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NUMBER OF LINEAR EXTENSIONS FOR A VARIANT OF UP-DOWN POSET

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Abstract. A variant of up-down posets described below and the number of their linear extensions were studied. We obtained the exponential generating functions which showed that how they are related to the Euler's up-down numbers.

1. Introduction

We consider the following poset:

$A_{m,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_{m-1} < \sigma_m < \tau_1 < \tau_2 > \tau_3 < \tau_4 > \cdots < (\text{ or } >)\tau_n\}$,
where $[n+m] = \{\sigma_i\}_{i=1}^m \cup \{\tau_j\}_{j=1}^n$. In other words, the orders between $\tau_i's$ in $A_{m,n}$ change alternatively. Let $c(m,n)$ be the number of linear extensions of the poset $A_{m,n}$. It is known that $c(0,n) = E_n$, Euler's updown number. $1, 1, 1, 2, 5, 16, 61, 272, 1385$ are the first few terms. (See [4] OEIS id A000111 about this.) Our goal here is to represent the number of linear extensions of $A_{m,n}$ using Euler's up-down numbers.

Let $F_m(y) := \sum_{n\geq 0} c(m,n)\frac{y^n}{n!}$. From the definition of $F_m(y)$, we obtain

$$F_0(y) = \sum_{k=0}^{\infty} E_{2k}\frac{y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} E_{2k+1}\frac{y^{2k+1}}{(2k+1)!} = \sec y + \tan y.$$

(See [2] for the zigzag poset.)

One way to get the formula $F_1(y)$ is as follows: Note that

$$A_{0,n+1} = \{\tau_1 < \tau_2 > \tau_3 < \tau_4 > \cdots < (\text{ or } >)\tau_{n+1}\}.$$

Consider another poset

$$B_{n+1} := \{\tau_0 > \tau_1 < \tau_2 > \tau_3 < \tau_4 > \cdots > (\text{ or } <)\tau_n\}.$$

The number of linear extensions of the poset $A_{0,n+1}$ (which is E_{n+1}) is same as that of the poset B_{n+1} because there is an obvious bijection between them.

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Since the sum of the number of linear extensions of B_{n+1} and that of $A_{1,n}$ is $(n+1)E_n$, we have the formula $c(1,n) = (n+1)E_n - E_{n+1}$ for $n \geq 1$. Thus

$$\begin{aligned} F_1(y) &= \sum_{n\geq 0} c(1,n)\frac{y^n}{n!} \\ &= 1 + \sum_{n\geq 1}\left((n+1)E_n - E_{n+1}\right)\frac{y^n}{n!} \\ &= 1 + \sum_{n\geq 1}(nE_n)\frac{y^n}{n!} + \sum_{n\geq 1} E_n\frac{y^n}{n!} - \sum_{n\geq 1} E_{n+1}\frac{y^n}{n!}. \end{aligned}$$

Since

$$\sum_{n\geq 1}(nE_n)\frac{y^n}{n!} = y\sum_{n\geq 1} E_n\frac{y^{n-1}}{(n-1)!} = y\frac{d}{dy}\sum_{n\geq 1} E_n\frac{y^n}{n!} = y(\sec^2 y + \sec y\tan y),$$

$$\sum_{n\geq 1} E_n\frac{y^n}{n!} = \sec y + \tan y - 1,$$

$$\begin{aligned} \sum_{n\geq 1} E_{n+1}\frac{y^n}{n!} &= \frac{d}{dy}\sum_{n\geq 1} E_{n+1}\frac{y^{n+1}}{n+1!} \\ &= \frac{d}{dy}(\sec y + \tan y - 1 - y) \\ &= \sec^2 y + \sec y\tan y - 1, \end{aligned}$$

we get

$$F_1(y) = 1 + (1 + (y-1)\sec y)(\sec y + \tan y).$$

First few terms of $c(1,n)$ are listed as follows:

$$1, 1, 1, 3, 9, 35, 155, 791, 4529, 28839, \ldots$$

(See [4] with id A034428. Note that $c(1,0) = 1$. There is certain relation with the sequences appeared in [1] and [3].)

