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ON THE ADAPTED CONNECTIONS ON KAEHLER-NORDEN SILVER MANIFOLDS

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Abstract. In this paper, we study almost complex Norden Silver manifolds and Kaehler-Norden Silver manifolds. We define adapted connections of first, second and third type to an almost complex Norden Silver manifold and establish the necessary and sufficient conditions for the integrability of almost complex Norden Silver structure. Moreover, we investigate that a complex Norden Silver map is a harmonic map between Kaehler-Norden Silver manifolds.

1. Introduction

In 1970, the notion of the polynomial structure on a manifold was introduced by Goldberg and Yano [7]. Suppose M denote a C^∞-differentiable real manifold. A tensor field F of type $(1,1)$ on M is said to define a polynomial structure of degree n, if F satisfies the following algebraic equation

$$Q(F) = F^n + a_n F^{n-1} + \cdots + a_2 F + a_1 I = 0$$

and F has constant rank on M. Here I denote $(1,1)$ identity tensor field and $a_1, a_2, \ldots, a_n$ are real numbers.

Gezer et al. [5] studied the properties of Riemannian manifolds equipped with the Hessian metric h and a complex Golden structure. In [6], Gezer et al. obtained a new sufficient condition of integrability for a Golden Riemannian structure. They also investigated some properties of twin Golden Riemannian metrics and the curvature properties of locally decomposable Golden Riemannian manifolds. Primo et al. [13] obtained some algebraic and geometric characterizations of the Silver ratio.

In [14], Salimov et al. investigated Norden metrics of Hessian type $h = \nabla^2 f$. Salimov [15] introduced anti-Hermitian metric connections of type I and type II, respectively. Moreover, Salimov considered the classes of anti-Hermitian manifolds associated with these connections. Sahin et al. [16] introduced the

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notion of a Golden map between Golden Riemannian manifolds and proved that such maps are harmonic. In [8], Hretcanu et al. introduced a structure on a class of Riemannian manifolds, known as a Golden structure. Also, they established some interesting properties of the Golden structure.

Etayo et al. [4] investigated adapted connections and obtained two special connections on almost Golden Riemannian structure, which measure the integrability of $(1,1)$-tensor field φ and the integrability of G-structure corresponding to an almost Golden Riemannian manifold (φ, g). Bilen et al. [2] studied the curvature properties of a pseudo-Riemannian manifold equipped with a Kaehler-Norden-Codazzi Golden structure, and defined special connections of type I and type II.

In [3], Crasmareanu et al. investigated the geometry of Golden structure on a manifold by using a corresponding almost product structure. Iscan et al. [9] investigated the geometry of Kaehler-Norden manifolds. Moreover, Iscan et al. used Tachibana operators to study the properties of curvature scalars and Riemannian curvature tensors of Kaehler-Norden manifolds. In [10], Kumar et al. studied the adapted connections on Kaehler-Norden Golden manifolds and almost complex Norden Golden manifolds.

Inspired by [10], we investigate Kaehler-Norden Silver manifolds and almost complex Norden Silver manifolds. Analogous to [2, 10], we define adapted connections of first, second and third type to an almost complex Norden Silver manifold and prove that a complex Norden Silver map is a harmonic map between Kaehler-Norden Silver manifolds.

2. Kaehler-Norden Silver Manifold

Suppose Θ_c be a tensor field of type $(1,1)$ on manifold M^{2n}. A tensor field Θ_c is said to be an almost complex Silver structure if it satisfies the following equation

$$\Theta_c^2 - 2\Theta_c + 3I = 0, \tag{1}$$

and the pair (M^{2n}, Θ_c) is called an almost complex Silver manifold. The complex number $1 + i\sqrt{2}$, which is a root of the equation $x^2 - 2x + 3 = 0$, is known as a complex Silver ratio [11].

Let J is an almost complex structure on M^{2n}, then

$$\Theta_c = (I \mp \sqrt{2}J), \tag{2}$$

is called an almost complex Silver structure on M^{2n}.

Conversely, let Θ_c denotes an almost complex Silver structure on M^{2n}, then

$$J^{\Theta_c} = \mp \frac{1}{\sqrt{2}} \left(\Theta_c - I\right), \tag{3}$$

is said to be an almost complex structure induced by Θ_c. Therefore, an almost complex Silver structure Θ_c defines a Θ_c-associated almost complex structure J^{Θ_c} and vice-versa. Obviously, $J^{\Theta_c} = J$ and $\Theta_c^{J^{\Theta_c}} = \Theta_c$.

Hence, there exist a one-to-one correspondence between almost complex structures and almost complex Silver structures on M^{2n}.

It is well known that the almost complex Silver manifold is integrable if Nijenhuis tensor N_{Θ_c} vanishes and is given by

$$N_{\Theta_c} = \Theta_c^2 [U,V] + [\Theta_c U, \Theta_c V] - \Theta_c [\Theta_c U, V] - \Theta_c [U, \Theta_c V]. \tag{4}$$

If the almost complex Silver structure Θ_c is integrable, then this structure is said to be complex Silver structure and (M^{2n}, Θ_c) is known as a complex Silver manifold.

