $\bigsqcup\limits_{[\lambda]\in\Gamma_k^m} VT_k^m([\lambda]) \times VT_k^m([\lambda])$ *and thus gives a combinatorial proof of identity* (7.1).

Proof. We prove the theorem by constructing the inverse of $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. First, we use $Q_{[\lambda]}$ followed by $P_{[\lambda]}$ in the reverse order to construct the sequence $[\lambda]^{(\frac{1}{2})}, [\lambda]^{(1)}, ..., [\lambda]^{(2k-\frac{1}{2})}, [\lambda]^{(2k)}$.
We initialize $T^{(2k)} = (\emptyset, \emptyset, ..., \emptyset)$.

We now present the process to construct $T^{(i+\frac{1}{2})}$ and E_{i+1} so that $T^{(i+1)} = (E_{i+1} \xrightarrow{RS} T^{(i+\frac{1}{2})})$. If $\lambda_1^{(i+\frac{1}{2})} = \lambda_1^{(i+1)}, \lambda_2^{(i+\frac{1}{2})} = \lambda_2^{(i+1)}, ..., \lambda_m^{(i+\frac{1}{2})} = \lambda_m^{(i+1)}$ then let $T_1^{(i+\frac{1}{2})} = T_1^{(i+1)}, T_2^{(i+\frac{1}{2})} = T_2^{(i+1)}, ..., T_m^{(i+\frac{1}{2})} = T_m^{(i+1)}$ and $E_{i+1} = (E_{i+1}^1, E_{i+1}^2, ..., E_{i+1}^m) = (\emptyset, \emptyset, ..., \emptyset)$. Otherwise, $\lambda_j^{(i+1)}/\lambda_j^{(i+\frac{1}{2})}$ is box z_j for all $j \in X \neq \emptyset, X \subseteq \{1, 2, ..., m\}$ and we use RS reverse insertion on the value in z_j to produce $T_j^{(i+\frac{1}{2})}$ and E_{i+1}^j such that $T_j^{(i+1)} = (E_{i+1}^j \xrightarrow{RS} T_j^{(i+\frac{1}{2})})$. Since, we uninserted the value in position of z_j, we know that $T_j^{(i+\frac{1}{2})}$ has shape $\lambda_j^{(i+\frac{1}{2})}$. Since, $\lambda_s^{(i+\frac{1}{2})} = \lambda_s^{(i+1)}$ for $s \in \{1, 2, ..., m\} \backslash X$ then let $T_s^{(i+\frac{1}{2})} = T_s^{(i+1)}$ and $E_{i+1}^s = \emptyset$.
Thus, $T^{(i+\frac{1}{2})} = \left[T_1^{(i+\frac{1}{2})}, T_2^{(i+\frac{1}{2})}, ..., T_m^{(i+\frac{1}{2})}\right]$ and $E_{i+1} = (E_{i+1}^1, E_{i+1}^2, ..., E_{i+1}^m)$, $E_{i+1}^j \neq \emptyset$ where $j \in X \neq \emptyset, X \subseteq \{1, 2, ..., m\}$ and $E_{i+1}^s = \emptyset$ where $s \in \{1, 2, ..., m\} \backslash X$.

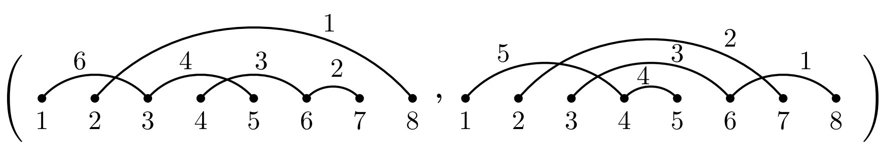
Next we discuss the method to construct $T^{(i)}$ and $E_{i+\frac{1}{2}}$ so that $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{jdt} T^{(i)})$. If $\lambda_1^{(i)} = \lambda_1^{(i+\frac{1}{2})}, \lambda_2^{(i)} = \lambda_2^{(i+\frac{1}{2})}, ..., \lambda_m^{(i)} = \lambda_m^{(i+\frac{1}{2})}$ then let $T_1^{(i)} = T_1^{(i+\frac{1}{2})}, T_2^{(i)} = T_2^{(i+\frac{1}{2})}, ..., T_m^{(i)} = T_m^{(i+\frac{1}{2})}$ and $E_{i+\frac{1}{2}} = (E_{i+\frac{1}{2}}^1, E_{i+\frac{1}{2}}^2, ..., E_{i+\frac{1}{2}}^m) = (\emptyset, \emptyset, ..., \emptyset)$. Otherwise, $\lambda_j^{(i)}/\lambda_j^{(i+\frac{1}{2})}$ is box z_j for all $j \in X \neq \emptyset, X \subseteq \{1, 2, ..., m\}$. Let $T_j^{(i)}$ be the tableau of shape $\lambda_j^{(i)}$ with the same entries as $T_j^{(i+1)}$ and having the entry $2k - i$ in box z_j. Let $E_{i+\frac{1}{2}}^j = 2k - i$. At any given step i, $2k - i$ is the largest value added to the tableau thus far, so that $T_j^{(i)}$ is standard. Further more, $T_j^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}}^j \xleftarrow{jdt} T_j^{(i)})$ since $E_{i+\frac{1}{2}}^j = 2k - i$ is already in a corner and thus jeu de taquin simply delete it. Since, $\lambda_s^{(i)} = \lambda_s^{(i+\frac{1}{2})}$ for $s \in \{1, 2, ..., m\} \backslash X$ then let $T_s^{(i)} = T_s^{(i+\frac{1}{2})}$ and $E_{i+\frac{1}{2}}^s = \emptyset$.

