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THREE-DIMENSIONAL LORENTZIAN PARA-KENMOTSU MANIFOLDS AND YAMABE SOLITONS

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Abstract. The aim of the present work is to study the properties of three-dimensional Lorentzian para-Kenmotsu manifolds equipped with a Yamabe soliton. It is proved that every three-dimensional Lorentzian para-Kenmotsu manifold is Ricci semi-symmetric if and only if it is Einstein. Also, if the metric of a three-dimensional semi-symmetric Lorentzian para-Kenmotsu manifold is a Yamabe soliton, then the soliton is shrinking and the flow vector field is Killing. We also study the properties of three-dimensional Ricci symmetric and η-parallel Lorentzian para-Kenmotsu manifolds with Yamabe solitons. Finally, we give a non-trivial example of three-dimensional Lorentzian para-Kenmotsu manifold.

1. Introduction

The notion of Yamabe flow and Ricci flow, introduced by Hamilton [13] on a Riemannian manifold, is capable to resolve many long standing problems of science and technology. For instance, it has applications in the string theory, thermodynamics, general relativity, cosmology, quantum field theory, etc (see, [1], [5], [6], [7], [11], [15]). A time dependent metric $g(t)$ on a Riemannian or pseudo-Riemannian manifold M is said to evolve by the Yamabe flow if the metric g satisfies the equation,

$$\frac{\partial g(t)}{\partial t} = -\tau g(t),$$

where τ is the scalar curvature of the metric g. In two-dimension the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial g(t)}{\partial t} = -2S$, where S denotes the Ricci tensor). However, for more than two-dimension the Yamabe flow do not agree with Ricci flow, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general. The Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation)

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in mathematical physics. A Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphism φ_t generated by a fixed vector field V on a manifold M, and homotheries, that is, $g(.,t) = \sigma(t)\varphi^*(t)g_0$. A (semi-)Riemannian manifold (M, g) is a Yamabe soliton [4, 8, 9, 10, 19, 21, 22, 23, 24] if it admits a smooth vector field V such that

$$\frac{1}{2}\mathcal{L}_V g = (\tau - \lambda)g \tag{1}$$

for a constant $\lambda \in \mathbb{R}$ (set of real numbers), where τ is the scalar curvature of g and $\mathcal{L}$ denotes the Lie-derivative operator. A Yamabe soliton is said to be *shrinking, steady* or *expanding* if it admits a soliton vector field for which $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, respectively. As a generalisation of Yamabe soliton, the η-Yamabe soliton [2, 20] on a (semi-)Riemannian manifold (M_n, g) is defined by

$$\frac{1}{2}\mathcal{L}_V g = (\tau - \lambda)g - \mu\eta \otimes \eta, \tag{2}$$

where $\lambda, \mu \in \mathbb{R}$ and η is a 1-form. In particular, the η-Yamabe soliton becomes Yamabe soliton for $\mu = 0$. Throughout the paper, we denote the Yamabe soliton by (g, V, λ) and η-Yamabe soliton by (g, V, λ, μ).

On the other hand, the Lorentzian manifold is playing an important role in mathematical physics specially in the development of the theory of relativity and cosmology. It is one of the most important subclass of pseudo-Riemannian manifolds [18]. In 1982, Matsumoto et al. gave the idea of Lorentzian para-Sasakian manifolds [16, 17]. Later, such manifolds have been studied by several authors. Recently, A. Haseeb et al. defined and studied the Lorentzian para-Kenmotsu manifold [14] as a subclass of Lorentzian para-contact manifold.

Motivated from the above studies, we are going to study the properties of three-dimensional Lorentzian para-Kenmotsu manifold and Yamabe solitons. Also, we give a non-trivial example of 3-dimensional Lorentzian para-Kenmotsu manifold and validate our some results.

