Honam Mathematical J. **43** (2021), No. 4, pp. 641–653
https://doi.org/10.5831/HMJ.2021.43.4.641

ENUMERATION OF FUSS-CATALAN PATHS BY TYPE AND BLOCKS

Suhyung An, JiYoon Jung, and Sangwook Kim*

Abstract. Armstrong enumerated the number of Fuss-Catalan paths with a given type and Rhoades provided the number of Dyck paths with a given type and a given number of blocks. In this paper we generalize those results to enumerate the number of Fuss-Catalan paths with a fixed type and a fixed number of blocks. We provide two proofs of this result. The first one uses the Chung-Feller theorem and a certain polynomial, while the second one is bijective. Also, we give a conjecture generalizing this result to the family of small Fuss-Schröder paths.

1. Introduction

A Dyck path of length n is a lattice path from $(0,0)$ to (n,n) using east steps $E=(1,0)$ and north steps $N=(0,1)$ such that it stays weakly above the diagonal line $y=x$. It is well-known that the number of Dyck paths of length n is given by the famous Catalan numbers

$$\frac{1}{n+1}\binom{2n}{n}.$$

For a Dyck path, its *type* is the integer partition formed by the lengths of E *runs*, the maximal adjacent east steps. For example, a path $NENNNNEENEENE$ has a type $\lambda=(2,2,1,1)$. The enumeration of Dyck paths by type was first done by Kreweras [4] in the context of noncrossing partitions.

Now we introduce the Fuss analogue of Dyck paths. Given a positive integer k, a k-Fuss-Catalan path of length n is a path from $(0,0)$ to (n,kn) using east steps E and north steps N such that it stays weakly above the line $y=kx$. The number of k-Fuss-Catalan paths of length n is given by the Fuss-Catalan

Received May 31, 2021. Revised September 7, 2021. Accepted September 8, 2021.
2020 Mathematics Subject Classification. 05A15, 05A19.
Key words and phrases. Dyck paths, Fuss-Catalan paths, type, block.
Suhyung An was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2019R1I1A1A01059433)
*Corresponding author

numbers

$$\frac{1}{kn+1}\binom{(k+1)n}{n}.$$

The type of a k-Fuss-Catalan path is also determined by its E runs and Armstrong [2] enumerated the number of k-Fuss-Catalan paths of a given type. A *block* of a k-Fuss-Catalan path is a section beginning with a north step N whose starting point is on the line $y = kx$ and ending with the first east step E that returns to the line $y = kx$ afterwards. Rhoades [6] provided the number of Dyck paths with a fixed type and a fixed number of blocks.

In this article, we provide the number of k-Fuss-Catalan paths with a given type and a given number of blocks, and two proofs of this result. In Section 2, we prove the case of k-Fuss-Catalan paths with only one block using Chung-Feller theorem and use a polynomial introduced by Zeng [7] to show the general case. The second proof is bijective and is provided in Section 3. We finish this paper with a conjecture about the number of small (k, r)-Fuss-Schröder paths with a given type and a given number of blocks, defined in Section 4.

2. Fuss-Catalan paths with a fixed type and a fixed number of blocks

We begin with prior work done on the topic and the necessary terminology to proceed. In what follows, we enumerate the number of k-Fuss-Catalan paths with a fixed type and exactly one block and generalize the result to the case with any fixed number of blocks.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) = 1^{m_1}2^{m_2}\cdots n^{m_n} \vdash n$, we have

$$\ell = m_1 + m_2 + \cdots + m_n,$$
$$n = 1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = \lambda_1 + \cdots + \lambda_\ell.$$

Armstrong [2] enumerated the number of k-Fuss-Catalan paths of a given type.

Theorem 2.1. *The number of k-Fuss-Catalan paths of length n with a type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is*

$$\frac{(\ell-1)!}{m_\lambda}\binom{nk}{\ell-1} = \frac{1}{\ell}\binom{\ell}{m_1, m_2, \dots, m_n}\binom{nk}{\ell-1}.$$

Here, $m_\lambda := m_1(\lambda)!m_2(\lambda)!m_3(\lambda)!\cdots m_n(\lambda)!$ where $m_i(\lambda)$ is the number of parts of λ equal to i.

Rhoades [6] provided the number of Dyck paths with a fixed type and a fixed number of blocks. Note that he used the notion of *connected components* instead of blocks.