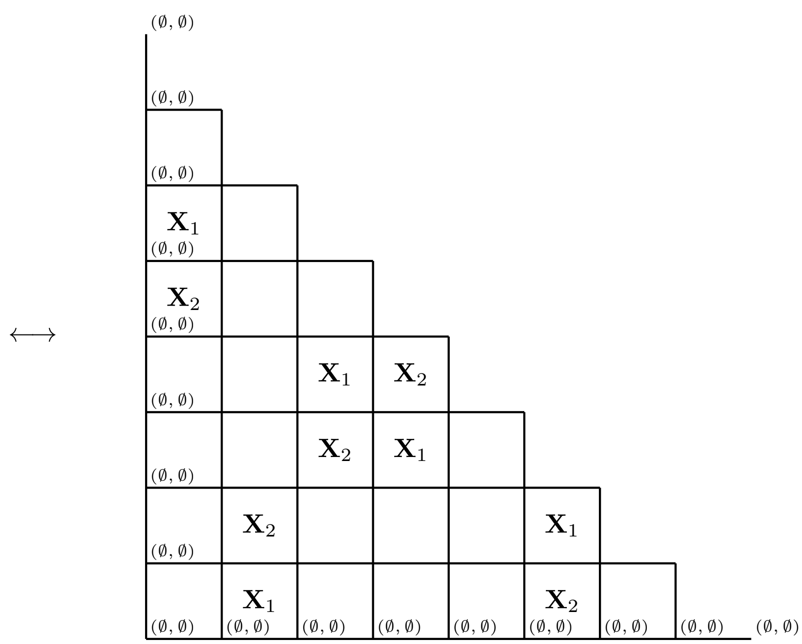
This iterative process will produce $E_{2k}, E_{2k-1}, ..., E_{\frac{1}{2}}$ which completely determines the diagram $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$. By this way we have constructed $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ and $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. □

Notice that in the m-tuple of standard representation of $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ a flip corresponds to a reflection over the vertical line between vertices k and $k+1$ in each component of a m-tuple. Our aim is to show that if $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \rightarrow (P, Q)$ then $flip(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \rightarrow (Q, P)$.

Given a tensor product partition diagram $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathcal{T}_k$ construct a triangular grid (as in the case of partition diagrams) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that contains the points in the triangular whose vertices are $(0, 0), (2k, 0)$ and $(0, 2k)$. Number the columns $1, 2, ..., 2k$ from left to right and the rows $1, 2, ..., 2k$ from bottom to top. Place an $\mathbf{X}_1$ in the box in column i and row j if and only if in the first one row diagram of m-tuple the vertex i is the left end point of edge j. Place an $\mathbf{X}_2$ in the box in column i and row j if and only if in the second one row diagram of m-tuple the vertex i is the left end point of edge j. Similarly, proceed in this way up to the m^{th} one row diagram of m-tuple. We then label the vertices of the diagram on the bottom row and left column with the m-tuple of empty partition $(\emptyset, \emptyset, ..., \emptyset)$.

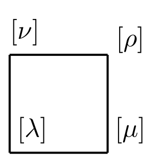
Example 7.4.





Note that the triangular array completely determines the tensor product partition diagram and vice-versa.

Now we inductively label the remaining vertices using the local rules of Fomin (as in the case of partition diagrams). If a box is labeled with $[\mu], [\nu], [\lambda]$ as given below then we add the label $[\rho]$ according to the following rule:

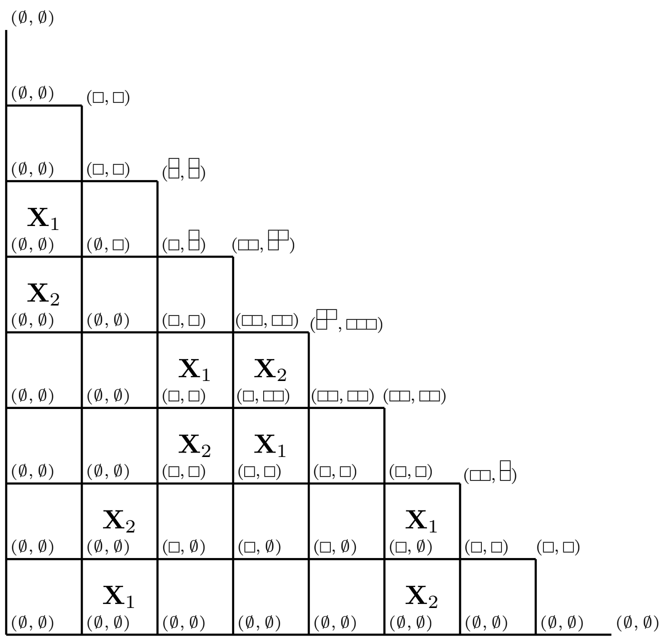


$[RL1]$ If $\mu_j \neq \nu_j, j \in \{1,2,...,m\}$ let $\rho_j = \mu_j \cup \nu_j$, i.e., $\rho_j^i = \max(\mu_j^i, \nu_j^i)$.