2. Some Preliminary results on Lorentzian para-Kenmotsu manifold and Yamabe soliton

In a $(2n+1)$-dimensional Lorentzian almost para-contact manifold M with the fundamental tensor field φ of type $(1,1)$, a unit time-like vector field ξ, a 1-form η and a Lorentzian metric g, we have

$$\varphi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1(\xi \text{ is time-like vector field}), \tag{3}$$

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad rank(\varphi) = 2n, \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{5}$$

$$g(X, \xi) = \eta(X) \tag{6}$$

for arbitrary vector fields X and Y on M, where I is the identity endomorphism of the tangent bundle of M. The structure (φ, ξ, η, g) on M is called a Lorentzian almost para-contact structure. A Lorentzian almost para-contact manifold M is called a Lorentzian para-Kenmotsu manifold if

$$(\nabla_X \phi)Y = -g\,(\phi X, Y)\,\xi - \eta\,(Y)\,\phi X, \tag{7}$$

for all vector fields X and Y. From equation (7), it follows that

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{8}$$

$$(\nabla_X \eta)Y = -g(\varphi X, \varphi Y) = -g(X,Y) - \eta(X)\eta(Y). \tag{9}$$

In line to achieve the purpose of the article, we require the following lemmas:

Lemma 2.1. [14] *On any* $(2n+1)$*-Lorentzian para-Kenmotsu manifold, we have*

1. $R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$
2. $R(X,\xi)\xi = -X - \eta(X)\xi,$
3. $R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X,$
4. $S(X,\xi) = 2n\eta(X), \quad S(\xi,\xi) = -2n$, *and* $Q(\xi) = 2n\xi,$

where R, S *and* Q *are the curvature tensor, Ricci tensor and Ricci operator of* M*, respectively.*

Lemma 2.2. *A Lorentzian para-Kenmotsu manifold of dimension three satisfies:*

$$QX = \left(\frac{\tau}{2} - 1\right) X + \left(\frac{\tau}{2} - 3\right) \eta(X)\xi, \tag{10}$$

$$S(X,Y) = \left(\frac{\tau}{2} - 1\right) g(X,Y) + \left(\frac{\tau}{2} - 3\right) \eta(X)\eta(Y), \tag{11}$$

and

$$\begin{aligned} R(X,Y)Z = &\left(\frac{\tau}{2} - 2\right) [g(Y,Z)X - g(X,Z)Y] \\ &+ \left(\frac{\tau}{2} - 3\right) [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\ &+ \left(\frac{\tau}{2} - 3\right) [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]\,. \end{aligned} \tag{12}$$

Proof. The Weyl conformal curvature tensor C of type $(1,3)$ on a $(2n+1)$-dimensional manifold M is defined by

$$\begin{aligned} C\,(X,Y)\,Z = &R\,(X,Y)\,Z - \frac{1}{2n-1}[S\,(Y,Z)\,X - S\,(X,Z)\,Y \\ &+ g\,(Y,Z)\,QX - g\,(X,Z)\,QY] \\ &+ \frac{\tau}{(2n)\,(2n-1)}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y]\,, \end{aligned} \tag{13}$$

where R, S, Q and τ are as defined above. It is know that the Weyl conformal curvature tensor vanishes in a three-dimensional (semi-)Riemannian manifold, therefore from equation (13) we infer

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & - S(X,Z)Y - \frac{\tau}{2}\left(g(Y,Z)X - g(X,Z)Y\right). \end{aligned} \tag{14}$$

On taking $X = Z = \xi$ in (14) and using equation (8) and Lemma 2.1 leads to equation (10). Again in view of $S(X,Y) = g(QX,Y)$ and equation (10), we get equation (11). The equations (10), (11) together with (14) provide equation (12). □

Furthermore the equation (11) can be concluded as:

Proposition 2.3. *Three-dimensional Lorentzian para-Kenmotsu manifolds are η-Einstein.*

Covariant differentiation of equation (10) along any vector field Y together with equation (8) provide

$$\begin{aligned} (\nabla_Y Q)X =& \frac{1}{2}Y(\tau)X + \frac{1}{2}Y(\tau)\eta(X)\xi \\ & - \left(\frac{\tau}{2} - 3\right)\left[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(X)Y\right]. \end{aligned}$$

Let us consider an orthonormal frame field on M. Contracting the above equation for Y and making use of well-known formula on semi-Riemannian manifolds $(divQ)(X) = \frac{1}{2}\nabla_X\tau$, we get

$$\xi(\tau)\eta(X) = 2(\tau - 6)\eta(X),$$

which leads to the following proposition:

Proposition 2.4. *On any three-dimensional Lorentzian para-Kenmotsu manifold, we have*

$$\xi(\tau) = 2(\tau - 6).$$

Let us recall the definition of conformal vector field:

Definition 2.5. *A vector field V on a $(2n+1)$-dimensional (semi-) Riemannian manifold (M,g) is said to be conformal if,*

$$\frac{1}{2}\mathcal{L}_V g = \rho g$$

for a smooth function ρ on M. The smooth function ρ is also known as conformal coefficient. In particular, a conformal vector field with a vanishing conformal coefficient reduces to a Killing vector field.

It is well-known that a conformal vector field satisfies

$$(\mathcal{L}_V S)(X,Y) = -(2n-1)g(\nabla_X D\rho, Y) + \Delta\rho g(X,Y),$$

$$\mathcal{L}_V \tau = -2\rho\tau + 4n\Delta\rho,$$

where D is the gradient operator and $\Delta = -divD$ is the Laplacian operator of g [25]. In view of Definition 2.5 and equation (1) (take $\rho = (\tau - \lambda)$), we can say that the Yamabe flow vector field V is conformal [21]. Hence, we have the following lemma for further use:

Lemma 2.6. *On any three-dimensional (semi-)Riemannian manifold equipped with a Yamabe soliton, we have*

$$(\mathcal{L}_V S)(X,Y) = -g(\nabla_X D\tau, Y) + \Delta\tau g(X,Y),$$

$$\mathcal{L}_V \tau = 4\Delta\tau - 2\tau(\tau - \lambda). \tag{15}$$

3. Ricci semi-symmetric Lorentzian para-Kenmotsu manifold and Yamabe soliton

A (semi-)Riemannian manifold M is said to be Ricci semi-symmetric if

$$R(X,Y) \cdot S = 0,$$

which is equivalent to

$$S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = 0$$

for all vector fields X, Y, Z and U on M. For a three-dimensional Lorentzian para-Kenmotsu manifold, the foregoing equation together with equation (11) can be written as

$$\left(\frac{\tau}{2} - 1\right)[g(R(X,Y)Z,U) + g(Z,R(X,Y)U)]$$
$$+ \left(\frac{\tau}{2} - 3\right)[\eta(R(X,Y)Z)\eta(U) + \eta(R(X,Y)U)\eta(Z)] = 0.$$

Making use of Lemma 2.1 and well-known property of Riemannian curvature tensor $g(R(X,Y)Z,U) = -g(R(X,Y)U,Z)$, the above equation can be written as

$$\left(\frac{\tau}{2} - 3\right)[g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)$$
$$+ g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z)] = 0.$$

Let $\{e_i, i = 1,2,3\}$ be an orthonormal basis of M. Substitute $X = U = e_i$ in equation (3) and summing over i $(1 \le i \le 3)$ leads to

$$\left(\frac{\tau}{2} - 3\right)[g(Y,Z) + 3\eta(Y)\eta(Z)] = 0 \tag{16}$$

for arbitrary vector fields Y and Z. Substituting $Y = Z = \xi$ in equation (16) and making use of equations (3) and (6), we obtain $\tau = 6$. Feeding equation (11) with the provided value of scalar curvature leads to

$$S(X, Y) = 2g(X, Y). \tag{17}$$

This shows that the Ricci semi-symmetric manifold is Einstein. The converse part is obvious. Hence, we have following theorem:

Theorem 3.1. *A three-dimensional Lorentzian para-Kenmotsu manifold is Ricci semi-symmetric if and only if it is Einstein.*

From equation (17), it is clear that the scalar curvature tensor of three-dimensional Lorentzian para-Kenmotsu manifold is constant. Thus we have

Corollary 3.2. *Every three-dimensional Ricci semi-symmetric Lorentzian para-Kenmotsu manifold possesses the constant scalar curvature.*

In view of $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$ and equation (8), we have

$$\frac{1}{2}\mathcal{L}_\xi g = -g - \eta \otimes \eta.$$

The above equation is identical to the equation (2) with $\mu = 1$ and $\lambda = 7$. Hence, we can write the following corollary:

Corollary 3.3. *Every three-dimensional Ricci semi-symmetric Lorentzian para-Kenmotsu manifold possesses an η-Yamabe soliton $(g, \xi, 7, 1)$.*

Feeding equation (15) with $\tau = 6$ provide $\lambda = 6$ and the resulting values of τ and λ together with equation (1) provide $(\mathcal{L}_V g)(X, Y) = 0$. Hence, we can write:

Corollary 3.4. *If the metric of a three-dimensional semi-symmetric Lorentzian para-Kenmotsu manifold M is a Yamabe soliton, then the Yamabe soliton is expanding with $\lambda = 6$ and the flow vector field V is Killing.*

It is well settled that all semi-symmetric manifolds are Ricci semi-symmetric. Let us find $R(X, Y) \cdot R$ under the assumption that the manifold under consideration is Ricci semi-symmetric. After feeding $\tau = 6$ in equation (12), we have

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{18}$$

As we know that

$$\begin{aligned}(R(X, Y) \cdot R)(Z, U)W = {} & R(X, Y)R(Z, U)W - R(R(X, Y)Z, U)W \\ & - R(Z, R(X, Y)U)W - R(Z, U)R(X, Y)W.\end{aligned} \tag{19}$$

In view of the foregoing equations (18) and (19), we have $(R(X, Y) \cdot R)(Z, U)W = 0$. Hence, we write the following theorem:

Theorem 3.5. *A three-dimensional Lorentzian para-Kenmotsu manifold is semi-symmetric if and only if it is Ricci semi-symmetric.*

The equation (18) can also be concluded as:

Corollary 3.6. *Three-dimensional semi-symmetric Lorentzian para-Kenmotsu manifolds are space forms.*

In view of the above discussion, we can write the following remark:

Remark 3.7. *The Theorem 3.1, Corollary 3.3 and Corollary 3.4 also hold for a three-dimensional semi-symmetric Lorentzian para-Kenmotsu manifold.*

4. Ricci symmetric Lorentzian para-Kenmotsu manifold and Yamabe soliton

The Ricci symmetric (semi-)Riemannian manifold M is characterized by

$$\nabla S = 0,$$

which can be re-written as

$$\nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(\nabla_X Y, Z) = 0 \tag{20}$$

for arbitrary vector fields X, Y and Z on M. For a three-dimensional Lorentzian para-Kenmotsu manifold, equation (20) together with equation (11) can be written as

$$\begin{aligned} &X(\tau)[g(Y,Z) + \eta(Y)\eta(Z)] \\ &- (\tau - 6)\,[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \end{aligned} \tag{21}$$

Let us take a local orthonormal frame $\{e_i : i = 1,2,3\}$ on a three-dimensional Lorentzian para-Kenmotsu manifold M. Substitute $Y = Z = e_i$ in equation (21) and summing over $i = 1,2,3$ provides $X(\tau) = 0$ for arbitrary vector field X. In view of Proposition 2.4 together with above discussion, we can state the following theorem:

Theorem 4.1. *A three-dimensional Ricci symmetric Lorentzian para-Kenmotsu manifold is of constant scalar curvature (i.e. $\tau = 6$).*

Let us find $\nabla_X R$ under the assumption that the manifold under consideration is semi-symmetric. After feeding $\tau = 6$ in the equation (12), we get $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Its covariant differentiation along any vector field W provides $(\nabla_W R)(X,Y)Z = 0$. Hence, we can state the following theorem:

Theorem 4.2. *In a three-dimensional Lorentzian para-Kenmotsu manifold, Ricci symmetric is the sufficient condition for symmetric manifold.*

Like as previous section, substituting $\tau = 6$ in equation (15) provides $\lambda = 6$ and the resulting values of τ and λ together with the equation (1) provide $\mathcal{L}_V g = 0$. Hence, we can write:

Corollary 4.3. *If the metric of a three-dimensional symmetric Lorentzian para-Kenmotsu manifold M is a Yamabe soliton, then the Yamabe soliton is expanding with $\lambda = 6$ as well as the flow vector field V is Killing.*