Let (M^{2n}, g) denote the pseudo-Riemannian manifold associated with an almost complex structure J. If the pseudo-Riemannian metric g is pure with respect to an almost complex structure J, i.e.

$$g(JU,V) = g(U,JV),$$

for any vector fields U and V on M, then (M^{2n}, J, g) is called an almost complex Norden manifold and M^{2n} is known as Norden manifold if J is integrable.

Suppose g be a pseudo-Riemannian metric equipped with an almost complex Silver structure Θ_c, then the triplet (M^{2n}, Θ_c, g) is said to be an almost complex Norden Silver manifold if it satisfies the following equation

$$g(\Theta_c U, V) = g(U, \Theta_c V), \tag{5}$$

for any vector fields U and V on M.

Thus, for an almost complex Norden Silver manifold (M^{2n}, Θ_c, g), we have

$$g(\Theta_c U, \Theta_c V) = 2g(\Theta_c U, V) - 3g(U,V). \tag{6}$$

Suppose ψ be a tensor field of type $(1,1)$ and $\Im_q^p(M)$ denote the set of all (p,q)-tensor fields on the smooth manifold M. A tensor field t of type $(0,s)$ is known as pure tensor field associated with ψ if

$$t(\psi U_1, U_2, \ldots, U_s) = t(U_1, \psi U_2, \ldots, U_s) = \cdots = t(U_1, U_2, \ldots, \psi U_s),$$

for any $U_1, U_2, \ldots, U_s \in \Im_0^1(M)$.

Let ψ be a tensor field of type $(1,1)$, then consider an operator

$$\phi_\psi : \Im_s^0(M) \to \Im_{s+1}^0(M),$$

operated on the pure tensor field t of type $(0,s)$ with respect to ψ and given by

$$\begin{aligned}(\phi_\psi t)\left(U,V_1,V_2,\ldots,V_s\right) = {} & (\psi U)\left(t(V_1,V_2,\ldots,V_s)\right)\\ & - U\left(t(\psi V_1,V_2,\ldots,V_s)\right)\\ & + t\left(\left(L_{V_1}\psi\right)U,V_2,\ldots,V_s\right)\\ & \cdots\\ & + t\left(V_1,V_2,\ldots,\left(L_{V_s}\psi\right)U\right),\end{aligned} \tag{7}$$

for any $U,V_1,V_2,\ldots,V_s \in \Im_0^1(M)$, where L_V denotes the Lie differentiation with respect to V [17].

If J is integrable then an almost complex structure J is called a complex structure. Suppose $\psi = J$ represents the complex structure on M^{2n} then a tensor field t is called as a holomorphic tensor field if

$$(\phi_J t)\left(U,V_1,V_2,\ldots,V_s\right) = 0.$$

Let (M^{2n},J,g) be the Norden manifold. If $\phi_J g = 0$ then a Norden metric g is called holomorphic and the Norden manifold is said to be a holomorphic Norden manifold [14, 9]. In some sense, holomorphic Norden manifolds are analogous to Kaehler manifolds due to the following theorem by Iscan et al. [9].

Theorem 2.1. *An almost Norden manifold is holomorphic Norden manifold if and only if the almost complex structure is parallel with respect to the Levi-Civita connection ∇^g.*

If M^{2n} is associated with a pseudo-Riemannian metric g and an almost complex structure J such that $\nabla^g J = 0$, then the triplet (M^{2n},J,g) is said to be a Kaehler-Norden manifold, where ∇^g be the Levi-Civita connection of g [9]. Therefore, there exists a one-one correspondence between Kaehler-Norden manifolds and Norden manifolds with holomorphic metric.

In the study of almost complex structure, the ϕ-operator method can be used for almost complex Silver structures because the almost complex structure J and almost complex Silver structure Θ_c are related to each other. Therefore, for the integrability of Θ_c on pseudo-Riemannian manifolds, we have the following result.

Theorem 2.2. *Suppose (M^{2n},Θ_c,g) is an almost complex Silver Norden manifold and ∇^g denotes the Levi-Civita connection of g. Then*

(i) if $\phi_{\Theta_c} g = 0$, then Θ_c is integrable,

(ii) the equality $\phi_{\Theta_c} g = 0$ is equivalent to $\nabla^g\Theta_c = 0$.

Proof. From (5) and $\nabla^g g = 0$, it follows that

$$g(U,(\nabla_W^g\Theta_c)V) = g((\nabla_W^g\Theta_c)U,V), \tag{8}$$

for any vector fields U, V and W on M.

