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SOME RESULTS ON INVARIANT SUBMANIFOLDS OF AN ALMOST KENMOTSU (κ, μ, ν)-SPACE

Mehmet ATÇEKEN and Gülsüm YÜCA*

Abstract. In the present paper, we study the geometric properties of the invariant submanifold of an almost Kenmotsu structure whose Riemannian curvature tensor has (κ, μ, ν)-nullity distribution. In this connection, the necessary and sufficient conditions are investigated for an invariant submanifold of an almost Kenmotsu (κ, μ, ν)-space to be totally geodesic under the behavior of functions κ, μ, and ν.

1. Introduction

It is well known that a $(2n+1)$-dimensional contact metric manifold $\widetilde{M}$ admits an almost contact metric structure (ϕ, ξ, η, g), i.e., it admits global vector field ξ, called the characteristic vector field or the Reeb vector field, its dual is η, a tensor ϕ of type $(1,1)$ and the Riemannian metric tensor g such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote set of the differentiable vector fields on $\widetilde{M}$[2].

The manifold $\widetilde{M}$ together with the structure tensor (ϕ, ξ, η, g) is called a contact metric manifold and we will denote it by $\widetilde{M}^{(2n+1)}(\phi, \xi, \eta, g)$ in the rest of this paper.

By $\widetilde{\nabla}$, we denote the Levi-Civita connection of g, then the Riemannian curvature tensor of $\widetilde{R}$ of $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is given by

$$\widetilde{R}(X, Y) = \widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]},$$

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*Corresponding author

for all $X, Y \in \Gamma(T\widetilde{M})$.

On the other hand, we define the tensor field (1,1)-type by h

$$2hX = (\ell_\xi \phi)X,$$

for all $X \in \Gamma(T\widetilde{M})$, where ℓ_ξ is the Lie-derivative in the direction of ξ. Then, the tensor field h is self-adjoint and satisfies

$$\phi h + h\phi = 0, \quad trh = tr\phi h = 0, \quad h\xi = 0. \tag{3}$$

We have also these formulas for the contact metric manifold

$$\widetilde{\nabla}_X \xi = -\phi^2 X - \phi hX, \quad \widetilde{\nabla}_\xi \phi = 0. \tag{4}$$

The contact metric manifold which for ξ is Killing vector field is called a K-contact manifold. It is well known that a contact metric manifold is K-contact if and only if $h = 0$.

The (κ, μ)-nullity distribution of a contact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ for the pair $(\kappa, \mu) \in \mathbb{R}^2$ is a distribution

$$\begin{aligned}&\widetilde{M}(\kappa,\mu) : p \longrightarrow \widetilde{M}_p(\kappa,\mu) = \{Z_p \in T_{\widetilde{M}}(p) : \\ &\widetilde{R}(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\} + \mu\{g(Y,Z)hX - g(X,Z)hY\}\},\end{aligned}$$

for all $X, Y \in \Gamma(T\widetilde{M})$. So, if the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, then

$$\widetilde{R}(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

and the manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is called (κ, μ)-contact metric manifold. If κ and μ are non-constant smooth functions on $\widetilde{M}$, then the manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is called generalized (κ, μ)-contact metric manifold[3].

Going beyond generalized (κ, μ)-space, T. Koufogiorgos, M. Markellos and V. J. Papantoniou introduced the notation of (κ, μ, ν)-contact metric manifold in [2], its Riemannian curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)\xi &= \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \\ &+ \nu\{\eta(Y)\phi hX - \eta(X)\phi hY\},\end{aligned} \tag{5}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where κ, μ, ν are smooth functions on $\widetilde{M}^{2n+1}$.

It is well known that an almost contact metric manifold is an almost Kenmotsu if $d\eta = 0$ and $d\Phi = 2\eta\Lambda\Phi$, where $\Phi(X,Y) = g(X, \phi Y)$ is the fundamental 2-form of $\widetilde{M}^{2n+1}$. If an almost Kenmotsu manifold $\widetilde{M}^{2n+1}$ satisfies (5), then it is called almost Kenmotsu (κ, μ, ν)-space [5].