Theorem 2.2. *The number of Dyck paths of length n with a type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and exactly m blocks is*

$$\frac{(\ell-1)!}{m_\lambda}\binom{n-m-1}{\ell-m}m = \frac{m}{\ell}\binom{\ell}{m_1, m_2, \dots, m_n}\binom{n-m-1}{\ell-m}.$$

To generalize Rhoades' result to k-Fuss-Catalan paths, we look at some terminology first. For a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, we define $\mathbf{r}(\lambda) = (m_1(\lambda), \dots, m_n(\lambda))$. For $\mathbf{v} \in \mathbb{N}^n$, let $A_\lambda^{(m)}(x;\mathbf{v}) \in \mathbb{R}[x]$ be a polynomial

$$\begin{aligned} A_\lambda^{(m)}(x;\mathbf{v}) &= \frac{(\ell-1)!}{(\ell-m)!}\frac{x}{x+\mathbf{r}(\lambda)\cdot\mathbf{v}}\frac{(x+\mathbf{r}(\lambda)\cdot\mathbf{v})_{\ell-m+1}}{m_\lambda} \\ &= \frac{x}{\ell}\binom{\ell}{m_1, m_2, \dots, m_n}\binom{x+\mathbf{r}(\lambda)\cdot\mathbf{v}-1}{\ell-m}, \end{aligned}$$

where $(y)_i = y(y-1)\cdots(y-i+1)$ is a falling factorial. This polynomial is introduced by Zeng [7] and he used the polynomial to prove various convolution identities involving multinomial coefficients. Rhoades [6] showed that, if $\mathbf{v} = (1, 2, \dots, n)$, then

(1) $A_\lambda^{(1)}(1;\mathbf{v})$ is the number of all Dyck paths of length n with a type λ, and
(2) $-A_\lambda^{(m)}(-m;\mathbf{v})$ is the number of all Dyck paths of length n with a type λ and m blocks.

One can easily check that, if $\mathbf{v} = (k, 2k, \dots, nk)$, then $A_\lambda^{(1)}(1;\mathbf{v})$ is the number of all k-Fuss-Catalan paths of a type λ. In Theorem 2.4, we show that

$$-A_\lambda^{(m)}(-m;\mathbf{v}) = \frac{(\ell-1)!}{m_\lambda}\binom{nk-m-1}{\ell-m}m$$

with $\mathbf{v} = (k, 2k, \dots, nk)$ is the number of k-Fuss-Catalan paths of a type λ with m blocks. First, we prove that $-A_\lambda^{(1)}(-1;\mathbf{v})$ with $\mathbf{v} = (k, 2k, \dots, nk)$ is the number of k-Fuss-Catalan paths with a type λ and only one block using the Chung-Feller theorem.

Lemma 2.3. *The number of k-Fuss-Catalan paths of length n with a type $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and only one block is*

$$\frac{(\ell-1)!}{m_\lambda}\binom{nk-2}{\ell-1} = \frac{1}{\ell}\frac{\ell!}{m_\lambda}\binom{nk-2}{\ell-1}.$$

Proof. Let L be the set of all lattice paths from $(0,0)$ to (n, nk) of a type λ with an east run of height nk and no east runs of height 0 and 1. Since there are $\binom{nk-2}{\ell-1}$ ways to assign the heights to $(\ell-1)$ remaining east runs and $\frac{\ell!}{m_\lambda}$ ways to arrange ℓ east runs, the total number of lattice paths in L is $\frac{\ell!}{m_\lambda}\binom{nk-2}{\ell-1}$.

The *flaws* of the paths in L are the east runs, not of height nk, whose right end points are lying on or below $y = kx$. Note that the paths with 0 flaws in L are k-Fuss-Catalan paths of length n with a type λ and one block. Now we provide algorithms showing the number of paths with u flaws in L is independent of u for every u such that $0 \le u \le \ell - 1$.

I. Algorithm for increasing the number of flaws: Let P be a path with $(u-1)$ flaws for $1 \leq u \leq \ell-1$. Then P can be uniquely decomposed into $NM_1M_2\cdots M_\ell$ by cutting P at a point $(0,1)$ and a right endpoint of each E run. Note that there are components whose right endpoints are above $y = kx$ since P has less than $\ell-1$ flaws, and let M_{i-1} be the highest of such components where $2 \leq i \leq \ell$. Then bring down $M_i\cdots M_\ell$ right before M_{i-1} and, if the moved ℓ-th E run becomes a new flaw, stop here. Otherwise, bring down $M_i\cdots M_\ell$ right before M_{i-2} and again, if the moved ℓ-th E run becomes a new flaw, we may stop. This process is continued until the moved ℓ-th E run of P becomes a new flaw for the first time, and let $Q = NM_1\cdots M_{j-1}M_i\cdots M_\ell M_j\cdots M_{i-1}$ denote the path obtained after the process.