Another way to get the same formula for $F_1(y)$ is as follows: Let

$$\begin{aligned} F_m^o(y) &= \left(F_m(y) - F_m(-y)\right)/2, \\ F_m^e(y) &= \left(F_m(y) + F_m(-y)\right)/2. \end{aligned}$$

Theorem 1.1. ($m = 1$) *The following system of differential equations holds:*

$$F_1^o(y) = y + \int_0^y F_1^o(s)\,ds\tan y \tag{1}$$

$$F_1^e(y) = 1 + \int_0^y F_1^o(s)\,ds\sec y \tag{2}$$

To prove the above theorem, we need the following lemma where a recurrence relation on $c(1, 2n+1)$ is introduced.

n	$c(n,0)$	$c(n,1)$	$c(n,2)$	$c(n,3)$	$c(n,4)$	$c(n,5)$	$c(n,6)$	$c(n,7)$	$c(n,8)$
0	1	1	1	2	5	16	61	272	1385
1	1	1	1	3	9	35	155	791	4529
2	1	1	1	4	14	64	323	1856	11796
3	1	1	1	5	20	105	595	3801	26586
4	1	1	1	6	27	160	1006	7072	53954
5	1	1	1	7	35	231	1596	12243	101178

TABLE 1. $c(m,n)$-table

Lemma 1.2. *The sequence $c(1,2n+1)$ satisfies the following recursive formula*

$$c(1,2n+1)=\sum_{k=0}^{n-1}\binom{1+2n}{2k+1}c(1,2n-2k-1)E_{2k+1}.$$

Proof. Consider the following poset:

$$A_{1,2n+1}:=\{\sigma_1<\tau_1<\tau_2>\tau_3<\tau_4>\cdots<\tau_{2n}>\tau_{2n+1}\}$$

To construct the linear extensions of the poset $A_{1,2n+1}$ we match each number in $[2n+2]$ with the element of the given poset so that the correspondence satisfies suitable conditions for cover relations. The largest number $2n+2$ in this correspondence match with one of $\tau_{2(n-k)}$ (where $0\le k\le n-1$) in the poset $A_{1,2n+1}$ so that the poset is decomposed into two parts. One part is $A_{1,2n-2k-1}$ and the other part is $A_{0,2k+1}$. Thus, first we choose $2k+1$ numbers from $[2n+1]$ for the part $A_{0,2k+1}$. Then we multiply $\binom{1+2n}{2k+1}$ by $c(1,2n-2k-1)$ and $c(0,2k+1)=E_{2k+1}$ to get the desired formula. □

Now we provide the proof of Theorem 1.1.

Proof of Theorem 1.1. First we separate odd and even terms of $F_1(y)$ as

$$\begin{aligned}F_1(y)&=\sum_{n\ge 0}c(1,n)\frac{y^n}{n!}=F_1^o(y)+F_1^e(y)\\&=\left[y+\sum_{n=1}^{\infty}c(1,2n+1)\frac{y^{2n+1}}{(2n+1)!}\right]+\left[1+\sum_{n=1}^{\infty}c(1,2n)\frac{y^{2n}}{(2n)!}\right].\end{aligned}$$

Then, as a result of the above lemma, $F_1^o(y)$ satisfies

$$\begin{aligned}F_1^o(y)-y&=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\binom{1+2n}{2k+1}c(1,2n-2k-1)E_{2k+1}\right)\frac{y^{2n+1}}{(2n+1)!}\\&=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\frac{(1+2n)!c(1,2n-2k-1)E_{2k+1}}{(1+2n-2k-1)!(2k+1)!}\right)\frac{y^{1+2n}}{(1+2n)!}\\&=\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\left(\frac{c(1,2n-2k-1)}{(2n-2k)!}y^{2n-2k}\right)\left(\frac{E_{2k+1}}{(2k+1)!}y^{2k+1}\right)\\&=\left(\sum_{l=0}^{\infty}\frac{c(1,2l+1)}{(2l+2)!}y^{2l+2}\right)\left(\sum_{k=0}^{\infty}\frac{E_{2k+1}}{(2k+1)!}y^{2k+1}\right).\end{aligned}$$

