Honam Mathematical J. **43** (2021), No. 4, pp. 667–678
https://doi.org/10.5831/HMJ.2021.43.4.667

NEW BLOW-UP CRITERIA FOR A NONLOCAL REACTION-DIFFUSION SYSTEM

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Abstract. Blow-up phenomena for a nonlocal reaction-diffusion system with time-dependent coefficients are investigated under null Dirichlet boundary conditions. Using Kaplan's method with the comparison principle, we establish new blow-up criteria and obtain the upper bounds for the blow-up time of the solution under suitable measure sense in the whole-dimensional space.

1. Introduction

We study a nonlocal reaction-diffusion system with time-dependent coefficients

$$\begin{cases} u_t = \Delta u + k_1(t)u^p \int_\Omega v^q dx, & (x,t) \in \Omega \times (0,t^*), \\ v_t = \Delta v + k_2(t)v^r \int_\Omega u^s dx, & (x,t) \in \Omega \times (0,t^*), \end{cases} \tag{1.1}$$

subject to null Dirchlet boundary and initial conditions

$$\begin{cases} u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a bounded region with smooth boundary $\partial\Omega$, $k_1(t), k_2(t)$ are bounded positive C^1-functions, p, $r \geq 0$, q, $s > 0$, t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonnegative initial data $u_0(x), v_0(x)$ are C^1-functions which satisfy compatibility conditions. Then the existence and uniqueness of nonnegative local classical solution to (1.1)-(1.2) are well known ([1,2] and [3, Chapter 14]). More precise conditions for other data will be given later.

Our nonlocal reaction system (1.1) serves as a typical model in chemical reactions, population dynamics and heat transfer, where u and v represent the thickness of two kinds of chemical reactants, the densities of two biological populations during a migration and the temperatures of two different materials during a propagation respectively [4].

Received June 11, 2021. Accepted July 9, 2021.
2020 Mathematics Subject Classification. 35K51, 35B30, 35B40.
Key words and phrases. nonlocal reaction-diffusion system, time-dependent coefficient, blow-up criteria, upper bounds for the blow-up time.

During the past decades, there have been many works to deal with the blow-up phenomena for the solutions of local or nonlocal reaction-diffusion equations (systems), we refer the reader to the monograph [3-5] as well as to the survey paper [2] and the references therein. Roughly, it has been seen that existence of global and nonglobal solutions and behavior of the solutions to reaction-diffusion equations (systems) depend on dimension, nonlinearity, initial data and boundary condition. In particular, Quittner and Souplet [5, Chapter 5] introduced the qualitative properties of the solution to a nonlocal reaction-diffusion equation with Dirichlet boundary in detail. In a sense, the nonlocal models are more close to the practical problems than the local models, and now many local theories are no longer holding. Therefore, they are more difficult and challenging. In this paper, we would like to investigate the blow-up phenomena of the solution for a nonlocal reaction system, and our main aim is to establish a new blow-up criteria. As we all know, there are only a few works about the blow-up criteria to the reaction systems.

In [6], Xu and Ye investigated the following weakly coupled local reaction-diffusion problem for large initial data and suitable parameters

$$\begin{cases} u_t = \Delta u + u^p v^q, & (x,t) \in \Omega \times (0,t^*)\,, \\ v_t = \Delta v + v^r u^s, & (x,t) \in \Omega \times (0,t^*)\,, \end{cases}$$

they derived the exact value of the blow-up time under null Dirichlet boundary conditions. Payne and Philippin [7] considered the semilinear parabolic system with time-dependent coefficients as follows

$$\begin{cases} u_t = \Delta u + k_1(t)\, f_1(v)\,, & (x,t) \in \Omega \times (0,t^*)\,, \\ v_t = \Delta v + k_2(t)\, f_2(u)\,, & (x,t) \in \Omega \times (0,t^*)\,, \end{cases}$$

under null Dirichlet boundary conditions, they obtained sufficient conditions for the solution blows up in finite time therefore derived the upper bounds for the blow-up time. Tao and Fang [8] investigated the weakly coupled local reaction-diffusion system with time-dependent coefficients as follows

