Honam Mathematical J. **43** (2021), No. 4, pp. 587–596
https://doi.org/10.5831/HMJ.2021.43.4.587

SOME INTEGRAL INEQUALITIES IN THE FRAMEWORK OF GENERALIZED K-PROPORTIONAL FRACTIONAL INTEGRAL OPERATORS WITH GENERAL KERNEL

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Abstract. In this article, using the concept proposed reciently by the author, of a Generalized k-Proportional Fractional Integral Operators with General Kernel, new integral inequalities are obtained for convex functions. It is shown that several known results are particular cases of the proposed inequalities and in the end new directions of work are provided.

One of the areas of greatest development in Mathematical Sciences today is Fractional Calculus. Although its history goes back, practically, to that of the classical calculus, only in the last 50 the number of researchers and investigations have multiplied significantly. This increase involves both theoretical and practical developments (see, for example, [18, 31, 32, 36]), and more and more results have been obtained in this area (see [9, 13, 18, 19, 22, 23, 30, 37, 38] and references cited therein). A more complete overview of the development of this area actually, with its overlapping with the Generalized Local Calculus, can be found at [3], [4], [5] and [39].

The classes of functions that we will consider will be defined below (see [17] and [40]).

Definition 0.1. *A function $h(\chi)$ is said to be in $L_{q,r}[0,+\infty)$ space if*

$$L_{q,r}[0,+\infty)=\{h:\|h\|_{L_{q,r}[0,+\infty)}=\left(\int_{a_1}^{a_2}|h(s)|^q s^r ds\right)^{\frac{1}{q}}<+\infty,\quad 1\le q<+\infty,\quad r\ge 0\}.$$

For $r=0$

Received March 22, 2021. Accepted September 8, 2021.
2020 Mathematics Subject Classification. 26A33, 26A51, 26D15.
Key words and phrases. Generalized k-Proportional Fractional Integral Operators with General Kernel, Fractional derivatives and integrals, Integral inequalities.
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$$L_q[0,+\infty) = \{h : \|h\|_{L_q[0,+\infty)} = \left(\int_{a_1}^{a_2} |h(s)|^q ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty\}.$$

In [16] the following functional class is defined.

Definition 0.2. *Let $h \in L_1[0,+\infty)$ and Ψ be an increasing and positive monotone function on $[0,+\infty)$ and also derivative Ψ' is continuous on $[0,+\infty)$ and $\Psi(0) = 0$. The space $X_\Psi^q(0,+\infty)(1 \leq q < +\infty)$ of those real-valued Lebesgue measurable functions h on $[0,+\infty)$ for which*

$$\|h\|_{X_q^\Psi} = \left(\int_{a_1}^{a_2} |h(s)|^q \Psi'(s) ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty.$$

Based on the previous definition, we have the functional class with which we will work.

Definition 0.3. *Let $h \in L_1[0,+\infty)$ and F continuous and positive function on $[0,+\infty)$ with $F(0) = 0$. The space $X_F^q(0,+\infty)(1 \leq q < +\infty)$ of those real-valued Lebesgue measurable functions h on $[0,+\infty)$ for which*

$$\|h\|_{X_F^q} = \left(\int_{a_1}^{a_2} |h(s)|^q F(s) ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty,$$

and for the case $q = +\infty$

$$\|h\|_{X_F^\infty} = ess \sup_{0 \leq s < \infty} \left[F(s)h(s)\right].$$

Remark 0.4. *If $F(t) = 1$, $1 \leq q < +\infty$ the space $X_F^q(0,+\infty)$ coincides with the $L_q[0,+\infty)$-space and also if we choose $F(t) = \frac{1}{t}$ the space $X_F^q(0,+\infty)$ coincides with $L_{q,r}[1,+\infty)$-space.*

Perhaps one of the most productive mathematical ideas lately, due to its variety of uses and interrelationships with different applications, is that of the convex function.

Definition 0.5. *A function $f : I \to \mathbb{R}$ is said to be **convex** on interval $I \subset \mathbb{R}$, if the inequality $f(tx + (1-t)y) \leq t(x) + (1-t)f(y)$, for $x, y \in I$ is fulfilled.*

We say that f is concave if $-f$ is convex. The consequent extensions of this concept, which have appeared lately, have transformed it into an extremely complex concept. To reflect on this, we suggest that the user read the work [28], where a fairly complete classification of most of the known definitions is made.

The generalized operator, the center of our work is presented in the following Definition (see [25]).

