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AN ELEMENTARY COMPUTATION OF HANKEL MATRICES ON THE UNIT DISC

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Abstract. In this paper, we compute directly the Hankel matrix representation of the Hankel operator on the Hardy space of the unit disc without using any classical kernel functions with respect to special orthonormal bases for the Hardy space and its orthogonal complement. This gives an elementary proof for the formula.

1. Introduction

Suppose that U is the unit disc in the complex plane and let bU be the boundary of U. For a function $\varphi \in L^\infty(bU)$, the Hankel operator with the symbol φ on the Hardy space $H^2(bU)$ is the bounded operator $H_\varphi : H^2(bU) \to H^2(bU)^\perp$ defined by

$$H_\varphi(f) = P^\perp(\varphi f)$$

where $H^2(bU)^\perp$ is the orthogonal complement of $H^2(bU)$ in $L^2(bU)$ and $P^\perp$ is the orthogonal projection of $L^2(bU)$ onto $H^2(bU)^\perp$. I would like to consider the matrix representation of the Hankel operator which is called the Hankel matrix with special orthonormal bases constructed by the author (see [1]). In fact, the author computed the Hankel matrices associated to more general bounded domains in the plain with very complicated formula using the classical kernel functions like the Sezgö kernel and the Garabedian kernel (see [3] and [2]).

In this paper, I would like to compute the Hankel matrices directly without using any kernel functions so that anyone who has studied complex analysis and matrix theory at the graduate level can easily understand it. First, I want to construct a special orthonormal basis for the Hardy space $H^2(bU)$, the orthogonal complement space $H^2(bU)^\perp$ and the L^2 space $L^2(bU)$ and then using such basis, I want to compute the Hankel matices associated to those bases. One of the greatest advantages when we develop the theory on a unit circle is that the points z and $1/z$ in the circle are conjugates to each other. And in this paper, instead of using the convention of expressing orthonormal

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basis in L^2 space as two (negagtive and positive) way infinite indexed functions, we use natural numbered functions for easier understanding.

In the section 2, we construct orthonormal bases for the Hardy space and the L^2 space with a given fixed point $a \in U$. And then in the section 3, we compute the Hankel matrix representation of the Hankel operator associated to those orthonormal bases for $H^2(bU)$ and $H^2(bU)^\perp$. We finally get a compact form of the Hankel matrix.

2. Orthonormal Bases

Throughout the paper, we assume that U is the unit disc. Let $L^2(bU)$ be the Hilbert space completion of $C^\infty(bU)$ with respect to the classical L^2 inner product $\langle\ ,\ \rangle$ defined by

$$\langle u, v\rangle = \int_{bU} u\,\overline{v}\; ds$$

where ds is the differential element of arc length on the boundary bU and let $H^2(bU)$ denote the classical Hardy space associated to U which is the space of holomorphic functions on U with L^2-boundary values in bU. Since $H^2(bU)$ can be regarded as the completion of the restrictions of holomorphic functions in $C^\infty(\overline{U})$ to bU in $L^2(bU)$, it follows from the inequality $|f(w)| \leq \|f\|_{L^2(bU)}\|C_w\|_{L^2(bU)}$ that the evaluation function at $a \in U$ is a continuous linear functional on $H^2(bU)$. Since $H^2(bU)$ is a closed subspace of $L^2(bU)$, there exists the orthogonal projection of $L^2(bU)$ onto $H^2(bU)$ which is denoted by

$$P : L^2(bU) \to H^2(bU).$$

In this section, I would like to construct an orthonormal basis for the $L^2(bU)$ without using any kernel functions. Let $a \in U$ be fixed.