$$\begin{cases} u_t = \Delta u + k_1(t) u^p v^q, & (x,t) \in \Omega \times (0,t^*)\,, \\ v_t = \Delta v + k_2(t) v^r u^s, & (x,t) \in \Omega \times (0,t^*)\,, \end{cases}$$

under null Dirichlet boundary conditions, they obtained the blow-up criteria and lower bounds for the blow-up time of the solution under two different measures in high-dimensional space ($N \geq 3$). Recently, there have been new developments in the study of nonlocal reaction systems. In [9], the authors considered our problem (1.1)-(1.2) for $k_1(t) = k_2(t) = 1$ and obtained the lower bound for the blow-up time using differential inequality technique in high-dimensional space ($N \geq 3$). At the same time, they derived the upper bounds for the blow-up time in the norm of L^2 and L^∞.

Inspired by [8] and [9], we will combine Kaplan's method with the comparison principle to seek the sufficient conditions to guarantee the solution of problem (1.1)-(1.2) exists globally or blows up in finite time, and then derive the upper bounds for the blow-up time in whole-dimensional space ($N \geq 1$).

2. Main result

We consider the fixed membrane problem

$$\Delta\varphi + \lambda\varphi = 0, \quad x \in \Omega,$$
$$\varphi(x) = 0, \quad x \in \partial\Omega,$$

where λ_1, φ_1 and μ_1, ψ_1 are the first eigenvalue and the corresponding eigenfunctions for region Ω and $\Omega_\varepsilon := \{x \in \Omega | \operatorname{dist}(\mathrm{x}, \partial\Omega) \geq \varepsilon\}$ respectively.

Theorem 2.1. *Suppose that $k_1(t), k_2(t)$ are bounded functions and let*

$$\underline{K} := \min\{k_1(t), k_2(t)\}, \quad \bar{K} := \max\{k_1(t), k_2(t)\},$$

and (u, v) is the nonnegative classical solution of problem (1.1)-(1.2).
(I) *For $\max\{p+q, r+s\} \leq 1$, if the initial data are small enough and satisfy (2.6), then the solution of problem (1.1)-(1.2) exists globally.*
(II) *For $\min\{p, r\} > 1$, if the initial data are small enough and satisfy (2.9), then the solution of problem (1.1)-(1.2) exists globally; Meanwhile,*
(i) *For $p > 1$, $0 < q \leq 1$, if the initial data are large enough and satisfy (2.16), then the solution of problem (1.1)-(1.2) blows up in finite time t^* with the following upper bound*

$$\frac{1}{(1-p)(\mu_1 - k_1)} \ln\left[1 - \frac{(\mu_1 - k_1) U_1{}^{1-p}(0)}{\underline{K}}\right],$$

where $k_1 = -\frac{\lambda_1}{p-1}$, $U_1(t) = \int_{\Omega_\varepsilon} \omega_1\psi_1 dx$, and ω_1 is defined in (2.12);
(ii) *For $p > 1$, $q > 1$, if the initial data are large enough and satisfy (2.22), then the solution of problem (1.1)-(1.2) blows up in finite time t^* with the following upper bound*

$$\frac{1}{(1-p)(\mu_1 - k_2)} \ln\left[1 - \frac{(\mu_1 - k_2) U_2{}^{1-p}(0)}{\underline{K}|\Omega|^{1-q}}\right],$$

where $k_2 = -\frac{\lambda_1}{p-1}q$, $U_2(t) = \int_{\Omega_\varepsilon} \omega_2\psi_1 dx$, and ω_2 is defined in (2.18);
(iii) *For $r > 1$, $0 < s \leq 1$, if the initial data are large enough, then the solution of problem (1.1)-(1.2) blows up in finite time t^* with the following upper bound*

$$\frac{1}{(1-r)(\mu_1 - k_3)} \ln\left[1 - \frac{(\mu_1 - k_3) V_1{}^{1-r}(0)}{\underline{K}}\right],$$

where $k_3 = -\frac{\lambda_1}{r-1}$, $V_1(t) = \int_{\Omega_\varepsilon} \omega_3\psi_1 dx$, $\omega_3 = e^{k_3 t}\underline{v}$;
(iv) *For $r > 1$, $s > 1$, if the initial data are large enough, then the solution of problem (1.1)-(1.2) blows up in finite time t^* with the following upper bound*

$$\frac{1}{(1-r)(\mu_1 - k_4)} \ln\left[1 - \frac{(\mu_1 - k_4) V_2{}^{1-r}(0)}{\underline{K}|\Omega|^{1-s}}\right],$$

where $k_4 = -\frac{\lambda_1}{r-1}s$, $V_2(t) = \int_{\Omega_\varepsilon} \omega_4\psi_1 dx$, $\omega_4 = e^{k_4 t}\underline{v}$.