Definition 0.6. *Let $h \in X_F^q(0,+\infty)$, F a continuous, positive function on $[0,+\infty)$ with $F(0)=0$. The right and left side Generalized k-Proportional Fractional Integral Operators with General Kernel of order γ of h are defined, respectively by*

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}h(\chi) = \frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,s),\lambda)F(s)h(s)}{(\mathbb{F}_+(\chi,s))^{1-\frac{\gamma}{k}}}ds, \tag{1}$$

and

$$J_{F,a_2-}^{\frac{\gamma}{k},\lambda}h(\chi) = \frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{\chi}^{a_2}\frac{G(\mathbb{F}_-(s,\chi),\lambda)F(s)h(s)}{(\mathbb{F}_-(s,\chi))^{1-\frac{\gamma}{k}}}ds, \tag{2}$$

with $\chi \in (a,b)$, $\mathbb{F}_+(\chi,s) = \int_s^\chi F(r)dr$, $\mathbb{F}_-(s,\chi) = \int_\chi^s F(r)dr$ *and* $G(\mathbb{F}_+(\chi,s),1) = G(\mathbb{F}_-(\chi,s),1) = 1$.

Remark 0.7. *Next, we will show how many integral operators are particular cases of* (1) *and* (2).

1. *Putting in Definition 0.6* $k=1$, $F=1$, *and* $\lambda=1$, *we obtain the classic Riemann-Liouville operators.*
2. *Under the above conditions, if* $k \neq 1$ *then the k-fractionals operators of* [24] *are obtained from Definition 0.6.*
3. *With* $F(s)=\frac{1}{s}$, $\lambda=1$ *and* $k=1$, *then the Hadamard fractional operator is reproduced, see* [14, 18, 36].
4. *With* $F(s)=\frac{1}{s^\rho}$, $\lambda=1$, *and* $k=1$, *then we obtain the Katugampola's fractional operator of* [17].
5. *If* $\lambda=1$, $F(s)=g'(s)$, *and* $k=1$, *then we obtain the integral operator of* [19].
6. *Putting* $F(s)=\frac{1}{s}$, $k\neq 1$ *and* $G(\mathbb{F}_+(\chi,s),\lambda) = exp\left[\frac{\lambda-1}{\lambda}\left(ln\frac{\chi}{s}\right)\right]$, *is obtained the integral operator of* [33].
7. *If* $\lambda \neq 1$, $F(s)=g'(s)$, $k=1$ *and* $G(\mathbb{F}_+(\chi,s),\lambda) = exp\left[\frac{\lambda-1}{\lambda}\left(g(\chi)-g(s)\right)\right]$, *then we obtain the called GFP integral operator of* [35].

The important role of inequalities in the development and evolution of mathematics is well known and has wide implications in various areas. The formalization of the mathematical theory of inequalities begins, essentially, in the 18th century with the studies carried out by Gauss and continued by Cauchy and Chebyshev, who had the idea of applying some inequalities to mathematical analysis. Later, the Russian mathematician Bunyakovsky, proved in 1859 the well-known Cauchy-Schwarz inequality for the case of infinite dimensions. In recent years, more and more researchers have devoted themselves to this area and it has expanded in multiple directions see, for example, the works [2, 6, 10, 11, 12, 16, 20, 26, 27, 28, 29] and the references cited therein.

In this paper using the operators of the Definition 0.6 we obtain several integral inequalities that contain as particular cases, several reported in the literature.

0.1. Main Results

In this section we present several integral inequalities, obtained under the Generalized k-Proportional Fractional Integral Operators with General Kernel of Definition 0.6.

Theorem 0.8. *Suppose that for $\lambda \in (0,1]$, $k>0$, $\gamma \in \mathbb{C}$ with $Re(\gamma)>0$ there are two positive continuous functions M and N such that $M \leq N$ on $[0,+\infty)$ satisfying $\frac{M}{N}$ is decreasing on $[0,+\infty)$ and M is increasing on $[0,+\infty)$. If ω is a convex function with $\omega(0)=0$ and F is a positive continuous functions defined on $[0,+\infty)$, then we have the following inequality*

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[M(\chi)]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[N(\chi)]} \geq \frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(M(\chi))]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(N(\chi))]}, \tag{3}$$

with $0 \leq a_1$.