If we define $\beta_1(y)$ as

$$\beta_1(y)=\sum_{l=0}^{\infty}\frac{c(1,2l+1)}{(2l+2)!}y^{2l+2}=\int_0^y\sum_{l=0}^{\infty}\frac{c(1,2l+1)}{(2l+1)!}s^{2l+1}\,ds=\int_0^y F_1^o(s)\,ds,$$

we get Equation (1). Similarly, we can obtain Equation (2). □

Theorem 1.3 ($m=1$)**.** *The exponential generating functions for odd terms and even terms of $c(1,n)$ are*

$$\begin{aligned}F_1^o(y)&=\tan y+y\sec^2 y-\sec y\tan y,\\F_1^e(y)&=1+y\sec y\tan y-\sec^2 y+\sec y.\end{aligned}$$

Thus, we get

$$F_1(y)=F_1^o(y)+F_1^e(y)=1+(1+(y-1)\sec y)(\sec y+\tan y).$$

Proof. From Equation (1), we have $\beta_1'(y)-y=\beta_1(y)\tan y$. Multiplying by the integrating factor $\cos y$ on both sides of the previous differential equation, we get

$$(\beta_1(y)\cos y)'=y\cos y.$$

Thus

$$\begin{aligned}\beta_1(y)&=\sec(y)\int_0^y s\cos(s)\,ds\\&=\sec(y)\,(y\sin(y)+\cos(y)-1)\\&=y\tan(y)+1-\sec(y).\end{aligned}$$

Using this, we get

$$\begin{aligned}F_1^o(y)&=\beta_1'(y)=\tan y+y\sec^2 y-\sec y\tan y,\\F_1^e(y)&=1+\beta_1(y)\sec y=1+y\sec y\tan y-\sec^2 y+\sec y.\end{aligned}$$

Hence

$$F_1(y)=F_1^o(y)+F_1^e(y)=1+(1+(y-1)\sec y)(\sec y+\tan y).$$

□

Now, we consider the case $m = 2$.

Theorem 1.4 ($m = 2$)**.** *The following system of differential equations holds:*

$$F_2^o(y) = y + \int_0^y F_2^o(s)\,ds \tan y + \int_0^y \int_0^t F_2^o(s)\,ds\,dt \sec^2 y \tag{3}$$

$$F_2^e(y) = 1 + \int_0^y F_2^o(s)\,ds \sec y + \int_0^y \int_0^t F_2^o(s)0\,ds\,dt \sec y \tan y \tag{4}$$

Let $\beta_2(y) = \int_0^y \int_0^t F_2^o(s)\,ds\,dt$. Then Equation (3) can be rewritten as

$$\beta_2''(y) = y + \beta_2'(y)\tan y + \beta_2(t)\sec^2 y.$$

Since this can be written as

$$\left(\beta_2'(y) - \frac{y^2}{2} - \beta_2(y)\tan y\right)' = 0,$$

we get

$$\beta_2'(y) - \frac{y^2}{2} - \beta_2(y)\tan y = c.$$

for some constant c. Since $\beta_2'(0) = 0 = \beta_2(0)$, c must be 0. Similar to what we did before, we get

$$(\beta_2(y)\cos y)' = \beta_2'(y)\cos y - \beta_2(y)\sin y = \frac{y^2}{2}\cos y.$$