(v) *For* $\max\{p,r\} \le 1$, *if* $qs \le (1-p)(1-r)$ *and the initial data are small enough and satisfy (2.27), then the solution to problem (1.1)-(1.2) exists globally; while* $qs > (1-p)(1-r)$ *and the initial data are large enough and satisfy (2.31), then the solution to problem (1.1)-(1.2) blows up in finite time* t^*, *and an upper bound for* t^* *is*

$$\frac{1}{\tau},$$

where positive constant τ *satisfies (2.30).*

Proof. We will prove the theorem in three cases.
Case 1. $\max\{p+q, r+s\} \le 1$.

Let $(\bar{u}, \bar{v})$ be the solution to the following problem

$$\tag{2.1} \begin{cases} \bar{u}_t = \Delta\bar{u} + \bar{K}\bar{u}^p \int_\Omega \bar{v}^q dx, & (x,t) \in \Omega \times (0,t^*), \\ \bar{v}_t = \Delta\bar{v} + \bar{K}\bar{v}^r \int_\Omega \bar{u}^s dx, & (x,t) \in \Omega \times (0,t^*), \\ \bar{u}(x,t) = \bar{v}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t^*), \\ \bar{u}(x,0) = u_0(x), \bar{v}(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

then by the comparison principle we can easily know that the solution $(\bar{u}, \bar{v})$ to problem (2.1) is a supersolution to the solution (u, v) to the problem (1.1)-(1.2). Considering the auxiliary function

$$\Phi(t) = \int_\Omega (\bar{u} + \bar{v})\varphi_1 dx,$$

where $\int_\Omega \varphi_1(x)\, dx = 1$.
Differentiating $\Phi(t)$ and using (2.1), Green's formula, we have

$$\begin{aligned} \Phi'(t) &= \int_\Omega (\bar{u}_t + \bar{v}_t)\varphi_1 dx \\ &= -\lambda_1 \int_\Omega \bar{u}\varphi_1 dx + \bar{K}\int_\Omega \bar{u}^p\varphi_1 dx \int_\Omega \bar{v}^q dx \\ &\quad - \lambda_1 \int_\Omega \bar{v}\varphi_1 dx + \bar{K}\int_\Omega \bar{v}^r\varphi_1 dx \int_\Omega \bar{u}^s dx \\ &= -\lambda_1\Phi(t) + \bar{K}\int_\Omega \bar{u}^p\varphi_1 dx \int_\Omega \bar{v}^q dx + \bar{K}\int_\Omega \bar{v}^r\varphi_1 dx \int_\Omega \bar{u}^s dx \end{aligned} \tag{2.2}$$

Applying Hölder's and Young's inequalities to the second and third terms on the right hand of (2.2), we derive

$$\begin{aligned}
&\bar{K}\int_\Omega \bar{u}^p\varphi_1 dx\int_\Omega \bar{v}^q dx\\
&\le \bar{K}(\int_\Omega \bar{u}^{p+q}\varphi_1 dx)^{\frac{p}{p+q}}(\int_\Omega \varphi_1 dx)^{\frac{q}{p+q}}(\int_\Omega \bar{v}^{p+q}\varphi_1 dx)^{\frac{q}{p+q}}(\int_\Omega \varphi_1^{-\frac{q}{p}} dx)^{\frac{p}{p+q}}\\
&\le \bar{K}\frac{p}{p+q}(\int_\Omega \bar{u}^{p+q}\varphi_1 dx)+\bar{K}\frac{q}{p+q}(\int_\Omega \varphi_1^{-\frac{q}{p}} dx)^{\frac{p}{q}}(\int_\Omega \bar{v}^{p+q}\varphi_1 dx)\\
&\le \bar{K}\frac{p}{p+q}(\int_\Omega \bar{u}\varphi_1 dx)^{p+q}+\bar{K}\frac{q}{p+q}(\int_\Omega \varphi_1^{-\frac{q}{p}} dx)^{\frac{p}{q}}(\int_\Omega \bar{v}\varphi_1 dx)^{p+q}\\
&\le D_1\bar{K}\Phi^{p+q}, \qquad (2.3)
\end{aligned}$$