   By construction, M_ℓ gives a new flaw, while $M_j\cdots M_{i-1}$ still contains no flaws, and so the new flaw is the highest flaw in Q. Since the right endpoint of M_ℓ is on or below the line $y = kx$, $M_i\cdots M_{\ell-1}$ still gives $(\ell-i)$ flaws in Q. Since $NM_1\cdots M_{j-1}$ stays the same, Q has exactly one more flaws than P.

II. Algorithm for decreasing the number of flaws: Conversely, let Q be a path with u flaws for $1 \leq u \leq \ell-1$. For a decomposition $Q = NM^1M^2\cdots M^\ell$ defined in a similar way to P, let M^t be the component including the last flaw and let M^s be the highest component such that $M^s\cdots M^t$ contains less N steps than a k-fold of E steps where $1 \leq s \leq t$. Then we can bring back $P = NM^1\cdots M^{s-1}M^{t+1}\cdots M^\ell M^s\cdots M^t$ from Q by moving $M^s\cdots M^t$ after M^ℓ.

   M^t becomes the last component of P after construction, and so it does not have a flaw anymore. As by M^s is defined in Q, $M^s\cdots M^{t-1}$ keeps $(t-s)$ flaws and $M^{t+1}\cdots M^\ell$ still contains no flaws in P. Since $NM^1\cdots M^{s-1}$ stays the same, P has exactly one less flaws than Q.

These algorithms provide a bijection between two sets of paths with u flaws and $(u-1)$ flaws respectively for $1 \leq u \leq \ell-1$. Thus the number of paths is independent of the number of flaws. □

Now we provide the main theorem. Although the proof is similar to the case of Dyck paths presented in [6] when using the base case $-A^{(1)}_\lambda(-1;\mathbf{v})$ where $\mathbf{v} = (k, 2k, \ldots, nk)$ in Lemma 2.3, we include the proof for completeness.

Theorem 2.4. *The number of k-Fuss-Catalan paths of length n with a type $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and m blocks is*

$$\frac{(\ell-1)!}{m_\lambda}\binom{nk-m-1}{\ell-m}m.$$

Proof. The polynomial $A^{(1)}_\lambda(mx;\mathbf{v})$ can be written as

$$A^{(1)}_\lambda(mx;\mathbf{v}) = \sum_{\mathbf{r}(\lambda^1)+\cdots+\mathbf{r}(\lambda^m)=\mathbf{r}(\lambda)} A^{(1)}_{\lambda^1}(x;\mathbf{v})\cdots A^{(1)}_{\lambda^m}(x;\mathbf{v})$$

where λ^i is a partitions into parts $\leq n$ for each i using a result of Raney [5, Theorem 2.2, 2.3] and induction. By Lemma 2.3 and the fact that $A^{(1)}_{\mathbf{0}}(x;\mathbf{v})=1$ where $\mathbf{0}$ is the empty partition, we can set $x = -1$ and $\mathbf{v} = (k, 2k, \ldots, nk)$ to get

$$\sum_{i=1}^{m}(-1)^i\binom{m}{i}C(n,i,\lambda) = A^{(1)}_{\lambda}(-m;\mathbf{v})$$

where $C(n, i, \lambda)$ is the number of k-Fuss-Catalan paths of length n with a type λ and i blocks. By the Principle of inclusion and exclusion, we have

$$C(n,m,\lambda) = \sum_{i=1}^{m}(-1)^i\binom{m}{i}A^{(1)}_{\lambda}(-i;\mathbf{v}).$$

It is enough to show that the right hand side of the above equation is equal to $-A^{(m)}_{\lambda}(-m;\mathbf{v})$. We start with the following identity:

$$\sum_{i=1}^{m}(-1)^{i+1}\,i\binom{m}{i}\binom{nk-i-1}{\ell-1} = m\binom{nk-m-1}{nk-\ell-1}.$$

Multiplying both sides by $\dfrac{(\ell-1)!}{m_\lambda}$ and using the definition of $A^{(1)}_{\lambda}(x;\mathbf{v})$, we get

$$\sum_{i=1}^{m}(-1)^i\binom{m}{i}A^{(1)}_{\lambda}(-i;\mathbf{v}) = \frac{(\ell-1)!}{m_\lambda}\binom{nk-m-1}{\ell-m}m.$$

□

3. Bijective proof of the main theorem

In this section, we provide a bijective proof of Theorem 2.4. In Subsection 3.1, we give a set of desired cardinality. In Subsections 3.2 and 3.3, we give a bijection between this set and the set of Fuss-Catalan paths with desired property. To clarify the path constructions in the proof, we begin by referencing some basic definitions and useful terms.