Proposition 1.1. *Given $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$ an almost Kenmotsu (κ, μ, ν)-space, then*

$$h^2 = (\kappa + 1)\phi^2, \quad \kappa < -1, \tag{6}$$

$$\xi(\kappa) = 2(\kappa + 1)(\nu - 2) \tag{7}$$

$$(\widetilde{\nabla}_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{8}$$

$$\widetilde{\nabla}_X \xi = -\phi^2 X - \phi hX \tag{9}$$

$$S(X, \xi) = 2n\kappa\eta(X) \tag{10}$$

$$\begin{aligned} \widetilde{R}(\xi, X)Y &= \kappa\{g(X, Y)\xi - \eta(Y)X\} + \mu\{g(hX, Y)\xi - \eta(Y)hX\} \\ &+ \nu\{g(\phi hX, Y)\xi - \eta(Y)\phi hX\}. \end{aligned} \tag{11}$$

They proved that this type of manifold is intrinsically related to the harmonicity of the Reeb vector on contact metric 3-manifolds. Some authors have studied manifolds satisfying condition (5) but a non-contact metric structure. In this connection, P. Dacko and Z. Olszak defined an almost cosymplectic (κ, μ, ν)-spaces as an almost cosymplectic manifold that satisfies (5), but with κ, μ and ν functions varying exclusively in the direction of ξ in[6]. Later examples have been given for this type manifold[7].

In modern analysis, the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. Also, the notion of geodesic plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds have also importance in mathematics as well as physical sciences. There have been several papers on contact metric manifolds which admit covector field ξ tangent to the submanifold. In this connection, we refer to [4, 11, 12, 14, 16].

We are especially interested in an invariant submanifold of an almost Kenmotsu (κ, μ, ν)-space to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-generalized pseudoparallel.

Pseudoparallel submanifolds have been studied in different structures and working on[8, 9, 10]. On the other hand, the study of the geometry of invariant submanifolds was introduced by Bejancu and Papaghuic[10]. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold.

In [4], authors show that an invariant submanifold of S-manifolds is totally geodesic under certain conditions.

In the present paper, we generalize the ambient space and investigate the conditions under which invariant pseudoparallel submanifolds of an almost Kenmotsu (κ, μ, ν)-space are totally geodesic.

Now, let M be an immersed submanifold of an almost Kenmotsu (κ, μ, ν)-space $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then, the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{12}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{13}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the induced connections on M, $\Gamma(T^{\perp}M)$, σ, and A are called the second fundamental form and shape operator of M, respectively, $\Gamma(TM)$ denotes the set differentiable vector fields on M. If $\sigma = 0$, then the submanifold M is called totally geodesic. They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{14}$$

The first covariant derivative of the second fundamental form σ is defined by

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{15}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then the submanifold is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation

$$\begin{aligned} \widetilde{R}(X, Y)Z &= R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X \sigma)(Y, Z) \\ &- (\widetilde{\nabla}_Y \sigma)(X, Z), \end{aligned} \tag{16}$$

for all $X, Y, Z \in \Gamma(TM)$.

$\widetilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\widetilde{R}(X, Y) \cdot \sigma)(U, V) &= R^{\perp}(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) \\ &- \sigma(U, R(X, Y)V), \end{aligned} \tag{17}$$

where the Riemannian curvature tensor of the normal bundle $\Gamma(T^{\perp}M)$ is given

$$R^{\perp}(X, Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}$$

On the other hand, the concircular curvature tensor of Riemannian manifold (M^{2n+1}, g) is given by

$$\mathcal{C}(X, Y)Z = \widetilde{R}(X, Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{18}$$

where τ denotes the scalar curvature of M.
Similarly, the tensor $\mathcal{C}\cdot\sigma$ is defined by

$$\begin{aligned}(\mathcal{C}(X,Y)\cdot\sigma)(U,V) &= R^{\perp}(X,Y)\sigma(U,V)-\sigma(\mathcal{C}(X,Y)U,V)\\ &- \sigma(U,\mathcal{C}(X,Y)V),\end{aligned} \tag{19}$$

for all $X,Y,U,V\in\Gamma(TM)$.