and

$$\bar{K}\int_\Omega \bar{v}^r\varphi_1 dx\int_\Omega \bar{u}^s dx \le D_2\bar{K}\Phi^{r+s}, \qquad (2.4)$$

where $D_1 = \max\{1, (\int_\Omega \varphi_1^{-\frac{q}{p}} dx)^{\frac{p}{q}}\}, D_2 = \max\{1, (\int_\Omega \varphi_1^{-\frac{s}{r}} dx)^{\frac{r}{s}}\}$. Substituting (2.3),(2.4) into (2.2), we can lead to the inequality

$$\begin{aligned}
\Phi'(t) &\le -\lambda_1\Phi(t) + D_1\bar{K}\Phi^{p+q}(t) + D_2\bar{K}\Phi^{r+s}(t)\\
&\le \Phi(t)\left(-\lambda_1 + \bar{K}D_1\Phi^{p+q-1} + \bar{K}D_2\Phi^{r+s-1}\right). \qquad (2.5)
\end{aligned}$$

By $p+q \le 1$ and $r+s \le 1$ we can easily get that the Φ^{p+q-1} and Φ^{r+s-1} in (2.5) are both monotone non-increasing about Φ.

Hence, if the initial data small enough satisfies

$$\bar{K}\left(D_1\Phi_0^{p+q-1} + D_2\Phi_0^{r+s-1}\right) < \lambda_1, \qquad (2.6)$$

where $\Phi_0 = \int_\Omega (u_0(x) + v_0(x))\varphi_1(x)\,dx$, then $\Phi(t)$ exists globally and by the comparison principle we can know that the solution (u, v) to problem (1.1)-(1.2) exists globally.

Case 2. $\min\{p, r\} > 1$.

(Global existence) Let

$$\bar{u} = \frac{\varphi_1(x)}{(A+t)^{l_1}}, \quad \bar{v} = \frac{\varphi_1(x)}{(A+t)^{l_2}},$$

where $\max_{\bar{\Omega}}\varphi_1(x) = 1$, A, l_1, l_2 are positive constants to be determined later.

By the condition $\min\{p,r\}>1$, we can directly compute

$$\begin{aligned}
&\bar{u}_t-\Delta\bar{u}-k_1(t)\,\bar{u}^p\int_\Omega \bar{v}^q dx\\
&=-l_1(A+t)^{-l_1-1}\varphi_1(x)+\lambda_1(A+t)^{-l_1}\varphi_1(x)\\
&\quad -k_1(t)\,\varphi_1^p(x)\,(A+t)^{-pl_1-ql_2}\int_\Omega \varphi_1^q dx\\
&\geq \varphi_1(x)\,(A+t)^{-l_1}\left[-l_1(A+t)^{-1}+\lambda_1-\bar{K}|\Omega|\,(A+t)^{-(p-1)l_1-ql_2}\right],
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\bar{v}_t-\Delta\bar{v}-k_2(t)\,\bar{v}^r\int_\Omega \bar{u}^s dx\\
&\geq \varphi_1(x)\,(A+t)^{-l_2}\left[-l_2(A+t)^{-1}+\lambda_1-\bar{K}|\Omega|\,(A+t)^{-(r-1)l_2-sl_1}\right].
\end{aligned}\tag{2.8}$$

For $l_1,l_2>0$, using $p>1$ and $r>1$ we know that $-(p-1)l_1-ql_2<0, -(r-1)l_2-sl_1<0$, it follows that $-l_1(A+t)^{-1}$, $-\bar{K}|\Omega|(A+t)^{-(p-1)l_1-ql_2}$, $-l_2(A+t)^{-1}$ and $-\bar{K}|\Omega|(A+t)^{-(r-1)l_2-sl_1}$ in (2.7) and (2.8) are monotone increasing about t respectively.