Now making use of (8) and $[U,V]=\nabla^g_U V-\nabla^g_V U$, we can transform (7) as given below:

(9)
$$(\phi_{\Theta_c} g)(U,V,W)=-g\left(\left(\nabla^g_U \Theta_c\right)V,W\right)+g\left(\left(\nabla^g_V \Theta_c\right)U,W\right)+g\left(V,\left(\nabla^g_W \Theta_c\right)U\right).$$

Similarly, we have

(10)
$$(\phi_{\Theta_c} g)(W,V,U)=-g\left(\left(\nabla^g_W \Theta_c\right)V,U\right)+g\left(\left(\nabla^g_V \Theta_c\right)W,U\right)+g\left(V,\left(\nabla^g_U \Theta_c\right)W\right).$$

Adding (9) and (10), we get

$$(\phi_{\Theta_c} g)(U,V,W)+(\phi_{\Theta_c} g)(W,V,U)=2g\left(U,(\nabla^g_V \Theta_c)W\right). \tag{11}$$

Now, substituting $\phi_{\Theta_c} g=0$ in equation (11), we get $\nabla^g\Theta_c=0$.

□

If g (pseudo-Riemannian metric) is pure associated with an almost complex Silver structure Θ_c, then g is pure along with the almost complex structure J. From equation (2), we have

$$\phi_{\Theta_c} g=\sqrt{2}\,\phi_J g.$$

As a consequence of the above equality and theorem (2.2), we have the following.

Theorem 2.3. *Let J be the almost complex structure of an almost complex Norden Silver manifold (M^{2n},Θ_c,g). If $\phi_J g=0$, then almost complex Silver structure Θ_c is integrable.*

Now, from the Theorem (2.2) and Theorem (2.3), a Kaehler-Norden Silver manifold can be defined as following.

Definition 2.4. *Let ∇^g be a Levi-Civita connection of g. A Kaehler-Norden Silver manifold is given by a triplet (M^{2n},Θ_c,g) which consists of a smooth manifold M^{2n} associated with a pseudo-Riemannian metric g and an almost complex Silver structure Θ_c such that $\nabla^g\Theta_c=0$. Here the metric g is supposed to be Nordenian, i.e. $g(\Theta_c U,V)=g(U,\Theta_c V)$.*

Let $\tilde{g}$ be a twin Norden Silver metric for an almost complex Norden Silver manifold (M^{2n},Θ_c,g) and is defined by

$$\tilde{g}(U,V)=(g\circ\Theta_c)(U,V)=g(\Theta_c U,V). \tag{12}$$

Obviously $\tilde{g}(\Theta_c U,V)=\tilde{g}(U,\Theta_c V)$, for any vector fields U and V on M. It should be noted that both the metrics g and $\tilde{g}$ are necessarily of signature (n,n).

Moreover, by equation (6), we can write

$$\tilde{g}(U,\Theta_c V)=2\tilde{g}(U,V)-3g(U,V).$$

Now, we have the following theorem.

Theorem 2.5. *An almost complex Silver structure* Θ_c *is an isomorphism on the tangent space* T_uM, $\forall\ u \in M$.

Proof. Let $U \in ker\,\Theta_c$ i.e. $\Theta_c U = 0$. Consequently, we have $\Theta_c^2 U = 0$. Now, from equation (1), it follows that $U = 0$, i.e. $ker\,\Theta_c = \{0\}$. Therefore, Θ_c is an isomorphism on T_uM. □

3. Adapted Connections on Almost Complex Norden Silver Manifolds

Etayo et al. [4] and Kumar et al. [10] investigated the connections which parallelize the Riemannian metric and almost Golden structures. Motivated by the work of Etayo et al. and Kumar et al., we study the connections which parallelize the pseudo-Riemannian metric and the almost complex Silver structures.

Definition 3.1. *Let* ∇^α *is a linear connection on* M *and* (M^{2n}, Θ_c, g) *is an almost complex Norden Silver manifold. If* ∇^α *parallelizes together* g *and* Θ_c, *i.e.,* $\nabla^\alpha g = 0$ *and* $\nabla^\alpha \Theta_c = 0$, *then* ∇^α *is an adapted connection to the almost complex Norden Silver structure* (Θ_c, g).

Theorem 3.2. *Suppose* ∇^α *is a linear connection on* M *and* (M^{2n}, Θ_c, g) *denotes the almost complex Norden Silver manifold. Then* ∇^α *is the adapted connection to the almost complex Norden structure* (J^{Θ_c}, g), *induced by* Θ_c *if and only if* ∇^α *be the adapted connection to almost complex Norden Silver structure* (Θ_c, g).

Proof. Let ∇^α be the adapted connection to almost complex Silver structure Θ_c, i.e. $\nabla^\alpha \Theta_c = 0$, then using equation (3), we have

$$\begin{aligned} \nabla^\alpha_U J^{\Theta_c} V &= \mp \frac{1}{\sqrt{2}} \nabla^\alpha_U \Theta_c V \pm \frac{1}{\sqrt{2}} \nabla^\alpha_U V \\ &= \mp \frac{1}{\sqrt{2}} [\Theta_c \nabla^\alpha_U V - \nabla^\alpha_U V] \\ &= J^{\Theta_c} \nabla^\alpha_U V, \end{aligned}$$

for any vector fields U, V on M and hence this implies that $\nabla^\alpha J^{\Theta_c} = 0$. Similarly, taking $\nabla^\alpha J^{\Theta_c} = 0$ and using (2), we get $\nabla^\alpha \Theta_c = 0$. □