Thus we have

$$\beta_2(y)\cos y = \int_0^y \frac{s^2}{2}\cos s\,ds = \left(\frac{y^2}{2} - 1\right)\sin y + y\cos y,$$

which gives

$$\beta_2(y) = \left(\frac{y^2}{2} - 1\right)\tan y + y.$$

Theorem 1.5 ($m = 2$)**.** *The exponential generating functions for odd terms and even terms of $c(2, n)$ are*

$$F_2^o(y) = \tan y + 2y\sec^2 y + (y^2 - 2)\sec^2 y \tan y,$$
$$F_2^e(y) = 1 + (2 - y^2/2)\sec y + 2y\sec y\tan y + (y^2 - 2)\sec^3 y.$$

Thus the exponential generating function for $c(2, n)$ is

$$F_2(y) = 1 + (1 - y^2/2)\sec y + (1 + 2y\sec y + (y^2 - 2)\sec^2 y)(\sec y + \tan y).$$

2. Main results

In this chapter we generalize the previous results for $m = 1$ and 2. The exponential generating function explains the situation that how they are mixed with the Euler's up-down numbers $\{E_n\}_{n\geq 0}$. Now

$$\begin{aligned} F_m(y) &= \sum_{n\geq 0} c(m,n)\frac{y^n}{n!} = F_m^o(y) + F_m^e(y) \\ &= \left[y + \sum_{n=1}^{\infty} c(m,2n+1)\frac{y^{2n+1}}{(2n+1)!}\right] + \left[1 + \sum_{n=1}^{\infty} c(m,2n)\frac{y^{2n}}{(2n)!}\right]. \end{aligned}$$

Then $F_m^o(y)$ satisfies

$$\begin{aligned} F_m^o(y) - y &= \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}\binom{m+2n}{2k+1}c(m,2n-2k-1)E_{2k+1}\right]\frac{y^{2n+1}}{(2n+1)!} \\ &= \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}\frac{(m+2n)!c(m,2n-2k-1)E_{2k+1}}{(m+2n-2k-1)!(2k+1)!}\right]\left(\frac{y^{m+2n}}{(m+2n)!}\right)^{(m-1)} \\ &= \left[\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\left(\frac{c(m,2n-2k-1)}{(m+2n-2k-1)!}y^{m+2n-2k-1}\right)\left(\frac{E_{2k+1}}{(2k+1)!}y^{2k+1}\right)\right]^{(m-1)} \\ &= \left[\left(\sum_{l=0}^{\infty}\frac{c(m,2l+1)}{(m+2l+1)!}y^{m+2l+1}\right)\left(\sum_{k=0}^{\infty}\frac{E_{2k+1}}{(2k+1)!}y^{2k+1}\right)\right]^{(m-1)} \\ &= \left[\beta_m(y)(\tan y)\right]^{(m-1)}, \end{aligned}$$

where

$$\beta_m(y) = \sum_{l=0}^{\infty}\frac{c(m,2l+1)}{(m+2l+1)!}y^{m+2l+1}.$$

Note that

$$(\beta_m(y))^{(m)} = F_m^o(y) - y.$$

Similar to Theorem 1.4, we get the general results:

Theorem 2.1. *The following system of differential equations holds:*

(5)

$$F^o_m(y) = y + \binom{m-1}{0}\beta_m^{(m-1)}(y)\tan y + \binom{m-1}{1}\beta_m^{(m-2)}(y)(\tan y)' + \cdots + \binom{m-1}{m-1}\beta_m(y)(\tan y)^{(m-1)},$$

(6)

$$F^e_m(y) = 1 + \binom{m-1}{0}\beta_m^{(m-1)}(y)\sec y + \binom{m-1}{1}\beta_m^{(m-2)}(y)(\sec y)' + \cdots + \binom{m-1}{m-1}\beta_m(y)(\sec y)^{(m-1)}.$$