Hence, choosing A large enough satisfies

$$-l_1A^{-1}+\lambda_1-\bar{K}|\Omega|A^{-(p-1)l_1-ql_2}\geq 0 \quad and \quad -l_2A^{-1}+\lambda_1-\bar{K}|\Omega|A^{-(r-1)l_2-sl_1}\geq 0.$$

Then we have

$$\bar{u}_t-\Delta\bar{u}-k_1(t)\,\bar{u}^p\int_\Omega \bar{v}^q dx\geq 0 \quad and \quad \bar{v}_t-\Delta\bar{v}-k_2(t)\bar{v}^r\int_\Omega \bar{u}^s dx\geq 0.$$

Obviously, if the initial data small enough satisfy

$$u_0(x)\leq \frac{\varphi_1(x)}{A^{l_1}},\quad v_0(x)\leq \frac{\varphi_1(x)}{A^{l_2}},\quad x\in\Omega. \tag{2.9}$$

Then $(\bar{u},\bar{v})$ exists globally and therefore by the comparison principle we can deduce that the solution (u,v) to problem (1.1)-(1.2) exists globally.

(Blow-up)

(i) $p>1$, $0<q\leq 1$.

Let

$$\underline{v}=e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}}(x), \tag{2.10}$$

where $\int_\Omega \varphi_1(x)dx=1$.

By direct computation we have

$$\begin{aligned}
\underline{v}_t-\Delta\underline{v}&=-\frac{\lambda_1}{q}e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}}(x)+\frac{\lambda_1}{q}e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}}(x)-\frac{1}{q}(\frac{1}{q}-1)e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}-2}(x)\,|\nabla\varphi_1|^2\\
&\leq -\frac{\lambda_1}{q}e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}}(x)+\frac{\lambda_1}{q}e^{-\frac{\lambda_1}{q}t}\varphi_1^{\frac{1}{q}}(x)=0\leq k_2(t)\underline{v}^r\int_\Omega \underline{u}^s dx,
\end{aligned}$$

where $\underline{u}$ is the solution to the following problem

$$\begin{cases} \underline{u}_t = \Delta \underline{u} + \underline{K}\, \underline{u}^p \int_\Omega \underline{v}^q dx, & (x,t) \in \Omega \times (0, t^*), \\ \underline{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0, t^*), \\ \underline{u}(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.11}$$

then the comparison principle yields the solution $(\underline{u}, \underline{v})$ to problem (2.10)-(2.11) is a subsolution to the solution (u, v) to problem (1.1)-(1.2).

Next, we need to prove that $\underline{u}$ blows up in finite time.
Let

$$\omega_1 = e^{k_1 t} \underline{u}, \tag{2.12}$$

where $k_1 = -\frac{\lambda_1}{p-1}$.
Therefore, ω_1 and $\underline{u}$ both exist globally, or both blow up.
Applying (2.11) and (2.12) we can compute that ω_1 satisfies

$$\begin{cases} \omega_{1t} - \Delta \omega_1 = k_1 \omega_1 + \underline{K} {\omega_1}^p, & (x,t) \in \Omega \times (0, t^*), \\ \omega_1(x,t) = 0, & (x,t) \in \partial\Omega \times (0, t^*), \\ \omega_1(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.13}$$

Considering the auxiliary function

$$U_1(t) = \int_{\Omega_\varepsilon} \omega_1 \psi_1 dx,$$

where $\int_{\Omega_\varepsilon} \psi_1(x)\, dx = 1$.
Differentiating $U_1(t)$ and using (2.13) and Hölder's inequality, we have

$$\begin{aligned} U_1'(t) + (\mu_1 - k_1) U_1(t) &= \underline{K} \int_{\Omega_\varepsilon} {\omega_1}^p \psi_1 dx \\ &\geq \underline{K} \left(\int_{\Omega_\varepsilon} \omega_1 \psi_1 dx \right)^p = \underline{K} {U_1}^p(t). \end{aligned} \tag{2.14}$$

Now, solving the ordinary differential equation (2.14) we derive

$${U_1}^{1-p}(t) \leq \frac{\underline{K}}{\mu_1 - k_1} + \left({U_1}^{1-p}(0) - \frac{\underline{K}}{\mu_1 - k_1} \right) e^{(p-1)(\mu_1 - k_1)t}, \tag{2.15}$$

Since $p > 1$ and $\lambda_1 > 0$ we can easily deduce that $k_1 = -\frac{\lambda_1}{p-1} < 0$ and $\mu_1 - k_1 > 0$.