A lattice path is said to *touch* a line $y = kx$ at a point (s, ks) if the path contains a solid E run having a right end point (s, ks). Every k-Fuss-Catalan path can be decomposed into *blocks* by cutting the path at each point where the path touches the diagonal line $y = kx$. A *circular shift of blocks* is the operation of rearranging the blocks of lattice paths by moving the final block to the first and shifting all other blocks to the next. Two k-Fuss-Catalan paths are *equivalent* if one can be obtained by repeatedly applying circular shifts to the other. Note that the size of the *equivalence class* of a k-Fuss-Catalan path is m if all the m blocks of the path are distinct.

Remind that $m_\lambda := m_1(\lambda)!\; m_2(\lambda)!\; m_3(\lambda)! \cdots m_n(\lambda)!$ where $m_i(\lambda)$ is the number of parts of λ equal to i. Because interchanging the east runs of the same size does not lead to a different path, and all the blocks of a k-Fuss-Catalan path are distinct if all the parts of λ are distinct, it is enough to show the number of equivalence classes on k-Fuss-Catalan paths of a type $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash n$ with m blocks is $(\ell-1)!\binom{nk-m-1}{\ell-m}$ in this proof.

3.1. A set $\mathfrak{L}$ of desired cardinality

We first consider the set of partially dotted lattice paths from $(0,0)$ to (n, nk) of a type $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash n$ with m blocks such that:

(A) The E run of length λ_ℓ is ended at a point (n, nk). See Figure 1.

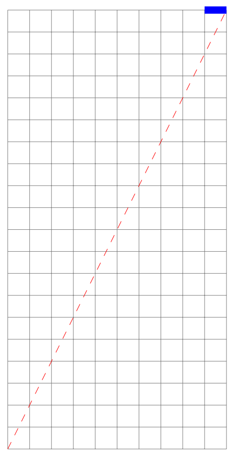

FIGURE 1. The shortest E run of length 1 is fixed as the last run of a path where $n = 10, k = 2$, and $\lambda = (4,3,2,1)$.

(B) Exactly $(\ell - m)$ E runs are drawn as dotted lines, and none of them has a height $0, 1, (n-m+1)k, \ldots, (n-1)k$ or nk. Here, the heights 0 and 1 are avoided because they produce trivial flaws. The heights $(n-m+1)k$, $\ldots, (n-1)k$ and nk are reserved for Step (a) below. There are $\binom{\ell-1}{\ell-m}$ different selections of these dotted E runs, $(\ell-m)!$ orderings for each selection, and $\binom{nk-m-1}{\ell-m}$ ways to assign heights to the dotted E runs. See Figure 2.

(C) There are $(m-1)!$ different orderings for the remaining $(m-1)$ solid E runs. For each ordering, the heights of the $(m-1)$ solid E runs are uniquely assigned so that the $(m-1)$ solid E runs touch the line $y = kx$ and the region under the generated lattice path is as large as possible. See Figure 3.

Note that there are $\binom{\ell-1}{\ell-m}(\ell-m)!\binom{nk-m-1}{\ell-m}(m-1)! = (\ell-1)!\binom{nk-m-1}{\ell-m}$ lattice paths satisfying the above conditions. On the lattice paths, if a dotted E run is ended at the point $(s, ks-q)$ for $1 \le s \le n-1$ and $0 \le q$, we say the

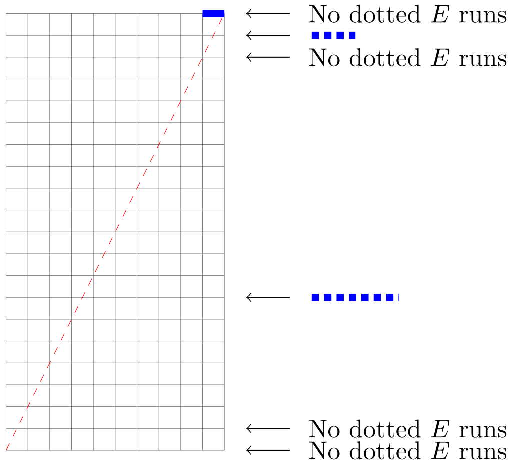

FIGURE 2. E runs of lengths 2 and 4 are selected as dotted lines, and their orders and heights are decided as shown where $n = 10, k = 2, m = 2$, and $\lambda = (4, 3, 2, 1)$.