For a $(0,k)$-type tensor field T, $k\geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-Tachibana tensor is defined by

$$\begin{aligned}Q(A,T)(X_1,X_2,...,X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,...,X_k)\\ ... &- T(X_1,X_2,...,X_{k-1},(X\wedge_A Y)X_k),,\end{aligned} \tag{20}$$

for all $X_1,X_2,...,X_k,X,Y\in\Gamma(TM)$[8], where

$$(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y. \tag{21}$$

Kowalczyk studied the semi-Riemannian manifolds satisfying $Q(S,R)=0$ and $Q(S,g)$=0[17]. Also De and Majhi investigated the invariant submanifolds of Kenmotsu manifolds and showed that geometric conditions of invariant submanifolds of Kenmotsu manifolds are totally geodesic[13]. Recently, Hu and Wang obtained the geometric conditions of invariant submanifolds of a trans-Sasakain manifolds to be totally geodesic[15]. Furthermore, the geometry of invariant submanifolds of different manifolds was studied by many geometers[see references].

Motivated by the above studies, we make an attempt to study the invariant submanifolds of an almost Kenmotsu (κ,μ,ν)-space satisfying some the geometric conditions such that $Q(S,\sigma)=0$, $Q(S,\widetilde{\nabla}\cdot\sigma)=0$, $Q(S,R\cdot\sigma)=0$, $Q(g,\mathcal{C}\cdot\sigma)=0$ and $Q(S,\mathcal{C}\cdot\sigma)=0$.

Finally, we show that the submanifold is either totally geodesic or functions κ,μ, and ν satisfy some conditions.

2. Some Results On Invariant Submanifolds of an Almost Kenmotsu (κ,μ,ν)-Space

Now, let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ be an almost an almost Kenmotsu (κ,μ,ν)-space and M be an immersed submanifold of $\widetilde{M}^{2n+1}$. If $\phi(T_xM)\subseteq T_xM$, for each point at $x\in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ with respect to ϕ. From (3), one can easily see that an invariant submanifold with respect to ϕ is also invariant with respect to h.

Proposition 2.1. *Let M be an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ such that ξ is tangent to M. Then the* *following equalities hold on M;*

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$$
$$+ \nu[\eta(Y)\phi hX - \eta(X)\phi hY] \tag{22}$$
$$(\nabla_X\phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{23}$$
$$\nabla_X\xi = -\phi^2X - \phi hX \tag{24}$$
$$\phi\sigma(X,Y) = \sigma(\phi X,Y) = \sigma(X,\phi Y), \quad \sigma(X,\xi) = 0, \tag{25}$$
$$S(X,\xi) = 2n\kappa\eta(X) \tag{26}$$

where ∇, σ and R denote the induced Levi-Civita connection on M, the second fundamental form and Riemannian curvature tensor of M, respectively.

Proof. We will omit the proof as it is a result of direct calculations. □

In the rest of this paper, we will assume that M is an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. In this case, from (3), we have

$$\phi hX = -h\phi X, \tag{27}$$

for all $X \in \Gamma(TM)$, that is, M is also invariant with respect to the tensor field h.

First, we look for the invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space satisfying $Q(S,\sigma) = 0$. For $A = S$ and $T = \sigma$ in (20), we have

$$\begin{aligned} -Q(S,\sigma)(U,V;X,Y) &= \sigma((X\wedge_S Y)U,V) + \sigma(U,(X\wedge_S Y)V) \\ &= S(Y,U)\sigma(X,V) - S(X,U)\sigma(Y,V) \\ &+ S(Y,V)\sigma(U,X) - S(X,V)\sigma(U,Y) = 0. \end{aligned}$$

for any $X,Y,U,V \in \Gamma(TM)$. Here, setting $X = V = \xi$ and taking into account of (25) and (26), we conclude that

$$2n\kappa\sigma(U,Y) = 0.$$

Therefore, we can state the following.