Therefore, by (2.15) we know that if initial data large enough satisfy

$$\begin{aligned} U_1(0) &= \int_{\Omega_\varepsilon} u_0(x) \psi_1(x)\, dx > \left(\frac{\underline{K}}{\mu_1 - k_1} \right)^{-\frac{1}{p-1}}, \\ v_0(x) &\geq \varphi_1^{\frac{1}{q}}(x), \quad x \in \Omega, \end{aligned} \tag{2.16}$$

then ω_1 blows up in finite time T_{ω_1} and

$$T_{\omega_1} \leq \frac{1}{(1-p)(\mu_1 - k_1)} \ln \left[1 - \frac{(\mu_1 - k_1) {U_1}^{1-p}(0)}{\underline{K}} \right].$$

Hence, $\underline{u}$ blows up in finite time and an upper bounds for t^* is

$$\frac{1}{(1-p)(\mu_1-k_1)}\ln\left[1-\frac{(\mu_1-k_1)\,{U_1}^{1-p}(0)}{\underline{K}}\right].$$

(ii) $p>1$, $q>1$.
Similarly, let

$$\underline{v}=e^{-\lambda_1 t}\varphi_1(x), \tag{2.17}$$

where $\int_\Omega \varphi_1(x)dx=1$.
By direct computation we have

$$\underline{v}_t-\Delta\underline{v}=-\lambda_1 e^{-\lambda_1 t}\varphi_1(x)+\lambda_1 e^{-\lambda_1 t}\varphi_1(x)=0\le k_2(t)\underline{v}^r\int_\Omega \underline{u}^s dx,$$

where $\underline{u}$ is the solution to the problem (2.11), then the comparison principle yields the solution $(\underline{u},\underline{v})$ to problem (2.17),(2.11) is a subsolution to the solution (u,v) to problem (1.1)-(1.2).

Next, we need to prove that $\underline{u}$ blows up in finite time.
Let

$$\omega_2=e^{k_2 t}\underline{u}, \tag{2.18}$$

where $k_2=-\frac{\lambda_1}{p-1}q$.
Therefore, ω_2 and $\underline{u}$ both exist globally, or both blow up.
Applying (2.11) and (2.18) we can compute that ω_2 satisfies

$$\begin{cases}\omega_{2t}-\Delta\omega_2=k_2\omega_2+\underline{K}{\omega_2}^p\int_\Omega\varphi_1^q dx, & (x,t)\in\Omega\times(0,t^*),\\ \omega_2(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,t^*), & \\ \omega_2(x,0)=u_0(x), \quad x\in\Omega. & \end{cases} \tag{2.19}$$

Considering the auxiliary function

$$U_2(t)=\int_{\Omega_\varepsilon}\omega_2\psi_1 dx,$$

where $\int_{\Omega_\varepsilon}\psi_1(x)\,dx=1$.
Differentiating $U_2(t)$ and using (2.19) and Hölder's inequality, we have

$$\begin{aligned}U_2'(t)+(\mu_1-k_2)\,U_2(t)&=\underline{K}\int_{\Omega_\varepsilon}{\omega_2}^p\psi_1 dx\int_\Omega\varphi_1^q dx\\ &\ge\underline{K}\left(\int_{\Omega_\varepsilon}\omega_2\psi_1 dx\right)^p|\Omega|^{1-q}=\underline{K}|\Omega|^{1-q}{U_2}^p(t).\end{aligned} \tag{2.20}$$

Now, solving the ordinary differential equation (2.20) we derive

$${U_2}^{1-p}(t)\le\frac{\underline{K}|\Omega|^{1-q}}{\mu_1-k_2}+\left({U_2}^{1-p}(0)-\frac{\underline{K}|\Omega|^{1-q}}{\mu_1-k_2}\right)e^{(p-1)(\mu_1-k_2)t}. \tag{2.21}$$

Since $p>1$ and $\lambda_1>0$ we can easily deduce that $k_2=-\frac{\lambda_1}{p-1}q<0$ and $\mu_1-k_2>0$.