Proof. Since the function M is increasing along with the function $\frac{\omega(M(\varsigma))}{M(\varsigma)}$, from properties of ω and N we have, for $\varsigma, s \in [a_1,+\infty)$

$$\left(\frac{\omega(M(\varsigma))}{M(\varsigma)} - \frac{\omega(M(s))}{M(s)}\right)\left(\frac{M(s)}{N(s)} - \frac{M(\varsigma)}{N(\varsigma)}\right) \geq 0. \tag{4}$$

From here we get

$$\frac{\omega(M(\varsigma))}{M(\varsigma)}M(s)N(\varsigma)+\frac{\omega(M(s))}{M(s)}M(\varsigma)N(s) \geq \frac{\omega(M(\varsigma))}{M(\varsigma)}M(s)N(s)+\frac{\omega(M(s))}{M(s)}M(\varsigma)N(\varsigma). \tag{5}$$

Multiplying the above result by $\frac{G(\mathbb{F}_+(\varsigma,\chi),\lambda)F(\varsigma)h(\varsigma)}{\lambda^\gamma k\Gamma_k(\gamma)(\mathbb{F}_+(\varsigma,\chi))^{1-\frac{\gamma}{k}}}$, then we integrate the resulting inequality with respect to ς over (a_1,χ), we get

$$\begin{aligned} & M(s)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)\right] + \frac{\omega(M(s))N(s)}{M(s)}J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[M(\chi)\right] \\ \geq\; & N(s)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\frac{\omega(M(\chi))}{M(\chi)}M(\chi)\right] + \frac{\omega(M(s))}{M(s)}M(s)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[N(\chi)\right]. \end{aligned} \tag{6}$$

Now, multiplying both sides of this inequality by $\frac{G(\mathbb{F}_+(s,\chi),\lambda)F(s)h(s)}{\lambda^\gamma k\Gamma_k(\gamma)(\mathbb{F}_+(s,\chi))^{1-\frac{\gamma}{k}}}$, and integrating the inequality obtained with respect to s over (a_1,ς), it holds that

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[M(\chi)\right]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[N(\chi)\right]} \geq \frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[M(\chi)\right]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)\right]} \tag{7}$$

Now, taking into account that $M \le N$ on $[0,+\infty)$ and $\frac{\omega(z)}{z}$ is an increasing on $[0,+\infty)$ we have $\frac{\omega[M(\varsigma)]}{M(\varsigma)} \ge \frac{\omega[N(\varsigma)]}{N(\varsigma)}$, using this inequality in (7) allows us to obtain the desired result. □

Remark 0.9. *If* $\lambda \neq 1$, $F(s) = g'(s)$, $k = 1$ *and* $G(\mathbb{F}_+(\chi, s), \lambda) = exp\left[\frac{\lambda-1}{\lambda}(g(\chi) - g(s))\right]$, *from this result we obtain the Theorem 3.1 of* [34].

Remark 0.10. *If we work with the Hadamard integral (see case 3 of Remark 0.7) then this result contains Theorem 3.1 of* [7] *and Theorem 3.1 of* [33].

In the following result, two orders of integration are used, allowing the Theorem 0.8 to be extended in a "double" formulation.

Theorem 0.11. *Suppose that for* $\lambda \in (0, 1]$, $k > 0$, $\gamma, \mu \in \mathbb{C}$ *with* $Re(\gamma), Re(\mu) > 0$ *there are two positive continuous functions* M *and* N *such that* $M \le N$ *on* $[0,+\infty)$ *satisfying* $\frac{M}{N}$ *is decreasing on* $[0,+\infty)$ *and* M *is increasing on* $[0,+\infty)$. *If* ω *is a convex function with* $\omega(0) = 0$ *and* F *is a positive continuous functions defined on* $[0,+\infty)$, *then we have the following inequality*

$$\begin{aligned} & J^{\frac{\mu}{k},\lambda}_{F,a_1+}\left[M(\chi)\right] J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)\right] + J^{\frac{\mu}{k},\lambda}_{F,a_1+}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)\right] J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left[M(\chi)\right] + \\ \ge\ & J^{\frac{\mu}{k},\lambda}_{F,a_1+}\left[N(\chi)\right] J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left[\omega(M(\chi))\right] + J^{\frac{\mu}{k},\lambda}_{F,a_1+}\left[\omega(M(\chi))\right] J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left[N(\chi)\right], \end{aligned} \tag{8}$$

with $0 \le a_1$.