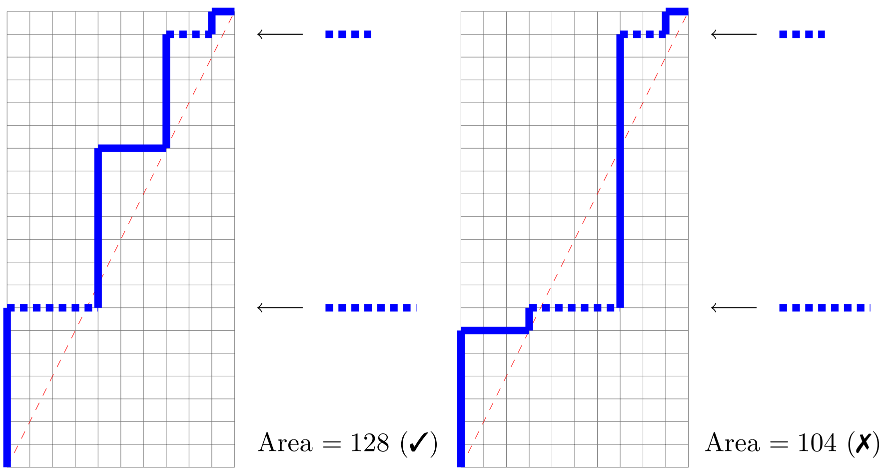

FIGURE 3. To generate the lattice path having a maximal area under the path, the remaining solid E run of length 3 should touch the main diagonal $y = 2x$ at height 14 where $n = 10, k = \ 2, m = 2$, and $\lambda = (4, 3, 2, 1)$.

E run has *q-flaw*. Note that, if q is a nonzero, then the q-flawed E runs are located in the first block. Otherwise, there exists a solid E run before a $q(\neq 0)$-flawed E run. Then, a lattice path with larger region under the path can be generated by moving the solid E run after the $q(\neq 0)$-flawed E run, and this is a contradiction to Condition (C).

Next, for 0-flawed E runs located after the first block, we need the following steps:

(a) If there are p 0-flawed E runs after the first block for $p \geq 1$, assign them new heights $(n-p)k, (n-p+1)k, \ldots, (n-1)k$ in order. See Figure 4.

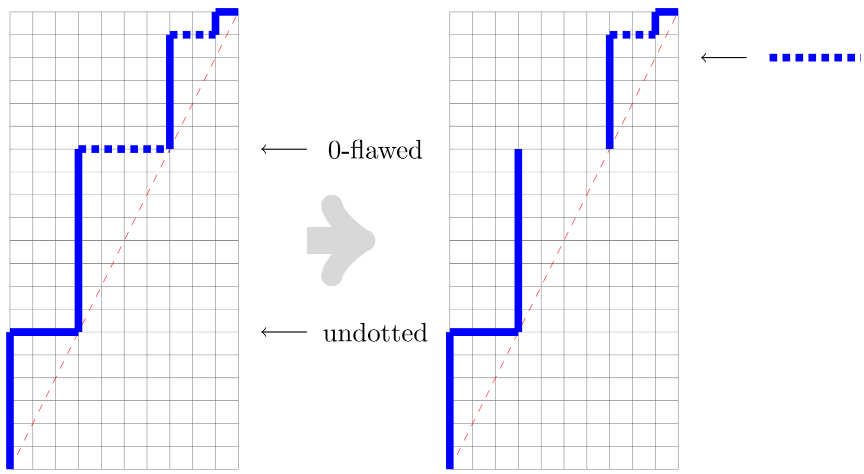

FIGURE 4. The first solid E run is of the height 6, and the 0-flawed E run of the height 14 is moved the line of the height 18 where $n = 10, k = 2, m = 2$, and $\lambda = (4, 3, 2, 1)$.

(b) Keeping the order of m solid E runs, the heights of the m E runs are uniquely reassigned so that the m E runs touch the line $y = kx$ and the region under the generated lattice path is as large as possible. See Figure 5.

Note that all the 0-flawed E runs after the first block are now moved to after the penultimate solid E run. That means newly rearranged lattice paths have 0-flawed E runs only in the first and last blocks. Each of these new lattice paths generated by Steps (a) and (b) has a dotted E run of height $(n-1)k$, but the former lattice paths have no dotted E run of height $(n-1)k$ since the existence of 0-flawed E runs after the first block means $m > 1$ in Condition (B). Therefore, there is no intersection between the set of new generated lattice paths and the set of former lattice paths. And, it is easy to check that two lattice paths generated from two different former lattice paths are also different. Hence, there still exist $(\ell-1)!\binom{nk-m-1}{\ell-m}$ lattice paths, and let $\mathfrak{L}$ be the set of these lattice paths.