Remark 2.1. *For $z \in bU$,*

$$\overline{\left(\frac{1-\overline{a}z}{z-a}\right)} = \frac{z-a}{1-\overline{a}z}. \tag{1}$$

Proof. Notice that $\overline{z} = 1/z$ for $z \in bU$. It is thus easy to see that

$$\overline{\left(\frac{1-\overline{a}z}{z-a}\right)} = \frac{1-a\overline{z}}{\overline{z}-\overline{a}} = \frac{1-a/z}{1/z-\overline{a}} = \frac{z-a}{1-\overline{a}z}.$$

□

Remark 2.2. *For $z \in bU$,*

$$\overline{\left(\frac{1}{z-a}\right)} = \frac{z}{1-\overline{a}z}. \tag{2}$$

Proof. It is straightforward from the identity $\overline{z} = 1/z$ for $z \in bU$. □

Remark 2.3. *For $z \in bU$,*

$$\overline{\left(\frac{1}{1-\overline{a}z}\right)} = \frac{z}{z-a}. \tag{3}$$

Proof. It is also straightforward from the identity $\overline{z} = 1/z$ for $z \in bU$. □

Remark 2.4. *For $z \in bU$,*

$$z\,ds = -i\,dz. \tag{4}$$

Proof. Let $z = z(t) = e^{it}, 0 \le t \le 2\pi$ be the parametrization of the unit circle with the positive orientation. Then we obtain the identities $dz = iz\,dt$ and $ds = |z'(t)|dt = dt$ and it follows that $z\,ds = z\,dt = -idz$. □

Now we can construct orthonormal bases for the Hardy space, the orthogonal complement and the L^2 space using previous remarks.

Theorem 2.5. *Let U be the unit disc and let $a \in U$ be fixed. For a positive integer p, we define the function x_p by*

$$x_p = \chi_o(p)\sqrt{\frac{1-|a|^2}{2\pi}}\frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} + \chi_e(p)\sqrt{\frac{1-|a|^2}{2\pi}}\frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}},$$

where χ_o and χ_e are the characteristic functions of the odd and even natural numbers, resp. Then

(i) The functions x_p where p runs over odd numbers form an orthonormal basis for the Hardy space $H^2(bU)$ on the unit disc.

(ii) The functions x_p where p runs over enen numbers form an orthonormal basis for the orthogonal complement $H^2(bU)^\perp$ of the Hardy space $H^2(bU)$ on the unit disc.

(iii) The functions x_p where p runs over all natural numbers form an orthonormal basis for the space $L^2(bU)$ on the unit disc.

Lemma 2.6. *For an odd number p,*

$$\left\langle \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}}, \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \right\rangle = \frac{2\pi}{1-|a|^2}. \tag{5}$$

Proof. It follows from Remark 2.1 and 2.3 that

$$\left\langle \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}}, \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \right\rangle = \int_{bU} \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \left(\frac{1-\overline{a}z}{z-a}\right)^{\frac{p-1}{2}} \frac{z}{z-a}\,ds.$$

Thus it equals from Remark 2.4

$$\frac{1}{i}\int_{bU} \frac{1}{(1-\overline{a}z)(z-a)}\,dz,$$

which proves the formula of Lemma 2.6 by the Cauchy integral Formula. □

Lemma 2.7. *For an even number p,*

$$\left\langle \frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}}, \frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}} \right\rangle = \frac{2\pi}{1-|a|^2}. \tag{6}$$

Proof. It follows from Remarks 2.1 and 2.2 that

$$\left\langle \frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}}, \frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}} \right\rangle = \int_{bU} \frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}} \left(\frac{z-a}{1-\overline{a}z}\right)^{\frac{p-2}{2}} \frac{z}{1-\overline{a}z}\, ds.$$

Thus it equals from Remark 2.4

$$\frac{1}{i}\int_{bU} \frac{1}{(z-a)(1-\overline{a}z)}\, dz,$$

which proves the formula of Lemma 2.7 by the Cauchy integral Formula. □

Lemma 2.8. *For an odd number p and an even number q,*

$$\left\langle \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}}, \frac{(1-\overline{a}z)^{\frac{q-2}{2}}}{(z-a)^{\frac{q}{2}}} \right\rangle = 0. \tag{7}$$