Proof. Multiplying both sides of (6) by $\frac{G(\mathbb{F}_+(v,\chi),\lambda)F(v)h(v)}{\lambda^\mu k\mu_k(\mu)(\mathbb{F}_+(v,\chi))^{1-\frac{\mu}{k}}}$, then we integrate the resulting inequality with respect to v over (a_1, ς), and proceeding as in the proof of the previous Theorem, we obtain the conclusion (13). □

Remark 0.12. *If in this Theorem we put* $\mu = \gamma$, *we obtain the Theorem 0.8.*

Remark 0.13. *If* $\lambda \neq 1$, $F(s) = g'(s)$, $k = 1$ *and* $G(\mathbb{F}_+(\chi, s), \lambda) = exp\left[\frac{\lambda-1}{\lambda}(g(\chi) - g(s))\right]$, *from this result we obtain the Theorem 3.2 of* [34].

Remark 0.14. *In the case 3 of Remark 0.7, this result contains Theorem 3.2 of* [7] *and Theorem 3.2 of* [33].

The following result generalizes Theorem 0.8 using a certain weight function h.

Theorem 0.15. *Suppose that for* $\lambda \in (0, 1]$, $k > 0$, $\gamma \in \mathbb{C}$ *with* $Re(\gamma) > 0$ *there are three positive continuous functions* M, N *and* h *such that* $M \le N$ *on* $[0,+\infty)$ *satisfying* $\frac{M}{N}$ *is decreasing on* $[0,+\infty)$ *and* M *is increasing on* $[0,+\infty)$. *If* ω *is a convex function with* $\omega(0) = 0$ *and* F *is a positive continuous functions defined on* $[0,+\infty)$, *then we have the following inequality*

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[M(\chi)]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[N(\chi)]} \geq \frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(M(\chi))h(\chi)]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(N(\chi))h(\chi)]}, \tag{9}$$

with $0 \leq a_1$.

Proof. Since the function M is increasing along with the function $\frac{\omega(M(\varsigma))}{M(\varsigma)}$, from properties of ω and N we have, for $\varsigma, s \in [a_1, +\infty)$

$$\frac{\omega(M(\varsigma))}{M(\varsigma)} \leq \frac{\omega(N(\varsigma))}{N(\varsigma)}.$$

From here we get

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)h(\chi)\right] \leq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\omega(N(\chi))h(\chi)\right]. \tag{10}$$

From the properties of M, N, ω we have

$$\left[\frac{\omega(M(\varsigma))}{M(\varsigma)} - \frac{\omega(M(s))}{M(s)}h(s)\right][M(s)N(\varsigma) - M(\varsigma)N(s)] \geq 0. \tag{11}$$

Following the procedure of the proof of Theorem 0.8, we obtain

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[M(\chi)]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[N(\chi)]} \geq \frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(M(\chi))h(\chi)]}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}\left[\frac{\omega(M(\chi))}{M(\chi)}N(\chi)h(\chi)\right]}, \tag{12}$$

using (10) in the above inequality, the conclusion of the Theorem is obtained. □

Remark 0.16. *This Theorem contains as particular cases Theorem 3.3 of* [34]*, Theorem 3.3 of* [7] *and Theorem 3.3 of* [33].

Our last result generalizes the previous one by incorporating two integration orders, thereby generalizing the previously obtained results.

Theorem 0.17. *Suppose that for* $\lambda \in (0,1]$, $k > 0$, $\gamma, \mu \in \mathbb{C}$ *with* $Re(\gamma), Re(\mu) > 0$ *there are three positive continuous functions* M, N *and* h *such that* $M \leq N$ *on* $[0, +\infty)$ *satisfying* $\frac{M}{N}$ *is decreasing on* $[0, +\infty)$ *and* M *is increasing on* $[0, +\infty)$. *If* ω *is a convex function with* $\omega(0) = 0$ *and* F *is a positive continuous functions defined on* $[0, +\infty)$, *then we have the following inequality*

$$\begin{aligned} &J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[M(\chi)]\, J_{F,a_1+}^{\frac{\mu}{k},\lambda}[\omega(M(\chi))h(\chi)] + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(M(\chi))h(\chi)]\, J_{F,a_1+}^{\frac{\mu}{k},\lambda}[M(\chi)] \\ \geq\ &J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[N(\chi)]\, J_{F,a_1+}^{\frac{\mu}{k},\lambda}[\omega(M(\chi))h(\chi)] + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[\omega(M(\chi))h(\chi)]\, J_{F,a_1+}^{\frac{\mu}{k},\lambda}[N(\chi)], \end{aligned} \tag{13}$$

with $0 \leq a_1$.

Proof. The proof follows the ideas of Theorem 0.11 and Theorem 0.15, so we leave the details to the reader. □

Remark 0.18. *If in this Theorem we put $\mu = \gamma$, we obtain the Theorem 0.15.*

Remark 0.19. *As before, there is no difficulty in obtaining Theorem 3.4 of* [7], *just like the homonyms of* [33] *and* [34].