Theorem 2.2. *Let M be an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,\sigma) = 0$ if and only if M is totally geodesic provided $\kappa \neq 0$.*

Next we demonstrate the invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space satisfying $Q(S,\widetilde{\nabla}\cdot\sigma) = 0$. Consider the invariant submanifold M of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ satisfying $Q(S,\widetilde{\nabla}\cdot\sigma) = 0$. This means that

$$-Q(S,\widetilde{\nabla}\cdot\sigma)(U,V,Z;X,Y) = (\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z) + (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)$$
$$+ (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z) = 0. \tag{28}$$

Putting $Y = V = \xi$ in (28) and by means of (10), (15), (20) and Proposition 2.1, we obtain

$$\begin{aligned}(\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(\xi, Z) &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S\xi)\xi, Z)\\ &+ (\widetilde{\nabla}_U\sigma)(\xi, (X\wedge_S\xi)Z) = 0.\end{aligned} \tag{29}$$

We will calculate each term. First,

$$\begin{aligned}(\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(\xi, Z) &= -\sigma(\nabla_{(X\wedge_S\xi)U}\xi, Z)\\ &= \sigma(\phi^2(X\wedge_S\xi)U + \phi h(X\wedge_S\xi)U, Z)\\ &= -\sigma(S(\xi,U)X - S(X,U)\xi, Z)\\ &+ \sigma(\phi h[S(\xi,U)X - S(X,U)\xi], Z)\\ &= 2n\kappa\eta(U)\{\phi\sigma(hX,Z) - \sigma(X,Z)\},\end{aligned} \tag{30}$$

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((X\wedge_S\xi)\xi, Z) &= (\widetilde{\nabla}_U\sigma)(S(\xi,\xi)X - S(X,\xi)\xi, Z)\\ &= 2n\kappa\{(\widetilde{\nabla}_U\sigma)(X,Z) - (\widetilde{\nabla}_U\sigma)(\eta(X)\xi, Z)\}\\ &= 2n\kappa\{(\widetilde{\nabla}_U\sigma)(X,Z) + \sigma(\nabla_U\eta(X)\xi, Z)\}\\ &= 2n\kappa\{(\widetilde{\nabla}_U\sigma)(X,Z) + \sigma(U[\eta(X)]\xi + \eta(X)\nabla_U\xi, Z)\}\\ &= 2n\kappa\{(\widetilde{\nabla}_U\sigma)(X,Z) - \eta(X)\sigma(\phi^2U + \phi hU, Z)\}\\ &= 2n\kappa\{(\widetilde{\nabla}_U\sigma)(X,Z) + \eta(X)\sigma(U,Z)\\ &- \eta(X)\sigma(\phi hU, Z)\}\end{aligned} \tag{31}$$

and

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(\xi, (X\wedge_S\xi)Z) &= (\widetilde{\nabla}_U\sigma)(\xi, S(\xi,Z)X - S(X,Z)\xi)\\ &= (\widetilde{\nabla}_U\sigma)(\xi, 2n\kappa\eta(Z)X) - (\widetilde{\nabla}_U\sigma)(\xi, S(X,Z)\xi)\\ &= -2n\kappa\sigma(\nabla_U\xi, \eta(Z)X)\\ &= 2n\kappa\eta(Z)\sigma(\phi^2U + \phi hU, X)\\ &= 2n\kappa\eta(Z)\{\phi\sigma(hU,X) - \sigma(U,X)\}.\end{aligned} \tag{32}$$