5. η-parallel Ricci tensor on Lorentzian para-Kenmotsu manifold and Yamabe soliton

A (semi-)Riemannian manifold is said to have η-parallel Ricci tensor if its non-vanishing Ricci tensor S satisfies the relation $(\nabla_Z S)(\phi X, \phi Y) = 0$. Replacing X, Y with $\phi X, \phi Y$, respectively, in equation (11) and making use of equations (4) and (5) leads to

$$S(\phi X, \phi Y) = \Big(\frac{\tau}{2} - 1\Big)[g(X, Y) + \eta(X)\eta(Y)].$$

The covariant differentiation of the foregoing equation along arbitrary vector field Z provides

$$\begin{aligned}(\nabla_Z S)(\phi X, \phi Y) &= \frac{1}{2}Z(\tau)[g(X, Y) + \eta(X)\eta(Y)]\\ &- \Big(\frac{\tau}{2} - 1\Big)[g(X, Z)\eta(Y) + g(Y, Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)].\end{aligned}$$

Hence, for a three-dimensional Lorentzian para-Kenmotsu manifold equipped with η-parallel Ricci tensor, we have

$$\begin{aligned}&\frac{1}{2}Z(\tau)[g(X, Y) + \eta(X)\eta(Y)]\\ &- \Big(\frac{\tau}{2} - 1\Big)[g(X, Z)\eta(Y) + g(Y, Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)] = 0.\end{aligned} \tag{22}$$

Setting $X = Y = e_i$ in equation (22) and summing over i $(1 \leq i \leq 3)$ provides $Z(\tau) = 0$ (*i.e.* $\tau = constant$). Immediate result together with Proposition 2.4 provides $\tau = 6$. Feeding $\tau = 6$ in equations (11) and (12) provide $S(X, Y) = 2g(X, Y)$ and $R(X, Y)Z = g(Y, Z)X - g(X, Z)Y$, respectively. Thus, we have:

Theorem 5.1. *If the Ricci tensor of a three-dimensional Lorentzian para-Kenmotsu manifold is η-parallel, then we can state the following:*

1. *The scalar curvature of the manifold is constant, i.e., $\tau = 6$.*
2. *The manifold is Einstein.*
3. *Sectional curvature of the manifold is constant $K = 1$, i.e., the manifold is locally isometric to $S^3(1)$.*

Let us investigate three-dimensional Lorentzian para-Kenmotsu manifold equipped with η-parallel Ricci tensor together with Yamabe soliton. Substituting $\tau = 6$ in equation (15) provides $\lambda = 6$. Hence, equation (1) leads to $\mathcal{L}_V g = 0$. Thus, we can state:

Corollary 5.2. *If the metric of a three-dimensional Lorentzian para-Kenmotsu manifold M equipped with η-parallel Ricci tensor is a Yamabe soliton, then the Yamabe soliton is expanding with $\lambda = 6$ as well as the flow vector field V is Killing.*

6. Three-dimensional Lorentzian para-Kenmotsu manifold admitting a non-null concircular vector field

A non vanishing vector field V on a three-dimensional Lorentzian para-Kenmotsu manifold is said to be concircular vector field [12] if

$$\nabla_X V = \sigma X, \tag{23}$$

where X is an arbitrary vector field and σ is a scalar function. It is noticed that a concircular vector field V with σ =constant is a concurrent vector field. Recently, Chaubey and Shaikh [3] studied the properties of non-null concircular vector field on $(LCS)_3$-manifold. It is obvious from (23) that $\nabla_Y \nabla_X V = \sigma\nabla_Y X + Y(\sigma)X$, which gives

$$'R(X,Y,V,Z) = X(\sigma)g(Y,Z) - Y(\sigma)g(X,Z), \tag{24}$$

where $'R(X,Y,V,Z) = g(R(X,Y)V,Z)$. Putting $Z = \xi$ in equation (24) and using equation (6) and Lemma 2.1(1), we obtain