Proof. It follows from Remarks 2.1 and 2.2 that

$$\left\langle \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}}, \frac{(1-\overline{a}z)^{\frac{q-2}{2}}}{(z-a)^{\frac{q}{2}}} \right\rangle = \int_{bU} \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \left(\frac{z-a}{1-\overline{a}z}\right)^{\frac{q-2}{2}} \frac{z}{1-\overline{a}z}\, ds.$$

Thus it equals from Remark 2.4

$$\frac{1}{i}\int_{bU} \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \left(\frac{z-a}{1-\overline{a}z}\right)^{\frac{q-2}{2}} \frac{1}{1-\overline{a}z}\, dz,$$

which is equal to zero because the integrand is holomorphic in a neighborhood of the unit disc. Notice that the two numbers $\frac{p-1}{2}$ and $\frac{q-2}{2}$ in the exponents are both nonnegative. □

Lemma 2.9. *The span of the functions x_p where p runs over all odd numbers is dense in the Hardy space $H^2(bU)$.*

Proof. Observe that the functions x_p when p is an odd number are obviously in $H^2(bU)$. In order to prove the density of the span of the functions x_p, it is enough to show that if k is in $H^2(bU)$ and is orthogonal to the span, then k is identically the zero function. We claim that if k is such a function, then it vanishes to infinite order at a and hence it proves Lemma 2.9.

Now suppose that $k \in H^2(bU)$ is orthogonal to the span. It follows from the Cauchy Integral formula and Remark 2.4 that

$$k(a) = \frac{1}{2\pi i}\int_{bU} \frac{k(z)}{z-a}\, dz = \frac{1}{2\pi}\int_{bU} \frac{k(z)}{z-a} z\, ds.$$

Thus it follows from Remark 2.3 and the definition of the inner product that the above identity is equal to

$$\frac{1}{2\pi}\int_{bU} k(z)\overline{\left(\frac{1}{1-\overline{a}z}\right)}\,ds = \frac{1}{2\pi}\left\langle k, \frac{1}{1-\overline{a}z}\right\rangle,$$

which by definition of the function x_1 equals

$$\frac{1}{2\pi}\sqrt{\frac{2\pi}{1-|a|^2}}\,\langle k, x_1\rangle$$

which vanishes by the assumption and hence we have proved $k(a) = 0$.

For the computation of $k'(a)$, it follows from Remarks 2.1, 2.3, 2.4 that

$$\begin{aligned}\langle k, x_3\rangle &= \sqrt{\frac{1-|a|^2}{2\pi}}\left\langle k, \frac{z-a}{(1-\overline{a}z)^2}\right\rangle \\ &= \sqrt{\frac{1-|a|^2}{2\pi}}\int_{bU} k(z)\frac{1-\overline{a}z}{z-a}\frac{z}{z-a}\,ds \\ &= \frac{1}{i}\sqrt{\frac{1-|a|^2}{2\pi}}\int_{bU}\frac{k(z)(1-\overline{a}z)}{(z-a)^2}\,dz,\end{aligned}$$

which by the Residue theorem is equal to

$$\begin{aligned}&\sqrt{2\pi(1-|a|^2)}\,\frac{d}{dz}\left[k(z)(1-\overline{a}z)\right]\Big|_{z=a} \\ &\qquad = \sqrt{2\pi(1-|a|^2)}\left[k'(a)(1-|a|^2) - k(a)\overline{a}\right]\end{aligned}$$

and hence the above identity equals

$$\sqrt{2\pi(1-|a|^2)}\left[k'(a)(1-|a|^2)\right]$$

because of the identity $k(a) = 0$. However the assumption of orthgonality of the function k yields $k'(a) = 0$.