Therefore, by (2.21) we know that if initial data large enough satisfy

$$U_2(0) = \int_{\Omega_\varepsilon} u_0(x)\,\psi_1(x)\,dx > \left(\frac{\underline{K}|\Omega|^{1-q}}{\mu_1 - k_2}\right)^{-\frac{1}{p-1}},$$

$$v_0(x) \geq \varphi_1(x), \quad x \in \Omega, \tag{2.22}$$

then ω_2 blows up in finite time T_{ω_2} and

$$T_{\omega_2} \leq \frac{1}{(1-p)(\mu_1 - k_2)} \ln\left[1 - \frac{(\mu_1 - k_2)\,{U_2}^{1-p}(0)}{\underline{K}|\Omega|^{1-q}}\right].$$

Hence, $\underline{u}$ blows up in finite time and an upper bounds for t^* is

$$\frac{1}{(1-p)(\mu_1 - k_2)} \ln\left[1 - \frac{(\mu_1 - k_2)\,U^{1-p}(0)}{\underline{K}|\Omega|^{1-q}}\right].$$

Similarly, for $r > 1$, when the initial data is large enough, the solution of problem (1.1)-(1.2) is blow-up in the finite time.
Case 3. $\max\{p, r\} \leq 1$,
(Global existence) For $qs \leq (1-p)(1-r)$, if $p = 1$ or $r = 1$, then $qs = 0$. We assume $q = 0$, while the discussion of $s = 0$ is similar. Therefore, by $p \leq 1$ and the comparison principle we can deduce that the solution u to equation (1.1)-(1.2) is global. It follows that for arbitrary $T > 0$, there exists $M(T)$ such that

$$u(x,t) \leq M(T), \quad x \in \Omega, t \leq T.$$

Suppose that V is the solution to the following problem

$$\begin{cases} V_t = \Delta V + \bar{K}\,[M(T)]^s\,|\Omega| V^r, & (x,t) \in \Omega \times (0,T), \\ V(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ V(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Then by $r \leq 1$ we can easily get that V is global and is a supersolution to the solution v to equation (1.1)-(1.2), hence the solution to problem (1.1)-(1.2) exists globally.

If $p < 1$ and $r < 1$, by $qs \leq (1-p)(1-r)$, we can compute

$$0 \leq \frac{qs}{(1-p)(1-r)} \leq 1. \tag{2.23}$$

Therefore, using (2.23) we know that there exists $\sigma > 0$ such that

$$0 \leq \frac{q}{(1-p)\,\sigma} \leq 1, \quad 0 \leq \frac{s\sigma}{1-r} \leq 1,$$

Hence,

$$\sigma p + q \leq \sigma, \quad \sigma s + r \leq 1. \tag{2.24}$$

Let

$$\bar{u} = c_1 e^{\sigma t}, \quad \bar{v} = c_2 e^t,$$

where $c_1, c_2 > 0$ are constants to be determined later.

Applying (2.24) we can compute

$$\bar{u}_t - \Delta\bar{u} - k_1(t)\,\bar{u}^p \int_\Omega \bar{v}^q dx = c_1\sigma e^{\sigma t} - k_1(t)\,c_1^p c_2^q e^{(\sigma p+q)t}|\Omega|$$

$$\geq \left(\sigma - \bar{K}c_1^{p-1}c_2^q|\Omega|\right)c_1 e^{\sigma t}. \tag{2.25}$$

$$\bar{v}_t - \Delta\bar{v} - k_2(t)\,\bar{v}^r \int_\Omega \bar{u}^s dx = c_2 e^t - k_2(t)\,|\Omega| c_1^s c_2^r e^{(\sigma s+r)t}$$

$$\geq \left(1 - \bar{K}c_1^s c_2^{r-1}|\Omega|\right)c_2 e^t. \tag{2.26}$$

Now, we choose c_1, c_2 satisfy

$$\sigma - \bar{K}c_1^{p-1}c_2^q|\Omega| \geq 0, \quad 1 - \bar{K}c_1^s c_2^{r-1}|\Omega| \geq 0.$$

Obviously, if initial data small enough satisfy

$$\max_{\bar{\Omega}} u_0(x) < c_1, \quad \max_{\bar{\Omega}} v_0(x) < c_2, \tag{2.27}$$

then by (2.25), (2.26) and the comparison principle we can deduce that $(\bar{u}, \bar{v})$ is a supersolution to the solution (u, v) to problem (1.1)-(1.2).