$$\eta(X)g(Y,V) - \eta(Y)g(X,V) = \eta(Y)X(\sigma) - \eta(X)Y(\sigma).$$

Setting $X = \phi^2 X$ and $Y = \xi$ in the foregoing equation leads to

$$X(\sigma) + \eta(X)\xi(\sigma) = -g(X,V) - \eta(X)\eta(V). \tag{25}$$

Since V is a non-null concircular vector field, $g(X,V) \neq 0$ and therefore equation (24) yields

$$X(\sigma)g(Y,V) = Y(\sigma)g(X,V). \tag{26}$$

Substituting Y with ξ in equation (26) leads to

$$X(\sigma)\eta(V) = \xi(\sigma)g(X,V).$$

The above equation can be re-written as

$$X(\sigma)\eta(V)\eta(X) = \xi(\sigma)g(X,V)\eta(X). \tag{27}$$

In consequence of (25), (26) and (27), one can find

$$\{g(X,V) + \eta(X)\eta(V)\}\{X(\sigma) + g(X,V)\} = 0,$$

which reveals that at least one of the multiple $\{g(X,V)+\eta(X)\eta(V)\}$ or $\{X(\sigma)+g(X,V)\}$ is equal to zero. Here we can come up with three conditions:

$$(i) \qquad g(X,V) + \eta(X)\eta(V) = 0, \qquad X(\sigma) + g(X,V) = 0.$$

$$(ii) \qquad g(X,V) + \eta(X)\eta(V) = 0, \qquad X(\sigma) + g(X,V) \neq 0.$$

$$(iii) \qquad g(X,V) + \eta(X)\eta(V) \neq 0, \qquad X(\sigma) + g(X,V) = 0.$$

Let us start with the common statement of (i) and (ii) ,*i.e.*, $g(X,V)+\eta(X)\eta(V) = 0$. Its covariant derivative along the vector field Y provides

$$(\nabla_Y\eta)(X)\eta(V) + (\nabla_Y\eta)(V)\eta(X) = 0.$$

The foregoing equation together with equation (9) and $g(X,V)+\eta(X)\eta(V) = 0$ yield $\eta(X)\eta(Y)\eta(V) = 0$, which shows that V is orthogonal to ξ. So under the assumption $V \not\perp \xi$, the only possibility is (iii). The equation (24) together with (iii) provides

$$'R(X,Y,V,Z) = g(X,Z)g(Y,V) - g(Y,Z)g(X,V). \tag{28}$$

Let $\{e_i, i = 1,2,3\}$ be an orthonormal frame field. Substitute $X = Z = e_i$ in equation (28) and summing over $i(1 \le i \le 3)$ leads to

$$S(Y,V) = 2g(Y,V), \tag{29}$$

where Y is an arbitrary vector field. With the help of equations (11) and (29), we observe that $(\tau - 6)[g(Y,V) + \eta(Y)\eta(V)] = 0$, which shows that $\tau = 6$. Thus we have

Theorem 6.1. *A three-dimensional Lorentzian para-Kenmotsu manifold equipped with a non-null concircular vector field, which is non-orthogonal to ξ, is a space form.*

The equation (1) together with well-known result $(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ can be re-written as:

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 2(\tau - \lambda)g(Y,V).$$

Let us suppose that the manifold under consideration is a three-dimensional Lorentzian para-Kenmotsu manifold and the soliton vector field of the Yamabe soliton is non-null concircular vector field. Hence, equation (23) and the earlier discussion yield $\sigma = 6 - \lambda$ and therefore σ is a constant. Thus we have

Corollary 6.2. *Let a three-dimensional Lorentzian para-Kenmotsu manifold admits a Yamabe soliton. If the soliton vector field of Yamabe soliton is a non-null concircular vector field, then it is a concurrent vector field.*

7. Example of three-dimensional Lorentzian para-Kenmotsu manifold

Example 7.1. *Consider a three-dimensional manifold $M^3 = \{(x,y,z) \in R^3 : z \neq 0\}$ with the standard coordinate system (x,y,z) of R^3. Let $e_1 = e^z\frac{\partial}{\partial x}$, $e_2 = e^z\frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z} = \xi$ be linearly independent vector fields at each point of M^3 and therefore they form a basis of the tangent space at each point of M^3.*