Now in order to show for all nonnegative integer j, $\displaystyle\frac{d^j}{dz^j}k(z)\Big|_{z=a} = 0$, we use the inductive method on j. Suppose that $j \geq 0$ is an integer and we have proved that $\displaystyle\frac{d^n}{dz^n}k(a) = 0$ for $n = 0, 1, \cdots, j$. Since k is orthogonal to the span,

$$\langle k, x_{2j+3}\rangle = 0. \tag{8}$$

Observe that for $p = 2j+3$, $(p-1)/2 = j+1$ and $(p+1)/2 = j+2$. Thus it follows from Remarks 2.1, 2.3, 2.4 that

$$\begin{aligned}0 = \left\langle k, \frac{(z-a)^{j+1}}{(1-\overline{a}z)^{j+2}}\right\rangle &= \int_{bU} k(z)\left(\frac{1-\overline{a}z}{z-a}\right)^{j+1}\frac{z}{z-a}\,ds \\ &= \frac{1}{i}\int_{bU}\frac{k(z)(1-\overline{a}z)^{j+1}}{(z-a)^{j+2}}\,dz,\end{aligned}$$

which by the Residue theorem and the Leibniz product rule equals

$$\begin{aligned}&\frac{2\pi}{(j+1)!}\frac{d^{j+1}}{dz^{j+1}}\left[k(z)(1-\overline{a}z)^{j+1}\right]\Big|_{z=a}\\&\qquad=\frac{2\pi}{(j+1)!}\sum_{n=0}^{j+1}\binom{j+1}{n}k^{(j+1-n)}(z)\left[(1-\overline{a}z)^{j+1}\right]^{(n)}\Big|_{z=a},\end{aligned}$$

which yields

$$\frac{2\pi}{(j+1)!}k^{(j+1)}(a)(1-|a|^2)^{j+1}$$

because of the inductive hypothesis and hence $k^{(j+1)}(a)=0$.

Theorefore we have proved by mathematical induction that the function k vanishes to infinite order and we have finished the proof of Lemma 2.9. □

Lemma 2.10. *The span of the functions x_p where p runs over all even numbers is dense in the orthogonal complement $H^2(bU)^\perp$ of the Hardy space $H^2(bU)$.*

Proof. The proof is very similar to the one of the previous lemma by considering functions x_q where q runs over even numbers. Thus, we omit the proof. □

Lemma 2.11. *The functions x_p with $p\in\mathbb{N}$ are linearly indepent.*

Proof. It is from Lemma 2.8 enough to show that for different odd numbers p_1,p_2 and different even numbers q_1,q_2,

$$\langle x_{p_1},x_{p_2}\rangle=\langle x_{q_1},x_{q_2}\rangle=0.$$

Suppose that p_1 and p_2 are odd numbers with $p_2>p_1$. Then we obtain from Remarks 2.1, 2.3, 2.4

$$\begin{aligned}\langle x_{p_1},x_{p_2}\rangle&=\frac{1-|a|^2}{2\pi}\left\langle\frac{(z-a)^{\frac{p_1-1}{2}}}{(1-\overline{a}z)^{\frac{p_1+1}{2}}},\frac{(z-a)^{\frac{p_2-1}{2}}}{(1-\overline{a}z)^{\frac{p_2+1}{2}}}\right\rangle\\&=\frac{1-|a|^2}{2\pi}\int_{bU}\frac{(z-a)^{\frac{p_1-1}{2}}}{(1-\overline{a}z)^{\frac{p_1+1}{2}}}\frac{(1-\overline{a}z)^{\frac{p_2-1}{2}}}{(z-a)^{\frac{p_2-1}{2}}}\frac{z}{z-a}\,ds\\&=\frac{1-|a|^2}{2\pi i}\int_{bU}(1-\overline{a}z)^{\frac{p_2-p_1-2}{2}}\frac{1}{(z-a)^{\frac{p_2-p_1+2}{2}}}\,dz,\end{aligned}$$