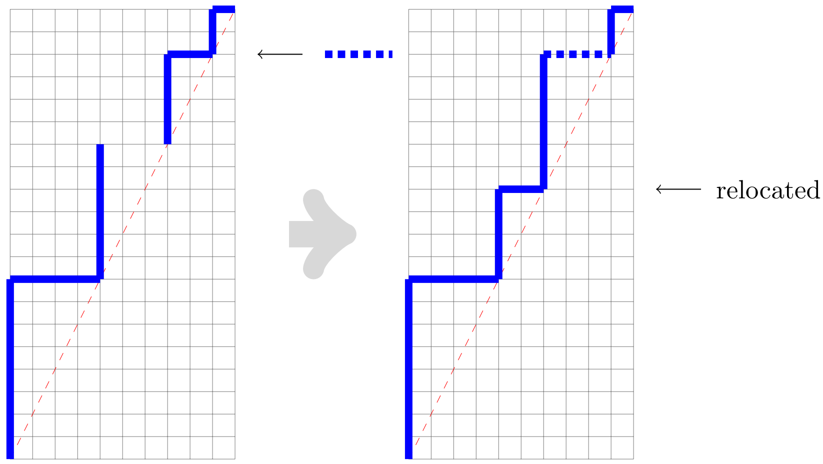

FIGURE 5. The solid E run of the height of 18 is relocated at the height of 12 where $n = 10, k = 2, m = 3$, and $\lambda = (4, 3, 2, 1)$.

3.2. A map from $\mathfrak{L}$ to $\mathfrak{F}$

In this subsection and the next, we provide a bijection from $\mathfrak{L}$ to the set $\mathfrak{F}$ of representatives of equivalence classes on k-Fuss-Catalan paths of a type $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash n$ with m blocks where representatives are k-Fuss-Catalan paths with the last block containing the E run of length λ_ℓ. For the map from $\mathfrak{L}$ to $\mathfrak{F}$, consider three cases:

(i) If there is no flawed E run, that means all the $(\ell - m)$ dotted E runs are strictly above the main diagonal $y = kx$, then let the dotted E runs be solid. We obtain k-Fuss-Catalan paths of a type λ with m blocks such that the E run of length λ_ℓ is fixed as the last step.

(ii) If flawed E runs exist only in the last block, Step (a) in Subsection 3.1 implies that λ_ℓ should be 1 and the flawed E run with the right end point $(n-1, (n-1)k)$ should be the only flawed E run. Then, let (t, tk) be the starting point of the last block, and switch the segment from $(t, tk+1)$ to $(n-1, (n-1)k)$ and the segment from $(n-1, (n-1)k)$ to (n, nk). See Figure 6. Remind that the E run of length λ_ℓ does not touch the diagonal $y = kx$ anymore since the N step from (t, tk) to $(t, tk+1)$ was fixed while switching segments, and the only flawed E run that was ended at the point $(n-1, (n-1)k)$ touches the main diagonal at (n, nk). Therefore, if all the dotted E runs are solid, then we obtain k-Fuss-Catalan paths of a type λ with m blocks such that the last block begins with the N run of length $(1 + k)$ followed by the E run of length $\lambda_\ell = 1$.

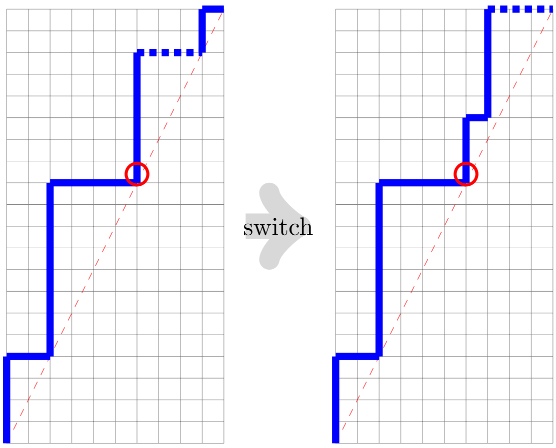

Figure 6. Switch the segment from $(6, 13)$ to $(9, 18)$ and the segment from $(9, 18)$ to $(10, 20)$ where $(t, tk) = (6, 12), (n-1, (n-1)k) = (9, 18), n = 10, k = 2, m = 2$, and $\lambda = (4, 3, 2, 1)$.