Suppose (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Let ∇^α is an adapted connection to the almost complex Norden Silver structure (Θ_c, g) and ∇^g denotes the Levi-Civita connection of g. Now, let $S \in \Im^1_2(M)$ denote the potential tensor of ∇^α associated with ∇^g and is given by

$$S(U, V) = \nabla^\alpha_U V - \nabla^g_U V, \tag{13}$$

for any vector fields U and V on M. Thus, the adapted connections for an almost complex Norden Silver manifold (M^{2n}, Θ_c, g) are given as follows:

Theorem 3.3. *Let (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then the set of linear connections adapted to the (Θ_c, g) is*

$$\left\{\nabla^g + S : \begin{array}{c} (\nabla^g_U \Theta_c)\, V = \Theta_c S\,(U,V) - S(U, \Theta_c V) \\ g\,(S(U,V),W) + g\,(S(U,W),V) = 0 \end{array}\right\}$$

for any vector fields U, V and W on M.

Proof. Let S denote the potential tensor of the adapted connection ∇^α with respect to ∇^g, then from equation (13), we have

$$\nabla^\alpha_U V = \nabla^g_U V + S(U,V).$$

By using above equation, we get

$$(\nabla^\alpha_U \Theta_c)\, V = (\nabla^g_U \Theta_c)\, V - \Theta_c S\,(U,V) + S(U, \Theta_c V),$$

and

$$(\nabla^\alpha_U g)\, V = -g\,(S(U,V),W) - g\,(S(U,W),V)\,.$$

Since ∇^α be the adapted connection to the almost complex Norden Silver structure (Θ_c, g). Hence, we get the desired result. □

For an arbitrary connection D from (12), we have

$$(D_U \tilde{g})(V,W) = (D_U g)(\Theta_c V, W) + g\,((D_U \Theta_c)V, W)\,. \tag{14}$$

Thus, we have the following result.

Theorem 3.4. *Suppose (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then $\nabla^\alpha g = 0$ and $\nabla^\alpha \tilde{g} = 0$, if and only if ∇^α is the adapted connection to the almost complex Norden Silver structure (Θ_c, g).*

Bilen et al. [2] introduced a special connection of the first type and second type for an almost complex Norden Golden manifold. Similarly, in case of an almost complex Norden Silver manifold (M^{2n}, Θ_c, g) the special connection of the first type and second type can be defined as follows:

Definition 3.5. *A linear connection $\nabla^i_U V = \nabla^g_U V + S^i(U,V)$ on an almost complex Norden Silver manifold (M^{2n}, Θ_c, g) satisfying the conditions $g\left(S^i(U,V), \Theta_c W\right) = g\left(S^i(U,W), \Theta_c V\right)$ and $\nabla^i \tilde{g} = 0$ is said to be a special connection of the first type, where S^i is a tensor field of type $(1,2)$.*

Taking the covariant derivative of the twin Norden Silver metric $\tilde{g}$ with respect to ∇^i, we get

$$\left(\nabla^i_U \tilde{g}\right)(V,W) = (\nabla^g_U \tilde{g})\,(V,W) - \tilde{g}\left(S^i(U,V),W\right) - \tilde{g}\left(S^i(U,W),V\right).$$

Now, using the definition of the twin Norden Silver metric and a special connection of the first type, we get the following equation

$$(\nabla^g_U \tilde{g})\,(V,W) = 2\, g\left(S^i(U,V), \Theta_c W\right)$$

or,

$$\Theta_c S^i(U,V) = \frac{1}{2}(\nabla^g_U \Theta_c)V.$$

By a direct calculation, we have

$$(15) \qquad \left(\nabla^i_U g\right)(V,W) = -\frac{1}{3} g\left(V, (\nabla^g_U \Theta_c)W\right) \neq 0.$$

Therefore from Definition (2.4), Definition (3.5), Theorem (3.4) and equation (15), we have the following result.

Theorem 3.6. *Suppose $\left(M^{2n}, \Theta_c, g\right)$ is an almost complex Norden Silver manifold. Then the special connection of the first type ∇^i is not an adapted connection to the almost complex Norden Silver structure (Θ_c, g) of M^{2n}. Furthermore, the special connection of the first type ∇^i is an adapted connection to (Θ_c, g) if almost complex Norden Silver manifold (M^{2n}, Θ_c, g) is a Kaehler-Norden Silver manifold.*

Now, we give the definition of special connection of the second type.

Definition 3.7. *A linear connection $\nabla^{ii}_U V = \nabla^g_U V + S^{ii}(U,V)$ on an almost complex Norden Silver manifold (M^{2n}, Θ_c, g) satisfying the conditions $g\left(S^{ii}(U,V), \Theta_c W\right) = g\left(S^{ii}(W,V), \Theta_c U\right)$ and $\nabla^{ii}\tilde{g} = 0$ is said to be a special connection of the second type, where S^{ii} denotes a tensor field of type $(1,2)$.*

From the definition (3.7), we have

$$\left(\nabla^{ii}_U g\right)(V,W) = (\nabla^g_U g)(V,W) - g\left(S^{ii}(U,V),W\right) - g\left(S^{ii}(U,W),V\right).$$