which by the Residue theorem and the Leibniz product rule equals

$$(1-|a|^2)\frac{1}{\left(\frac{p_2-p_1}{2}\right)!}\frac{d^{\frac{p_2-p_1}{2}}}{dz^{\frac{p_2-p_1}{2}}}\left[(1-\overline{a}z)^{\frac{p_2-p_1-2}{2}}\right]\Big|_{z=a}$$

which vanishes because the number of times of differentiating is bigger than the degree of the polynomial. Similarly we can show the orthogonality of the functions x_q's indexed with even numbers. In fact, suppose that q_1 and q_2 are even numbers with $q_2 > q_1$. Then we obtain from Remarks 2.1, 2.3, 2.4

$$\begin{aligned}
\langle x_{q_2}, x_{q_1}\rangle &= \frac{1-|a|^2}{2\pi}\left\langle \frac{(1-\overline{a}z)^{\frac{q_2-2}{2}}}{(z-a)^{\frac{q_2}{2}}}, \frac{(1-\overline{a}z)^{\frac{q_1-2}{2}}}{(z-a)^{\frac{q_1}{2}}}\right\rangle \\
&= \frac{1-|a|^2}{2\pi}\int_{bU} \frac{(1-\overline{a}z)^{\frac{q_2-2}{2}}}{(z-a)^{\frac{q_2}{2}}} \frac{(z-a)^{\frac{q_1-2}{2}}}{(1-\overline{a}z)^{\frac{q_1-2}{2}}} \frac{z}{1-\overline{a}z}\, ds \\
&= \frac{1-|a|^2}{2\pi i}\int_{bU} (1-\overline{a}z)^{\frac{q_2-q_1-2}{2}} \frac{1}{(z-a)^{\frac{q_2-q_1+2}{2}}}\, dz,
\end{aligned}$$

which by the Residue theorem and the Leibniz product rule equals

$$(1-|a|^2)\frac{1}{\left(\frac{q_2-q_1}{2}\right)!}\frac{d^{\frac{q_2-q_1}{2}}}{dz^{\frac{q_2-q_1}{2}}}\left[(1-\overline{a}z)^{\frac{q_2-q_1-2}{2}}\right]\Big|_{z=a}$$

which vanishes because the number of times of differentiating is bigger than the degree of the polynomial. □

Proof of Theorem 2.5. We combine all lemmas from Lemma 2.6 through Lemma 2.11 to finish the proof of the Theorem 2.5. □

3. Hankel Matrices

Now we are ready to compute the Hankel matix of the Hardy space associated to the orthonormal bases constructed in the section 2.

Lemma 3.1. *If p and m are odd numbers and l is an even number, then*

$$\left\langle \frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} \frac{(z-a)^{\frac{m-1}{2}}}{(1-\overline{a}z)^{\frac{m+1}{2}}}, \frac{(1-\overline{a}z)^{\frac{l-2}{2}}}{(z-a)^{\frac{l}{2}}}\right\rangle = 0.$$

Proof. This is trivial because the first component of the inner product is in the Hardy space $H^2(bU)$ and the second one is in it's complement. □

Suppose that p and l are even numbers and m is an odd number. As before, it follows from Remarks 2.1, 2.3, 2.4 that

$$\begin{aligned}
&\left\langle \frac{(z-a)^{\frac{p-2}{2}}}{(1-\overline{a}z)^{\frac{p}{2}}} \frac{(z-a)^{\frac{m-1}{2}}}{(1-\overline{a}z)^{\frac{m+1}{2}}}, \frac{(1-\overline{a}z)^{\frac{l-2}{2}}}{(z-a)^{\frac{l}{2}}}\right\rangle \\
&\qquad\qquad = \frac{1}{i}\int_{bU} \frac{(1-\overline{a}z)^{\frac{p-m-l-3}{2}}}{(z-a)^{\frac{p-m-l+3}{2}}}\, dz.
\end{aligned} \tag{9}$$

Lemma 3.2. *If p and l are even numbers and m is an odd number with $p \leq m+l-3$, then the identity (9) is equal to zero.*