(iii) If flawed E runs exist in the first block, let (v, w) and (a, b) be right end points of the rightmost flawed one and the leftmost worst-flawed one of the flawed E runs respectively. Here, a worst-flawed E run is the q-flawed E run with maximal q. Then, apply $\langle -v \pmod{n}, -vk \pmod{nk}\rangle$-shift on the lattice path. That means, the segment from $(0, 0)$ to (v, vk) is moved right after a point (n, nk), and a point (v, vk) becomes a new origin. After that, cut off the segment from $(n - v, nk - vk)$ to $(n - v, nk - vk + 1)$ that was the N step from $(0, 0)$ to $(0, 1)$ and the segment from $(a-v+n, b-vk+nk)$ to (n, nk) that was the segment from (a, b) to (v, vk), and place them in order right after the starting point of the block containing the E run of length λ_ℓ. See Figure 7. Since flawed E runs were located in the first and last blocks only, and (v, w) was the right end point of the rightmost flawed E run in the first block, all the flawed E runs were shifted after the starting point of the block containing the E run of length λ_ℓ. Because we moved the N step from $(n - v, nk - vk)$ to $(n - v, nk - vk + 1)$ and the segment from $(a - v + n, b - vk + nk)$ to (n, nk) right after the starting point of the block containing the E run of length λ_ℓ, and (a, b) was a right end point of the leftmost worst-flawed E run, there is no more flawed E runs when all the arrangements are finished. And, every rearranged lattice path still consists of m blocks since E run that was ended at a point (a, b) touches the main diagonal at (n, nk) while the E run of length λ_ℓ does not touch the diagonal $y = kx$ anymore. Therefore, if all the dotted E runs are solid, then we obtain k-Fuss-Catalan paths of a type λ with m blocks such that the last block

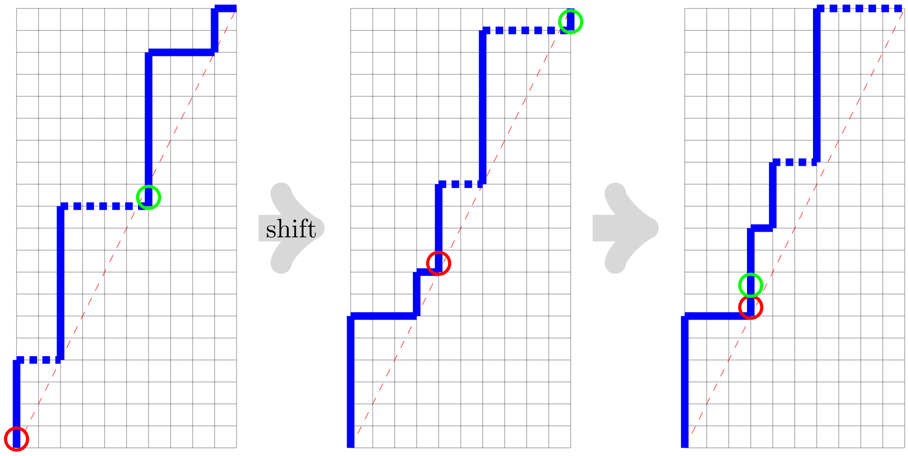

FIGURE 7. Take $\langle -6 \pmod{10}, -12 \pmod{20} \rangle$ shift where $(a, b) = (v, w) = (6, 11)$, $n = 10, k = 2, m = 2$, and $\lambda = (4, 3, 2, 1)$. After that, move the segments from $(4, 8)$ to $(4, 9)$ and from $(10, 19)$ to $(10, 20)$ right after a point $(3, 6)$.

contains the E run of length λ_ℓ, but the E run of length λ_ℓ is not the last run of the path and the last block doesn't begin with the N run of length $(1 + k)$ followed by the E run of length 1.

3.3. A map from $\mathfrak{F}$ to $\mathfrak{L}$

Inversely, take a k-Fuss-Catalan path of a type $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash n$ with m blocks which has the E run of length λ_ℓ in the last block. We also consider three cases:

(i) If the E run of length λ_ℓ is the last run of the path, we get a path in $\mathfrak{L}$ by letting E runs not touching the line $y = kx$ as dotted lines.

(ii) If the last block begins with the N run of length $(1 + k)$ followed the E run of length 1, decompose the last block into 3 components by cutting right after the first N step of the last block and the E run of length 1. After the second component is switched with the third component, if the E runs not touching the line $y = kx$ and the E run of height of $(n-1)k$ become dotted, we get a path in $\mathfrak{L}$ such that the E run of height $(n-1)k$ is the only dotted line touching the main diagonal $y = kx$.