Substituting (30), (31) and (32) in (29) provided $\kappa \neq 0$, we have

$$\begin{aligned}\eta(U)\{\phi\sigma(hX,Z) &- \sigma(X,Z)\} + \{(\widetilde{\nabla}_U\sigma)(X,Z) + \eta(X)\sigma(U,Z)\\ &- \eta(X)\phi\sigma(hU,Z)\} + \eta(Z)\{\phi\sigma(hU,X) - \sigma(U,X)\}\\ &= 0.\end{aligned} \tag{33}$$

Setting $Z = \xi$ in (33) and by virtue of Proposition 2.1, we can infer

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(X,\xi) + \phi\sigma(U,X) - \sigma(hU,X) &= -\sigma(\nabla_U\xi, X)\\ &+ \phi\sigma(hU,X) - \sigma(U,X) = 0,\end{aligned}$$

that is,

$$\phi\sigma(hU,X) - \sigma(U,X) = 0. \tag{34}$$

If hU is written instead of U in (34) and taking into account Proposition 2.1 and (6), we have

$$\phi\sigma(h^2U,X)-\sigma(hU,X)=-(\kappa+1)\phi\sigma(U,X)-\sigma(hU,X)=0. \tag{35}$$

From (34) and (35), we conclude that

$$\kappa\sigma(U,Z)=0.$$

Thus, we have the following.

Theorem 2.3. *An invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ satisfies $Q(S,\widetilde{\nabla}\cdot\sigma)=0$ if and only if it is totally geodesic provided $\kappa\neq 0$.*

Now, we will calculate the condition $Q(S,\widetilde{R}\cdot\sigma)=0$. (17) and (20) implies that

$$(\widetilde{R}(X,Y)\cdot\sigma)((Z\wedge_S W)U,V)+(\widetilde{R}(X,Y)\cdot\sigma)(U,(Z\wedge_S W)V)=0,$$

for all $X,Y,U,V,Z,W\in\Gamma(TM)$. Expanding the last equality and inserting $Y=U=V=W=\xi$, we have

$$(\widetilde{R}(X,\xi)\cdot\sigma)(S(\xi,\xi)Z,\xi)-(\widetilde{R}(X,\xi)\cdot\sigma)(S(Z,\xi)\xi,\xi)=0.$$

By making use of (5) and (17), it follows that

$$2n\kappa\sigma(\widetilde{R}(X,\xi)\xi,Z)=2n\kappa\sigma(\kappa[X-\eta(X)\xi]+\mu hX+\nu\phi hX,Z)=0,$$

that is,

$$\kappa\sigma(X,Z)+\mu\sigma(hX,Z)+\nu\phi\sigma(hX,Z)=0. \tag{36}$$

If hX is written instead of X in (36) and taking into account Proposition 2.1, we have

$$\kappa\sigma(hX,Z)-(\kappa+1)[\mu\sigma(X,Z)+\nu\phi\sigma(X,Z)]=0. \tag{37}$$

From (36) and (37), we conclude that

$$[\kappa^2+(\kappa+1)(\mu^2-\nu^2)]\sigma(X,Z)+2(\kappa+1)\mu\nu\phi\sigma(X,Z)=0.$$

Hence, we mention the following theorem.

Theorem 2.4. *Let M be an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ satisfying $Q(S,\widetilde{R}\cdot\sigma)=0$. Then, M is either totally geodesic or $\kappa^2+(\kappa+1)(\mu^2-\nu^2)=0$, and $(\kappa+1)\mu.\nu=0$.*

Next, we research the condition $Q(g,\mathcal{C}\cdot\sigma)=0$. By means of (19) and (20), we have

$$\begin{aligned}-Q(g,\mathcal{C}(X,Y)\cdot\sigma)(U,V,Z,W) &= (\mathcal{C}(X,Y)\cdot\sigma)((Z\wedge_g W)U,V)\\ &+ (\mathcal{C}(X,Y)\cdot\sigma)(U,(Z\wedge_g W)V)=0,\end{aligned}$$

for all $X, Y, U, V, Z, W \in \Gamma(TM)$, which implies that

$$\begin{aligned}(\mathcal{C}(X,Y)\cdot\sigma)(g(U,W)Z,V) &- (\mathcal{C}(X,Y)\cdot\sigma)(g(Z,U)W,V)\\ (\mathcal{C}(X,Y)\cdot\sigma)(U,g(V,W)Z) &- (\mathcal{C}(X,Y)\cdot\sigma)(U,g(Z,V)W)\\ &= 0.\end{aligned}$$