Proof. In this case, since the number $(p-m-l+3)/2$ is nonpositive, the integrand of (9) is holomorphic in a neighborhood of the unit disc and hence the line integral vanishes. □

Lemma 3.3. *If p and l are even numbers and m is an odd number with $p = m + l - 1$, then the identity (9) is equal to*

$$\frac{2\pi}{(1-|a|^2)^2}. \tag{10}$$

Proof. In this case, observe that $(p-m-l+3)/2 = 1$ and $p-m-l-3)/2 = -2$. It thus follows from the Cauchy integral formula that the identity (9) equals

$$\frac{1}{i}\cdot 2\pi i \cdot \left.\frac{1}{(1-\overline{a}z)^2}\right|_{z=a} = \frac{2\pi}{(1-|a|^2)^2} \tag{11}$$

and we are done. □

Lemma 3.4. *If p and l are even numbers and m is an odd number with $p = m + l + 1$, then the identity (9) is equal to*

$$\frac{2\pi\,\overline{a}}{(1-|a|^2)^2}. \tag{12}$$

Proof. In this case, observe that $(p-m-l+3)/2 = 2$ and $p-m-l-3)/2 = -1$. It thus follows from the Residue theorem that the identity (9) equals

$$\frac{1}{i}\cdot 2\pi i \cdot \left.\frac{d}{dz}\left[\frac{1}{1-\overline{a}z}\right]\right|_{z=a} = \frac{2\pi\overline{a}}{(1-|a|^2)^2} \tag{13}$$

and we are done. □

Lemma 3.5. *If p and l are even numbers and m is an odd number with $p = m + l + 3$, then the identity (9) is equal to zero.*

Proof. In this case, observe that $(p-m-l+3)/2 = 3$ and $(p-m-l-3)/2 = 0$. Thus the integrand is the function $(z-a)^{-3}$ which is meromorphic in a neighborhood of the unit disc with a single pole of order 3 at a and hence by the fundamental theorem of calculus, the identity (9) equals zero. □

Lemma 3.6. *If p and l are even numbers and m is an odd number with $p \geq m + l + 5$, then the identity (9) is equal to zero.*

Proof. In this case, observe that $(p-m-l+3)/2 \geq 4$ and $(p-m-l-3)/2 \geq 1$. Thus the integrand is meromorphic with the single pole at $z = a$ of order greater than or equal to 4. It thus follows from the Residue theorem that the identity (9) is equal to

$$\frac{2\pi}{\left(\frac{p-m-l+1}{2}\right)!}\frac{d^{\frac{p-m-l+1}{2}}}{dz^{\frac{p-m-l+1}{2}}}\left.\left[(1-\overline{a}z)^{\frac{p-m-l-3}{2}}\right]\right|_{z=a} \tag{14}$$

which vanishes because the number $(p-m-l+1)/2$ of times of integration is bigger than the degree $(p-m-l-3)/2$ of the polynomial. □

Now we collect all lemmas in this section and are ready to prove the following main theorem.

Theorem 3.7. *Let U be the unit disc and let $a \in U$ be fixed. Let the set $\{x_p \,|\, p = 1, 2, \cdots\}$ be the orthonormal basis of $L^2(bU)$ defined as in Theorem 2.5 by*

$$x_p = \chi_o(p)\sqrt{\frac{1-|a|^2}{2\pi}}\,\frac{(z-a)^{\frac{p-1}{2}}}{(1-\overline{a}z)^{\frac{p+1}{2}}} + \chi_e(p)\sqrt{\frac{1-|a|^2}{2\pi}}\,\frac{(1-\overline{a}z)^{\frac{p-2}{2}}}{(z-a)^{\frac{p}{2}}}$$