(iii) If the E run of length λ_ℓ is not the last run of the path and the last block doesn't begin with the N run of length $(1 + k)$ followed by the E run of length 1, decompose the last block into 5 segments $P_1P_2P_3P_4P_5$ in order where P_1 is the first N step of the last block, P_3 is a sequence containing $k\lambda_\ell$ N steps such that the first and the last are N steps, and P_4 is the E run of length of λ_ℓ. Then, rearrange the last block in order

$P_3P_4P_1P_5P_2$, and apply circular shifts so that the E run of length of λ_ℓ can be the last run of a path.

It is not hard to see that each case is the inverse of corresponding case in Subsection 3.2.

4. Future work about small Fuss-Schröder paths

In this section, we provide a conjecture which generalizes Theorem 2.4 to the case of small (k, r)-Fuss-Schröder paths.

For k and r such that $1 \leq r \leq k$, a small (k, r)-Fuss-Schröder path of length n is a path from $(0, 0)$ to (n, kn) using east steps, north steps, and diagonal steps $D = (1, 1)$ that satisfies the following three conditions:

(1) the path never passes below the line $y = kx$,
(2) the diagonal steps are only allowed to go from the line $y = kj + r - 1$ to the line $y = kj + r$, for some j such that $0 \leq j \leq n - 1$, and
(3) no diagonal steps touch the main diagonal line $y = kx$.

Eu and Fu [3] showed that the number of small (k, r)-Fuss-Schröder paths with a fixed length and a fixed number of diagonal steps is independent of r.

The *type* of a small (k, r)-Fuss-Schröder path is determined by its E runs. Note that the type of a small (k, r)-Fuss-Schröder path of length n is a partition of some number less than or equal to n. An, Jung, and Kim [1] enumerated the number of small (k, r)-Fuss-Schröder paths of a given type.

Theorem 4.1. *The number of small (k, r)-Fuss-Schröder paths of length n with a type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ for $1 \leq r \leq k$ is*

$$\frac{(\ell - 1)!}{m_\lambda} \binom{nk}{\ell - 1} \binom{n - 1}{|\lambda| - 1},$$

where $|\lambda|$ is the sum of the parts of λ.

A *block* of a small (k, r)-Fuss-Schröder path is a section beginning with a north step N whose starting point is on the line $y = kx$ and ending with the first east step E that returns to the line $y = kx$ afterwards. Eu and Fu [3] provided the number of small (k, r)-Fuss-Schröder paths with d diagonal steps and m blocks.

Theorem 4.2. *The number of small (k, r)-Fuss-Schröder paths of length n with d diagonal steps and m blocks is*

$$\frac{m}{n} \binom{n}{d} \binom{kn + n - d - m - 1}{n - d - m},$$

independent of r.

This work moves towards the goal of proving the following conjecture about the number of small (k, r)-Fuss-Schröder paths with a fixed type and a fixed number of blocks.

Conjecture 4.3. *The number of small (k, r)-Fuss-Schröder paths of length n with a type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with m blocks is*

$$\frac{(\ell-1)!}{m_\lambda}\binom{nk-m-1}{\ell-m} m \binom{n-1}{|\lambda|-1},$$

independent of r.

References

[1] Suhyung An, JiYoon Jung, and Sangwook Kim. Enumeration of Fuss-Schröder paths. *Electron. J. Combin.*, 24(2):Paper 2.30, 12, 2017.

[2] Drew Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202(949):x+159, 2009.

[3] Sen-Peng Eu and Tung-Shan Fu. Lattice paths and generalized cluster complexes. *J. Combin. Theory Ser. A*, 115(7):1183–1210, 2008.

[4] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[5] George N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94:441–451, 1960.

[6] Brendon Rhoades. Enumeration of connected Catalan objects by type. *European J. Combin.*, 32(2):330–338, 2011.

[7] Jiang Zeng. Multinomial convolution polynomials. *Discrete Math.*, 160(1-3):219–228, 1996.

Suhyung An
Department of Mathematics, Yonsei University
Seoul 03722, Republic of Korea
E-mail: hera1973@yonsei.ac.kr

JiYoon Jung
Department of Mathematics, Marshall University
Huntington, WV 25755, U.S.A.
E-mail: jungj@marshall.edu

Sangwook Kim
Department of Mathematics, Chonnam National University
Gwangju 61186, Republic of Korea
E-mail: swkim.math@chonnam.ac.kr