Since ∇^g represents the Levi-Civita connection of g, above equation yields

$$(16) \qquad \left(\nabla^{ii}_U g\right)(V,W) = -g\left(S^{ii}(U,V),W\right) - g\left(S^{ii}(U,W),V\right).$$

For a special connection of the second type, by taking the covariant derivative of the twin Norden Silver metric $\tilde{g}$ with respect to ∇^{ii}, we obtain

$$(17) \quad 2g\left(\Theta_c S^{ii}(U,V),W\right) = (\nabla^g_U \tilde{g})(V,W) - (\nabla^g_V \tilde{g})(W,U) + (\nabla^g_W \tilde{g})(U,V).$$

From equation (14), we get

$$(\nabla^g_U \tilde{g})(V,W) = (\nabla^g_U g)(\Theta_c V, W) + g\left((\nabla^g_U \Theta_c)V, W\right).$$

Since ∇^g is the Levi-Civita connection of g, it follows that

$$(\nabla^g_U \tilde{g})(V,W) = g\left((\nabla^g_U \Theta_c)V, W\right).$$

Using above equation in equation (17), so we have

(18)

$$2g\left(\Theta_c S^{ii}(U,V),W\right) = g\left((\nabla^g_U \Theta_c)V,W\right) - g\left((\nabla^g_V \Theta_c)W,U\right) + g\left((\nabla^g_W \Theta_c)U,V\right).$$

Suppose (M^{2n}, Θ_c, g) represents the Kaehler-Norden Silver manifold. Now, making use of $\nabla^g \Theta_c = 0$ in equations (16) and (18), we get

$$\nabla^{ii} g = 0 \text{ and } S^{ii}(U,V) = 0.$$

Therefore from Theorem (3.4) and the Definition (3.7), we have

Theorem 3.8. *Let (M^{2n}, Θ_c, g) be a Kaehler-Norden Silver manifold. Then the special connection of the second type ∇^{ii} is an adapted connection to the almost complex Norden Silver structure (Θ_c, g).*

Suppose $\tilde{g}$ is a twin Norden Silver metric and $\widetilde{\nabla}^{\tilde{g}}$ denote the Levi-Civita connection of $\tilde{g}$. Let $\widetilde{\nabla}^{\alpha}$ is an adapted connection to (Θ_c, g) and then $\tilde{S}$ is the potential tensor of $\widetilde{\nabla}^{\alpha}$ associated with $\widetilde{\nabla}^{\tilde{g}}$ given by

$$\tilde{S}\,(U,V) = \widetilde{\nabla}^{\alpha}_U V - \widetilde{\nabla}^{\tilde{g}}_U V,$$

for any vector fields U, V on M and $\tilde{S} \in \Im^1_2(M)$.

If $\widetilde{\nabla}^{\alpha}$ be the adapted connection to the almost complex Norden Silver structure (Θ_c, g) of M, then by the Theorem (3.4), we have $\widetilde{\nabla}^{\alpha}\Theta_c = 0$ and $\widetilde{\nabla}^{\alpha}\tilde{g} = 0$. Now, analogous to the Theorem (3.3), we have

Theorem 3.9. *Let (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then the set of linear connections adapted to the (Θ_c, g) is*

$$\left\{ \widetilde{\nabla}^{\tilde{g}} + \tilde{S} : \begin{array}{c} (\widetilde{\nabla}^{\tilde{g}}_U \Theta_c)V = \Theta_c\, \tilde{S}(U,V) - \tilde{S}\,(U, \Theta_c V) \\ g(\tilde{S}(U,V), \Theta_c W) + g(\tilde{S}(U,W), \Theta_c V) = 0 \end{array} \right\}$$

for any vector fields U, V and W on M.

Now, operating the ϕ_{Θ_c}-operator to the twin Norden Silver metric $\tilde{g}$ and then making use of equation (7), we get

$$\begin{aligned} (\phi_{\Theta_c}\tilde{g})\,(U,V,W) &= (\Theta_c\, U)\,(\tilde{g}(V,W)) - U\,(\tilde{g}\,(\Theta_c V, W)) + \tilde{g}\,((L_V \Theta_c)\, U, W) \\ &\quad + \tilde{g}\,(V, (L_W \Theta_c)\, U) \\ &= (L_{\Theta_c U}\tilde{g} - L_U\,(\tilde{g} \circ \Theta_c))\,(V,W) + \tilde{g}\,(V, \Theta_c L_U W) \\ &\quad - \tilde{g}\,(\Theta_c V, L_U W) \\ &= (\phi_{\Theta_c} g)\,(U, \Theta_c V, W) + g\,(N_{\Theta_c}\,(U,V)\,, W)\,. \end{aligned} \tag{19}$$

Now, we have the following theorem.

Theorem 3.10. *Suppose (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold and $\widetilde{\nabla}^{\tilde{g}}$ denotes the Levi-Civita connection of twin Norden Silver metric $\tilde{g}$. Then*

(i) if $\phi_{\Theta_c}\tilde{g} = 0$, then Θ_c is integrable,

(ii) the equality $\phi_{\Theta_c}\tilde{g} = 0$ is equivalent to $\widetilde{\nabla}^{\tilde{g}}\Theta_c = 0$.