where χ_o and χ_e are the characteristic functions of the odd and even natural numbers, resp. Suppose that $\varphi = \sum_{p=1}^{\infty} \alpha_p x_p \in L^\infty(bU)$ be the Fourier series representation associated to the basis $\{x_p \,|\, p = 1, 2, \cdots\}$. Let m and l be an odd and an even positive number, resp. Then the l-th and m-th entry of the Hankel matrix $[H_\varphi]$ associated to the Hankel operator $H_\varphi : H^2(bU) \to H^2(bU)^\perp$ with symbol φ with respect to the bases $\{x_p \,|\, p$ runs over odd numbers$\}$ and $\{x_p \,|\, p$ runs over even numbers$\}$ is given by

$$[H_\varphi]_{lm} = \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\alpha_{m+l-1} + \overline{a}\,\alpha_{m+l+1}). \tag{15}$$

Proof. Since the bases $\{x_p \,|\, p$ runs over odd numbers$\}$ and $\{x_p \,|\, p$ runs over even numbers$\}$ are orthonormal, it follows that

$$[H_\varphi]_{lm} = \langle H_\varphi(x_m), x_l\rangle = \sum_{p=1}^{\infty} \alpha_p \langle x_p x_m, x_l\rangle .$$

By Lemma 3.1, the above identity is equal to

$$[H_\varphi]_{lm} = \langle H_\varphi(x_m), x_l\rangle = \sum_{\substack{p=1 \\ p \text{ even}}}^{\infty} \alpha_p \langle x_p x_m, x_l\rangle . \tag{16}$$

On the other hand, it follows from Lemmas 3.2 through 3.6 that the (possibly) nonzero remaining terms in the summation of the identity (16) is only when p is $m + l - 1$ or $m + l + 1$. In the mean time, Lemma 3.3 and Lemma 3.4 yield that

$$[H_\varphi]_{lm} = \alpha_{m+l-1} \langle x_{m+l-1} x_m, x_l\rangle + \alpha_{m+l+1} \langle x_{m+l+1} x_m, x_l\rangle$$
$$= \alpha_{m+l-1}\left(\frac{1-|a|^2}{2\pi}\right)^{3/2} \frac{2\pi}{(1-|a|^2)^2} + \alpha_{m+l+1}\left(\frac{1-|a|^2}{2\pi}\right)^{3/2} \frac{2\pi\,\overline{a}}{(1-|a|^2)^2}$$

which is exactly equal to the formula (15) and hence we are done. □

On the other hand, it is easy to see from the identity (15) that the Hankel matrix $[H_\varphi]$ is computed by

$$[H_\varphi] = \frac{1}{\sqrt{2\pi(1-|a|^2)}} \left(\begin{bmatrix} \alpha_2 & \alpha_4 & \alpha_6 & \cdots \\ \alpha_4 & \alpha_6 & \alpha_8 & \cdots \\ \alpha_6 & \alpha_8 & \alpha_{10} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} + \overline{a} \begin{bmatrix} \alpha_4 & \alpha_6 & \alpha_8 & \cdots \\ \alpha_6 & \alpha_8 & \alpha_{10} & \cdots \\ \alpha_8 & \alpha_{10} & \alpha_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \right).$$

Therefore by defining the infinite matrices A and L by

$$(17) \qquad A_- = \begin{bmatrix} \alpha_2 & \alpha_4 & \alpha_6 & \cdots \\ \alpha_4 & \alpha_6 & \alpha_8 & \cdots \\ \alpha_6 & \alpha_8 & \alpha_{10} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \; L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

we have proved the following corollary.

Corollary 3.8. *With the same notation as Theorem 3.7 and (17), the Hankel matrix $[H_\varphi]$ associated to the Hankel operator $H_\varphi : H^2(bU) \to H^2(bU)^\perp$ with symbol φ with respect to the bases $\{x_p \,|\, p$ runs over odd numbers$\}$ and $\{x_p \,|\, p$ runs over even numbers$\}$ is given by*

$$(18) \qquad [H_\varphi] = \frac{1}{\sqrt{2\pi(1-|a|^2)}}(A + \overline{a}\, AL).$$

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