From Equation (5), we have

$$\beta_m^{(m)}(y) = \left(\frac{y^m}{m!} + \beta_m(y)\tan y\right)^{(m-1)}$$

which implies that

$$\beta'_m(y) = \beta_m(y)\tan y + \frac{y^m}{m!} + c_0 + c_1 y + c_2 y^2 + \cdots + c_{m-1}y^{m-1}.$$

Since $\beta_m(0) = \beta'_m(0) = \cdots = \beta_m^{(m-1)}(0) = 0$, we get

$$\beta'_m(y) = \beta_m(y)\tan y + \frac{y^m}{m!}.$$

Similar to the previous case, we obtain the solution $\beta_m(y)$ as follows:

(7) $$\beta_m(y) = \sec y \int_0^y \frac{s^m}{m!}\cos(s)\,ds.$$

Therefore we get the following theorem.

Theorem 2.2. *The exponential generating function for $c(m,n)$ is*

$$F_m(y) = (1+y) + \left[\beta_m(y)(\sec y + \tan y)\right]^{(m-1)}.$$

Proof. We have

$$\begin{aligned} F_m(y) &= F^o_m(y) + F^e_m(y) \\ &= \left[\frac{y^m}{m!} + \beta_m(y)\tan y\right]^{(m-1)} + \left[\frac{y^{m-1}}{(m-1)!} + \beta_m(y)\sec y\right]^{(m-1)} \\ &= (1+y) + \left[\beta_m(y)(\sec y + \tan y)\right]^{(m-1)}. \end{aligned}$$

□

3. Further Analysis of $\beta_m(y)$

In this section, we provide further analysis of $\beta_m(y)$ and several questions. First, we provide a recurrence relation of $\beta_m(y)$.

Corollary 3.1.

$$\beta_m(y) = \frac{y^m}{m!}\tan(y) + \frac{y^{m-1}}{(m-1)!} - \beta_{m-2}(y).$$

Proof. From Equation (7),

$$\begin{aligned}\beta_m(y)\cos(y) &= \int_0^y \frac{s^m}{m!}\cos(s)\,ds \\ &= \left[\frac{s^m}{m!}\sin(s)\right]_{s=0}^y - \int_0^y \frac{s^{m-1}}{(m-1)!}\sin(s)\,ds \\ &= \frac{y^m}{m!}\sin(y) + \frac{y^{m-1}}{(n-1)!}\cos(y) - \beta_{m-2}(y)\cos(y).\end{aligned}$$

□

The next result provides the ordinary generating function of $\beta_m(y)$.

Corollary 3.2. *Let* $B(x,y) := \sum_{m\geq 0}\beta_m(y)x^m$. *Then*

$$B(x,y) = \frac{1}{1+x^2}\left[e^{xy}(x+\tan(y)) - x\sec(y)\right].$$

Proof. From Equation (7),

$$\begin{aligned}B(x,y) &:= \sum_{m\geq 0}\beta_m(y)x^m \\ &= \sec(y)\int_0^y \sum_{m\geq 0}\frac{(sx)^m}{m!}\cos(s)\,ds \\ &= \sec(y)\int_0^y \exp(sx)\cos(s)\,ds \\ &= \frac{e^{xy}(x\cos(y)+\sin y) - x}{(1+x^2)\cos(y)} \\ &= \frac{e^{xy}(x+\tan(y)) - x\sec(y)}{1+x^2}.\end{aligned}$$

□

We close this section with a couple of questions and a problem.

Question 3.3. *Now, if we let* $G(x,y) = \sum_{m\geq 0}F_m(y)\frac{x^m}{m!}$, *what is the shape of* $G(x,y)$*?*

Question 3.4. *If we represent* $G(x,y) = \sum_{n=0}^{\infty} H_n(x) \frac{y^n}{n!}$, *it is curious what* $H_n(x)$ *looks like. How will it be tied to* $\beta_m(y)$*'s?*

Problem 3.5. *One might consider the order polytope of the poset* $A_{m,n}$ *and its discrete volume.*

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