Therefore from the second condition of the Theorem (3.10) and equation (19), we infer that; if $\widetilde{\nabla}^{\tilde{g}}\Theta_c = 0$, then an almost complex Norden Silver manifold (M^{2n}, Θ_c, g) becomes a Kaehler-Norden Silver manifold.

Theorem 3.11. *Let (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. The linear connection $\widetilde{\nabla}^{\alpha}_U V = \tilde{S}(U,V) + \widetilde{\nabla}^{\tilde{g}}_U V$ on M satisfying $g(\tilde{S}(U,V), W) = g(\tilde{S}(U,W), V)$ and $\widetilde{\nabla}^{\alpha} g = 0$. Then, $\widetilde{\nabla}^{\alpha}$ is not an adapted connection on the almost complex Norden Silver structure (Θ_c, g). Furthermore, $\widetilde{\nabla}^{\alpha}$ be an adapted* *connection on the almost complex Norden Silver structure if* (M^{2n}, Θ_c, g) *is a Kaehler-Norden Silver manifold.*

Proof. Let U, V and W be any vector fields on M, then by the definition of $\widetilde{\nabla}^{\alpha}$, we get

$$(\widetilde{\nabla}^{\alpha}_{U}\tilde{g})(V,W) = (\widetilde{\nabla}^{\tilde{g}}\tilde{g})(V,W) - \tilde{g}(\tilde{S}(U,V),W) - \tilde{g}(\tilde{S}(U,W),V),$$

Since $\widetilde{\nabla}^{\tilde{g}}$ is a Levi-Civita connection of $\tilde{g}$, above equation yields

$$\begin{aligned}(\widetilde{\nabla}^{\alpha}_{U}\tilde{g})(V,W) &= -\tilde{g}(\tilde{S}(U,V),W) - \tilde{g}(\tilde{S}(U,W),V)\\ &= -g(\Theta_c\tilde{S}(U,V),W) - g(\Theta_c\tilde{S}(U,W),V). \quad (20)\end{aligned}$$

Taking covariant derivative of the metric g corresponding to $\widetilde{\nabla}^{\alpha}$, we obtain

$$(\widetilde{\nabla}^{\alpha}_{U}g)(V,W) = (\widetilde{\nabla}^{\tilde{g}}_{U}g)(V,W) - g(\tilde{S}(U,V),W) - g(\tilde{S}(U,W),V).$$

By the assumptions in theorem, above equation becomes

$$(\widetilde{\nabla}^{\tilde{g}}_{U}g)(V,W) = 2g(\tilde{S}(U,V),W),$$

Now replacing W by $\Theta_c W$ in above equation, yields

$$(\widetilde{\nabla}^{\tilde{g}}_{U}g)(V,\Theta_c W) = 2g(\tilde{S}(U,V),\Theta_c W). \quad (21)$$

By definition of $\tilde{g}$ and equation (14), we have

$$(\widetilde{\nabla}^{\tilde{g}}_{U}\tilde{g})(V,W) = (\widetilde{\nabla}^{\tilde{g}}_{U}g)(V,\Theta_c W) + g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V,W),$$

Since $\widetilde{\nabla}^{\tilde{g}}$ be the Levi-Civita connection of $\tilde{g}$, we get

$$(\widetilde{\nabla}^{\tilde{g}}_{U}g)(V,\Theta_c W) = -g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V,W),$$

Using above equality in (21), we have

$$-g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V,W) = 2g(\Theta_c\tilde{S}(U,V),W),$$

or,

$$\Theta_c\tilde{S}(U,V) = -\frac{1}{2}(\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V. \quad (22)$$

Now, putting equation (22) in equation (20), we have

$$\begin{aligned}(\widetilde{\nabla}^{\alpha}_{U}\tilde{g})(V,W) &= \frac{1}{2}g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V,W) + \frac{1}{2}g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)W,V)\\ &= g((\widetilde{\nabla}^{\tilde{g}}_{U}\Theta_c)V,W) \neq 0.\end{aligned}$$

Hence, the result follows. □

Kumar et al. [10] established a special connection of the type three for an almost complex Norden Golden manifold. Analogously, we introduce a special connection of the type three for an almost complex Norden Silver manifold as given below:

Definition 3.12. *Suppose ∇^g denotes the Levi-Civita connection of g and (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then, we define the special connection of the type three ∇^{iii} of (M^{2n}, Θ_c, g) by the following relation*

$$\nabla^{iii}_U V = \nabla^g_U V + \frac{1}{2}\left(\nabla^g_U J^{\Theta_c}\right) J^{\Theta_c} V, \tag{23}$$

where U, V be any vector fields on M and J^{Θ_c} denote the almost complex structure induced from the Silver structure Θ_c.