Here taking $Y = V = U = Z = \xi$, by view of Proposition 2.1 (5), (18) and (19), we have

$$\begin{aligned}(\mathcal{C}(X,\xi)\cdot\sigma)(\eta(W)\xi - W,\xi) &= R^{\perp}(X,\xi)\sigma(\eta(W)\xi - W,\xi)\\ &- \sigma(\mathcal{C}(X,\xi)(\eta(W)\xi - W),\xi)\\ &- \sigma(\eta(W)\xi - W,(\mathcal{C}(X,\xi)\xi) = 0,\end{aligned}$$

that is,

$$\begin{aligned}R^{\perp}(X,\xi)\sigma(\eta(W)\xi,\xi) &- R^{\perp}(X,\xi)\sigma(W,\xi) - \eta(W)\sigma(\mathcal{C}(X,\xi)\xi,\xi)\\ &+ \sigma(\mathcal{C}(X,\xi)W,\xi) - \eta(W)\sigma(\mathcal{C}(X,\xi)\xi,\xi)\\ &+ \sigma(\mathcal{C}(X,\xi)\xi,W) = 0.\end{aligned} \tag{38}$$

By virtue of Proposition 2.1, non-vanishing components of (38) give us

$$\begin{aligned}\sigma(\mathcal{C}(X,\xi)\xi,W) &= \left(\kappa - \frac{\tau}{2n(2n+1)}\right)\sigma(X,W) + \mu\sigma(hX,W)\\ &+ \nu\phi\sigma(hX,W) = 0.\end{aligned} \tag{39}$$

In (39), If hX is written instead of X in (39) and taking into account that Proposition 2.1, (6), we have

$$\begin{aligned}\left(\kappa - \frac{\tau}{2n(2n+1)}\right)\sigma(hX,W) &- (\kappa+1)(\mu\sigma(X,W)\\ &+ \nu\phi\sigma(X,W)) = 0.\end{aligned} \tag{40}$$

Thus, (39) and (40) imply that

$$\left[\left(\kappa - \frac{\tau}{2n(2n+1)}\right)^2 + (\kappa+1)(\mu^2-\nu^2)\right]\sigma(X,W) + 2(\kappa+1)\mu\nu\phi\sigma(X,W) = 0.$$

Therefore, the following theorems may be noted.

Theorem 2.5. *Let M be an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then, $Q(g,\mathcal{C}\cdot\sigma) = 0$ if and only if M either is totally geodesic or the scalar curvature τ of the ambient manifold is given by*

$$\tau = 2n(2n+1)\left[\kappa \mp \sqrt{(\kappa+1)(\nu^2-\mu^2)}\right], \quad (\kappa+1)\mu\nu = 0.$$

Theorem 2.6. *Let M be an invariant submanifold of an almost Kenmotsu (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then, $Q(S,\mathcal{C}\cdot\sigma) = 0$ if and only if M either is totally geodesic or the scalar curvature τ of the ambient manifold is given by*

$$\tau = 2n(2n+1)\left[\kappa \mp \sqrt{(\kappa+1)(\nu^2-\mu^2)}\right], \quad \mu\nu = 0, \quad \kappa \neq 0.$$

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Mehmet ATÇEKEN
Department of Mathematics, Aksaray University, 68100, Turkey.
E-mail: mehmet.atceken382@gmail.com

Gülsüm YÜCA
Department of Mathematics, Aksaray University, 68100, Turkey.
E-mail: gulsumbicer@gmail.com