Let g be a Lorentzian metric of M^3 defined by

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and ϕ is a $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = 0.$$

Using linearity of ϕ and g, we have

$$\eta(e_3) = -1, \quad \phi^2 X = X + \eta(X)e_3 \text{ and } g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for any $X, Y \in TM$. Here η is a 1-form on M^3 defined by $\eta(X) = g(X, e_3)$ for any $X \in TM$. Hence for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost Lorentzian almost para-contact metric structure on M^3. The Lie-bracket for the example can be calculated by using the definition $[X,Y]f = X(Yf) - Y(Xf)$. All possible Lie-brackets for the example are as follows:

$$\begin{array}{lll} [e_1, e_1] = 0, & [e_1, e_2] = 0, & [e_1, e_3] = -e_1, \\ [e_2, e_1] = 0, & [e_2, e_2] = 0, & [e_2, e_3] = -e_2, \\ [e_3, e_1] = e_1, & [e_3, e_2] = e_2, & [e_3, e_3] = 0 . \end{array}$$

Let ∇ denote the Levi-Civita connection with respect to the semi-Riemannian metric g. Using the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned}$$

and the semi-Riemannian metric g, we can easily calculate

$$\begin{array}{lll} \nabla_{e_1} e_1 = -e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0 . \end{array}$$

Now, for $X = X^1 e_1 + X^2 e_2 + X^3 e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \nabla_X \xi &= \nabla_{(X^1 e_1 + X^2 e_2 + X^3 e_3)} e_3 \\ &= X^1 \nabla_{e_1} e_3 + X^2 \nabla_{e_2} e_3 + X^3 \nabla_{e_3} e_3 \\ &= -X^1 e_1 - X^2 e_2, \end{aligned} \tag{30}$$

$$\begin{aligned} -X - \eta(X)\xi &= -(X^1 e_1 + X^2 e_2 + X^3 e_3) \\ &\quad - g(X^1 e_1 + X^2 e_2 + X^3 e_3, e_3) e_3 \\ &= -X^1 e_1 - X^2 e_2 - X^3 e_3 + X^3 e_3 \\ &= -X^1 e_1 - X^2 e_2, \end{aligned} \tag{31}$$

where X^1, X^2, X^3 are scalars. In view of equations (30) and (31), we can say that the structure (ϕ, ξ, η, g) is a Lorentzian para-Kenmotsu structure on M^3. *Consequently $M^3(\phi, \xi, \eta, g)$ is a Lorentzian para-Kenmotsu manifold.* In view of equation $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ along with $R(e_i, e_i)X =$

0; $\forall\, i = 1, 2, 3$, non-zero curvature tensors are as under:

$$R(e_1, e_2)e_1 = -e_2 = -R(e_2, e_1)e_1, \quad R(e_1, e_3)e_1 = -e_3 = -R(e_3, e_1)e_1,$$

$$R(e_1, e_2)e_2 = e_1 = -R(e_2, e_1)e_2, \quad R(e_2, e_3)e_2 = -e_3 = -R(e_3, e_2)e_2,$$

$$R(e_1, e_3)e_3 = -e_1 = -R(e_3, e_1)e_3, \quad R(e_2, e_3)e_3 = -e_2 = -R(e_3, e_2)e_3.$$

In the given example, the vector fields e_1, e_2 are space-like and the vector field e_3 is time-like. Hence, the Ricci tensor and scalar curvature can be written as:

$$\begin{aligned} S(X, Y) &= \sum_{i=1}^{3} \varepsilon_i g(R(e_i, X)Y, e_i) \\ &= g(R(e_1, X)Y, e_1) + g(R(e_2, X)Y, e_2) - g(R(\xi, X)Y, \xi), \end{aligned}$$

$$\tau = \sum_{i=1}^{3} \varepsilon_i S(e_i, e_i) = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3). \tag{32}$$

Hence we have the following:

$$S(e_1, e_1) = 2, \quad S(e_2, e_2) = 2, \quad S(e_3, e_3) = -2.$$

Feeding equation (32) with the foregoing values of Ricci tensors provides the scalar curvature $\tau = 6$.

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