Theorem 3.13. *Let (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then the special connection of the type three ∇^{iii} is an adapted connection to the almost complex Norden Silver structure (Θ_c, g).*

Proof. Let U, V and W be any vector fields on M, by using (23), we get

$$\nabla^{iii}_U J^{\Theta_c} V = \frac{1}{2}\nabla^g_U J^{\Theta_c} V + \frac{1}{2} J^{\Theta_c} \nabla^g_U V, \tag{24}$$

and

$$\begin{aligned} J^{\Theta_c}\nabla^{iii}_U V &= J^{\Theta_c}\nabla^g_U V + \frac{1}{2} J^{\Theta_c}\left[-\nabla^g_U V - J^{\Theta_c}\nabla^g_U J^{\Theta_c} V\right] \\ &= \frac{1}{2}\nabla^g_U J^{\Theta_c} V + \frac{1}{2} J^{\Theta_c}\nabla^g_U V. \end{aligned} \tag{25}$$

From (24) and (25), we have $\nabla^{iii} J^{\Theta_c} = 0$.
By using the Theorem (3.2), we obtain

$$\nabla^{iii}\Theta_c = 0.$$

Now, from (23), we have

$$\left(\nabla^{iii}_U g\right)(V, W) = \left(\nabla^g_U g\right)(V, W) - \frac{1}{2}\left[g\left(\left(\nabla^g_U J^{\Theta_c}\right) J^{\Theta_c} V, W\right) + g\left(\left(\nabla^g_U J^{\Theta_c}\right) J^{\Theta_c} W, V\right)\right].$$

As we know, ∇^g be the Levi-Civita connection of g, above equation yields

$$\begin{aligned} \left(\nabla^{iii}_U g\right)(V, W) &= \frac{1}{2}\Big[g\left(\nabla^g_U V, W\right) + g\left(\nabla^g_U J^{\Theta_c} V, J^{\Theta_c} W\right) + g\left(\nabla^g_U W, V\right) \\ &\quad + g\left(\nabla^g_U J^{\Theta_c} W, J^{\Theta_c} V\right)\Big] \\ &= \frac{1}{2}\left[U g(V, W) + U g\left(J^{\Theta_c} V, J^{\Theta_c} W\right)\right] \\ &= 0. \end{aligned}$$

Hence, the proof. □

As we know, if the Nijenhuis tensor $N_{J^{\Theta_c}}$ vanishes then almost complex structure J^{Θ_c} is integrable and is given by

$$N_{J^{\Theta_c}}(U, V) = [J^{\Theta_c} U, J^{\Theta_c} V] - J^{\Theta_c}[J^{\Theta_c} U, V] - J^{\Theta_c}[U, J^{\Theta_c} V] + \left(J^{\Theta_c}\right)^2 [U, V], \tag{26}$$

for any vector fields U and V in M. It is well-known that Nijenhuis tensor $N_{J^{\Theta_c}}$ is also given by
(27)
$$N_{J^{\Theta_c}}(U,V) = (\nabla^g_U J^{\Theta_c})J^{\Theta_c}V + (\nabla^g_{J^{\Theta_c}U} J^{\Theta_c})V - (\nabla^g_V J^{\Theta_c})J^{\Theta_c}U - (\nabla^g_{J^{\Theta_c}V} J^{\Theta_c})U.$$

Theorem 3.14. *Let $N_{J^{\Theta_c}}$ and N_{Θ_c} are the Nijenhuis tensors of J^{Θ_c} and Θ_c, respectively. Suppose (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold, then the following relation holds*

(28) $$N_{J^{\Theta_c}} = \frac{1}{2} N_{\Theta_c},$$

where U and V are any vector fields on M.

Proof. Using (3) in equation (26) and then making use of (4), the result follows.

□

Suppose T^{iii} be the torsion tensor of the special connection of the type three ∇^{iii} on (M^{2n}, Θ_c, g), then using equation (23), we get

(29) $$T^{iii}(U,V) = \frac{1}{2}\left(\nabla^g_U J^{\Theta_c}\right) J^{\Theta_c}V - \frac{1}{2}\left(\nabla^g_V J^{\Theta_c}\right) J^{\Theta_c}U,$$

and

(30) $$T^{iii}\left(J^{\Theta_c}U, J^{\Theta_c}V\right) = -\frac{1}{2}\left(\nabla^g_{J^{\Theta_c}U} J^{\Theta_c}\right)V + \frac{1}{2}(\nabla^g_{J^{\Theta_c}V} J^{\Theta_c})U.$$

Subtracting equation (30) from equation (29) and using (27), we obtain

(31) $$T^{iii}(U,V) - T^{iii}\left(J^{\Theta_c}U, J^{\Theta_c}V\right) = \frac{1}{2} N_{J^{\Theta_c}}(U,V).$$

Therefore from equation (28) and (31), we have the following result.

Theorem 3.15. *Suppose (M^{2n}, Θ_c, g) is an almost complex Norden Silver manifold. Then Θ_c is integrable if and only if*

$$T^{iii}(U,V) = T^{iii}\left(J^{\Theta_c}U, J^{\Theta_c}V\right),$$

for any vector fields U and V on M.