(Blow-up) For $qs > (1-p)(1-r)$, let

$$\underline{u} = \frac{C_1\varphi_1(x)}{(1-\tau t)^\alpha}, \quad \underline{v} = \frac{C_2\varphi_1(x)}{(1-\tau t)^\beta}, \quad x \in \Omega, 0 \leq t < \frac{1}{\tau},$$

where $\max_{\bar{\Omega}} \varphi_1(x) = 1, \alpha = \frac{q+1-r}{qs-(1-p)(1-r)}, \beta = \frac{s+1-p}{qs-(1-p)(1-r)}, \tau, C_1, C_2 > 0$ are constants to be determined later.

Let

$$\gamma = \min\{-p\alpha - q\beta + \alpha + 1, -s\alpha - r\beta + \beta + 1\},$$

then by the definition of α, β we can easily know $\gamma = 0$.

By directly computation we deduce

$$\begin{aligned}
&\underline{u}_t - \Delta\underline{u} - k_1(t)\,\underline{u}^p \int_\Omega \underline{v}^q dx \\
&= \alpha\tau C_1\varphi_1(1-\tau t)^{-\alpha-1} + \lambda_1 C_1\varphi_1(1-\tau t)^{-\alpha} \\
&\quad - k_1(t)\,C_1^p C_2^q \varphi_1^p (1-\tau t)^{-p\alpha-q\beta} \int_\Omega \varphi_1^q dx \\
&\leq C_1\varphi_1(1-\tau t)^{-\alpha-1}\left[\alpha\tau + \lambda_1(1-\tau t) - \underline{K}C_1^{p-1}C_2^q C_{\varphi_q}(1-\tau t)^\gamma\right] \\
&= C_1\varphi_1(1-\tau t)^{-\alpha-1}[\alpha\tau + \lambda_1(1-\tau t) - \underline{K}C_1^{p-1}C_2^q C_{\varphi_q}],
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
&\underline{v}_t - \Delta\underline{v} - k_2(t)\,\underline{v}^r \int_\Omega \underline{u}^s dx \\
&\leq C_2\varphi_1(1-\tau t)^{-\beta-1}\left[\beta\tau + \lambda_1(1-\tau t) - \underline{K}C_1^s C_2^{r-1} C_{\varphi_s}\right],
\end{aligned} \tag{2.29}$$

where $C_{\varphi_q} = \int_\Omega \varphi_1^q dx, C_{\varphi_s} = \int_\Omega \varphi_1^s dx$.

Choosing $C_1, C_2 > 0$ satisfy

$$\begin{cases} \lambda_1 < \underline{K}\delta^{p-1}C_1^{p-1}C_2^qC_{\varphi_q}, \\ \lambda_1 < \underline{K}\delta^{r-1}C_1^sC_2^{r-1}C_{\varphi_s}, \end{cases}$$

and

$$\tau = \min\left\{\frac{\underline{K}C_1^{p-1}C_2^qC_{\varphi_q} - \lambda_1}{\alpha}, \frac{\underline{K}C_1^sC_2^{r-1}C_{\varphi_s} - \lambda_1}{\beta}\right\} > 0. \tag{2.30}$$

Therefore, if initial data large enough satisfy

$$u_0(x) \geq C_1\varphi_1(x), \quad v_0(x) \geq C_2\varphi_1(x), \tag{2.31}$$

then using (2.28)-(2.31) and the comparison principle we obtain that $(\underline{u}, \underline{v})$ is a subsolution to problem (1.1)-(1.2). Therefore, the solution (u, v) to problem (1.1)-(1.2) blows up in finite time t^*, and an upper bound for t^* is

$$\frac{1}{\tau},$$

which completes the proof. □

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