1. Conclusions

In this work we have obtained variousintegral inequalities using a generalized fractional operator, which allows obtaining as particular cases, several of those reported in the literature.

We want to point out, in addition to the observations made throughout the work, that with different choices of the F kernel we can obtain, as particular cases, several well-known integral operators. So, for example, if

1. The classic Riemann integral is obtained with $F(t) = t^{\alpha-1}$, $\alpha = 1$, $\lambda = 1$ and $\beta = k$ (with notation changed).
2. If $\lambda = 1$, $F(t) = t^{\alpha-1}$ and $\beta = k$ we obtain the fractional Riemann-Liouville integral.
3. The k-Riemann-Liouville fractional integral of Mubeen and Habibullah (see [24]), we can get it by doing $F(t) = t^s$ with $s = 1$, and $\lambda = 1$.
4. The Katugampola fractional integral of [17] is obtained, putting $\lambda = 1$ and $F(t) = t^{-\alpha}$ (the notation is changed).
5. If we put $F = t^{-s}$ with $s = 1$, then we get the right sided Hadamard fractional integral of [15].
6. An integral operator with non-singular kernel can also be obtained from our Definition 0.6. Thus, considering $\gamma = k = 1$, $F(t) = 1$, and $G(\mathbb{F}_+(x,s),\alpha) = exp\left[-\frac{1-\alpha}{\alpha}(x-s)\right]$, a slight modification of the operator of [1].

The aforementioned details allow us to add some additional observations. If $F(t) = t^{\alpha-1}$, $\alpha = 1$, $\lambda = 1$ and $\beta = k$, that is, if we work with the classical Riemann integral, From the Theorems 0.8 and 0.15 we obtain the following results.

Theorem 1.1 (Theorem 9, [21])**.** *Suppose that there are two positive continuous functions f and g such that $f \leq g$ on $[a,b]$ satisfying $\frac{f}{g}$ is decreasing on $[a,b]$ and f is increasing on $[a,b]$. If φ is a convex function with $\varphi(0) = 0$, then we have the following inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b g(x)dx} \geq \frac{\int_a^b \varphi(f(x))dx}{\int_a^b \varphi(g(x))dx}.$$

Theorem 1.2 (Theorem 10, [21]). *Suppose that there are three positive continuous functions f, g and h such that $f \leq g$ on $[a,b]$ satisfying $\frac{f}{g}$ is decreasing on $[a,b]$ and f is increasing on $[a,b]$. If φ is a convex function with $\varphi(0) = 0$, then the inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b g(x)dx} \geq \frac{\int_a^b \varphi(f(x))h(x)dx}{\int_a^b \varphi(g(x))h(x)dx}$$

holds.

In the case that $\lambda = 1$, $F(t) = t^{\alpha-1}$ and $\beta = k$, that is, the definition operator 0.6 is reduces to the fractional Riemann-Liouville integral, we obtain the following versions of Theorems 0.8 and 0.15.

Theorem 1.3 (Theorem 3.1, [8]). *Suppose that there are two positive continuous functions f and g such that $f \leq g$ on $[a,b]$ satisfying $\frac{f}{g}$ is decreasing on $[a,b]$ and f is increasing on $[a,b]$. If φ is a convex function with $\varphi(0) = 0$, then the inequality*

$$\frac{J^{\gamma} f(t)}{J^{\gamma} g(t)} \geq \frac{J^{\gamma} \varphi(f(t))}{J^{\gamma} \varphi(g(t))}, a < t \leq b, \gamma > 0$$

is valid.

Theorem 1.4 (Theorem 3.5, [8]). *Suppose that there are three positive continuous functions f, g and h such that $f \leq g$ on $[a,b]$ satisfying $\frac{f}{g}$ is decreasing on $[a,b]$ and f is increasing on $[a,b]$. If φ is a convex function with $\varphi(0) = 0$, then the inequality*

$$\frac{J^{\alpha} f(t)}{J^{\alpha} g(t)} \geq \frac{J^{\alpha} \left[\varphi(f(t))h(t)\right]}{J^{\alpha} \left[\varphi(g(t))h(t)\right]}, a < t \leq b, \gamma > 0$$

holds.

These particular cases show the strength of the Definition 0.6 and leave open several possibilities for future work. By other hand, the proven results are direct contributions to the theory of integral inequalities and fractional calculus, and as we pointed out above, it is expected that they lead to some applications, for example, in the study of the existence and uniqueness of solutions in differential equations. fractional.

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