4. Complex Silver Maps

Suppose $\theta : M \to N$ be a smooth mapping from M to N and (M, g), (N, g') be two pseudo-Riemannian manifolds. Then the differential $d\theta$ of θ is a section of the bundle $Hom(TM, \theta^{-1}TN) \to M$, where $\theta^{-1}TN$ is the pullback bundle with fibres $\left(\theta^{-1}TN\right)_u = T_{\theta(u)}N,\ u \in M$.

The second fundamental form of θ is given by

(32) $$\nabla d\theta(U,V) = \nabla^\theta_U\, d\theta(V) - d\theta(\nabla^M_U V),$$

for any vector fields U and V on M. It is also known that the second fundamental form is symmetric. Here ∇^M be the Levi-Civita connection on TM, ∇^θ denotes the pull-back of the Levi-Civita connection on N to the bundle $\theta^{-1}TN$, and ∇ be the connection on $Hom(TM, \theta^{-1}TN)$ induced from these connections.

If $trace \nabla d\theta = 0$, then the mapping θ is said to be harmonic. Let $\tau(\theta)$ be the tension field of θ and is defined by

$$\tau(\theta) = trace \nabla\, d\theta = \sum_{i=1}^{m} \nabla d\theta(e_i, e_i),$$

where $e_1, e_2, \ldots, e_m$ be the local orthonormal frame on M and $\tau(\theta) = 0$ if and only if θ is harmonic [1].

Definition 4.1. *Suppose $\theta : (M, \Theta_c, g) \to (N, \Theta'_c, g')$ is a smooth map from a complex Norden Silver manifold (M, Θ_c, g) to a complex Norden Silver manifold (N, Θ'_c, g'). Then, θ is said to be a complex Norden Silver map if*

$$d\theta\, \Theta_c = \Theta'_c\, d\theta. \tag{33}$$

From above definition, we have the following result.

Theorem 4.2. *Let θ is a complex Norden Silver map between Kaehler-Norden Silver manifolds (M, Θ_c, g) and (N, Θ'_c, g'). Then θ is a harmonic map.*

Proof. Let $U, V \in \Gamma(TM)$ then by (1), (32) and (33), we have

$$(\nabla d\theta)(U, \Theta_c V) = \nabla^\theta_U \Theta'^2_c d\theta(V) - 2d\theta(\nabla^M_U \Theta_c V) + 3\nabla^\theta_U d\theta(V),$$

Now, using (1) in above equation, yields

$$(\nabla d\theta)(U, \Theta_c V) = 2\nabla^\theta_U \Theta'_c d\theta(V) - 2d\theta(\nabla^M_U \Theta_c V). \tag{34}$$

Since, $(\nabla^M_U \Theta_c)V = 0$ for a Kaehler-Norden Silver manifold (M, Θ_c, g). i.e.

$$\nabla^M_U \Theta_c V = \Theta_c \nabla^M_U V.$$

Analogously, for a Kaehler-Norden Silver manifold (N, Θ'_c, g'), it follows that

$$\nabla^\theta_U \Theta'_c V = \Theta'_c \nabla^\theta_U V.$$

Substituting (33) in (34) and using $\nabla^M_U \Theta_c V = \Theta_c \nabla^M_U V$ and $\nabla^\theta_U \Theta'_c V = \Theta'_c \nabla^\theta_U V$, we get

$$(\nabla d\theta)(U, \Theta_c V) = 2\Theta'_c(\nabla^\theta_U d\theta(V) - d\theta(\nabla^M_U V)) = 2\Theta'_c(\nabla d\theta)(U, V). \tag{35}$$

By the symmetry of the second fundamental form, we obtain

$$(\nabla d\theta)(U, \Theta_c V) = (\nabla d\theta)(\Theta_c U, V). \tag{36}$$

From (35) and (36), we get

$$(\nabla d\theta)\left(\Theta_c U, \Theta_c V\right) = 2\Theta_c'^2 (\nabla d\theta)(U, V). \tag{37}$$

Suppose $\{e_1, e_2, \ldots, e_{2n}\}$ is an orthonormal basis of $T_u M,\ u \in M$, then by the Theorem (2.5), Θ_c is an isomorphism. Then $\{\Theta_c e_1, \Theta_c e_2, \ldots, \Theta_c e_{2n}\}$ is also an orthonormal basis of $T_u M$. From (1) and (37), it follows that

$$\sum_{i=1}^{2n} (\nabla d\theta)\left(\Theta_c e_i, \Theta_c e_i\right) = 2\left[2\sum_{i=1}^{2n} \Theta_c'(\nabla d\theta)(e_i, e_i) - 3\sum_{i=1}^{2n}(\nabla d\theta)(e_i, e_i)\right],$$

a simple calculation gives

$$\tau(\theta) = 4\ \Theta_c'\, \tau(\theta) - 6\,\tau(\theta),$$

or

$$\Theta_c'\, \tau(\theta) = \frac{7}{4}\tau(\theta). \tag{38}$$

Now, operating Θ_c' on equation (38) and making use of equation (1), we get

$$\Theta_c' \tau(\theta) = \frac{97}{32}\tau(\theta). \tag{39}$$

Subtracting (38) from (39), we have $\tau(\theta) = 0$.
Hence the assertion follows. □

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