$[RL2]$ If $\mu_j = \nu_j, \lambda_j \subset \mu_j$ and $\lambda_j \neq \mu_j, j \in \{1,2,...,m\}$ then this will automatically imply that μ_j can be obtained from λ_j by adding a box to λ_j^i. Let ρ_j can be obtained from μ_j by adding a box to μ_j^{i+1}.

$[RL3]$ If $\mu_j = \nu_j = \lambda_j, j \in \{1,2,...,m\}$ then if the square does not contain a $\mathbf{X}_j$, let $\rho_j = \lambda_j$ and if the square does contain a $\mathbf{X}_j$ then ρ_j be obtained from λ_j by adding 1 to λ_j^1.

Using these rules we can uniquely label every corner one step at a time. The resulting diagram is called the *growth diagram* $G_{\underline{d}}$ for $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$. The growth diagram for Example 7.4. is



Let $P_{\underline{d}}$ denote the chain of m-partitions that follows the staircase path on the diagonal of $G_{\underline{d}}$ from $(0,2k)$ to (k,k) and $Q_{\underline{d}}$ denote the chain of m-partitions that follows the staircase path on the diagonal of $G_{\underline{d}}$ from $(2k,0)$ to (k,k). The pair $(P_{\underline{d}}, Q_{\underline{d}})$ represents a pair of m-vacillating tableaux whose shape is the partition at (k,k). From the above example

$$Q_{[\lambda]} = \Big((\emptyset,\emptyset),(\emptyset,\emptyset),(\square,\square),(\square,\square),(\begin{smallmatrix}\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}),(\square\square,\square\square),(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square\square)\Big)$$

$$P_{[\lambda]} = \Big((\emptyset,\emptyset),(\emptyset,\emptyset),(\square,\square),(\square,\square),(\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\square,\square),(\square\square,\square\square),(\square\square,\square\square),(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square\square)\Big)$$

Theorem 7.5. *Let $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ with $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \to (P,Q)$. Then $P_{\underline{d}} = P$ and $Q_{\underline{d}} = Q$.*

Proof. The proof is based on [4]. Turn each diagram $d_s, s \in \{1, 2, ..., m\}$ of $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ into a diagram d'_s on $4k$ vertices by splitting each vertex i into two vertices labeled by $i - \frac{1}{2}$ and i. If there is an edge from vertex j to vertex i in d_s with $j < i$, let j be adjacent to $i - \frac{1}{2}$ in d'_s. If there is an edge from vertex j to vertex i in d_s with $j > i$, let i be adjacent to $j - \frac{1}{2}$ in d'_s. A key advantage of the use of growth diagrams is that the symmetry of the algorithm is nearly obvious. We have that i is the left end point of the edge labeled j in diagram d_s of $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ if and only if j is the left point of the edge labeled i in diagram d_s of $flip(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$. Thus the growth diagram of $G_{\underline{d}}$ is the reflection over the line $y = x$ of the growth diagram of $G_{flip(\underline{d})}$ and so $P_{\underline{d}} = Q_{flip(\underline{d})}$ and $Q_{\underline{d}} = P_{flip(\underline{d})}$. □

Corollary 7.6. *If* $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \to (P, Q)$ *then* $flip(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \to (Q, P)$.

Corollary 7.7. *A diagram* $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k$ *is symmetry if and only if* $(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \to (P, P)$.

Proof. The proof is based on [4]. If $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ is symmetry then by the above corollary we must have $P = Q$. To prove the converse part, let $P = Q$ and place the m-vacillating tableaux on the staircase border of the growth diagram. The local rules we have defined above are invertible. Given $[\mu], [\nu]$ and $[\rho]$, one can follow the rules backwards to uniquely find $[\lambda]$ and determine whether there is an $\mathbf{X}_i, i \in \{1, 2, ..., m\}$ in the box. Thus, the interior of the growth diagram is uniquely determined. By the symmetry of having $P = Q$ along the staircase the growth diagram must have a symmetry interior and a symmetric placement of the $\mathbf{X}'_i s$. This forces $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$ to be symmetric. □

This corollary tells us that the number of symmetry diagrams in $\mathfrak{T}_k$ is equal to the number of m-vacillating tableaux of length $2k$ or the number of paths to level k in the Bratteli diagram of $\mathbb{C}\{P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)\}$. Thus,

$$\mathrm{Card}(\{(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \in \mathfrak{T}_k | (d_1 \otimes d_2 \otimes \cdots \otimes d_m) \textit{ is symmetry}\}) = \sum_{[\lambda] \in \Gamma_k^m} m_k^{[\lambda]}.$$

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