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ϵ-PERTURBATION METHOD FOR VOLUME OF HYPERCUBES CLIPPED BY TWO OR THREE HYPERPLANES

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Abstract. The first author suggested an exact volume formula of the hypercubes $[0,1]^n$ clipped by several hyperplanes expressed directly in terms of linear coefficients of the hyperplanes. However, it requires awkward assumptions to apply the formula to various situations. We suggest a concrete method to overcome those restrictions for two or three hyperplanes using ϵ-perturbation, which gives an exact value applicable for any kind of arrangement of hyperplanes with no consideration.

1. Introduction

Many application problems require evaluating the volume of a hypercube clipped by several hyperplanes. The hyperplanes correspond to linear constraints in machine learning [6] and computational statistics problems and metamer mismatch volume [4] [5].

There are exact volume formulas (see [1] for clipped by one hyperplane and [2] for more than one hyperplanes) of $[0,1]^n$ clipped by hyperplanes and for a general convex polytope [3], which are expressed in a sufficiently concrete manner. Those volume formulas give mathematically exact volumes, however that formulas need some critical conditions. There are several assumptions which originate in Lawrence's original formula [3] for a general convex polytope, and those are transformed to *good clipping conditions* of a clipped hypercube (see Section 4.2 and [2] in detail). The conditions are mathematically reasonable because it essentially means that the vertices are in general position. Nevertheless, these give rise to subtle difficulties under various practical situations

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since it is not so easy to check a violation and to devise a detour way. In this paper, we give a complete expression of the exact volume for $[0,1]^n$ clipped by two or three hyperplanes using ϵ-perturbation (see also Section 6.2 and Appendix B, C, D in [2] for specially easy ϵ-perturbation 4 examples). Although we only consider two and three hyperplanes, our idea can be extended to arbitrary number of hyperplanes by using of suitable ϵ-perturbation.

2. Notations

The hypercube $[0,1]^n$ is an n-cell of a cubical complex K, and we define F^i as the open d-skeleton $K^i \setminus K^{i-1}$, where K^i is the i-skeleton of K. For example, $[0,1]^2$ consists of four points F^0, four open intervals F^1 and one open rectangle F^2. Let

$$
\begin{aligned}
g_1(\mathbf{x}) &= \mathbf{a}\cdot\mathbf{x} + r_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n + r_1,\\
g_2(\mathbf{x}) &= \mathbf{b}\cdot\mathbf{x} + r_2 = b_1x_1 + b_2x_2 + \cdots + b_nx_n + r_2,\\
g_3(\mathbf{x}) &= \mathbf{c}\cdot\mathbf{x} + r_3 = c_1x_1 + c_2x_2 + \cdots + c_nx_n + r_3,\\
H_i &= \{\mathbf{x}|g_i(\mathbf{x}) = 0\} \text{ and } H_i^+ = \{\mathbf{x}|g_i(\mathbf{x}) \geq 0\}.
\end{aligned}
$$

We denote that $|\cdot|$ denote the cardinality and specially $|0_{\mathbf{v}}|$ is the number of indices i of vertex $\mathbf{v} = (v_1,\ldots,v_n)$ for which $v_i = 0$, and $* : F^1 \to \{1,2,\ldots,n\} =: [n]$ is the function that assigns $\mathbf{v}$ the index i for which $v_i \neq 0,1$, and $*_1 : F^2 \to [n]$ (resp. $*_2 : F^2 \to [n]$) is the function that assigns v the first index i (resp. the second index i) for which $v_i \neq 0,1$, and $(\mathbf{ab})_{ij}$, $(\mathbf{ac})_{ij}$, $(\mathbf{bc})_{ij}$ and $(\mathbf{abc})_{ijk}$ denote the determinants $\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$, $\begin{vmatrix} a_i & c_i \\ a_j & c_j \end{vmatrix}$, $\begin{vmatrix} b_i & c_i \\ b_j & c_j \end{vmatrix}$ and $\begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix}$, respectively. To help understanding, let us see an example with $\mathbf{w} = (0,1,0,\frac{7}{8},0,0,1) \in F^1 \subset [0,1]^7$ and $\mathbf{v} = (0,1,\frac{1}{3},0,0,1,\frac{3}{5}) \in F^2 \subset [0,1]^7$, then we get

$$
\begin{aligned}
&|0_{\mathbf{w}}| = 4,\ \ |0_{\mathbf{v}}| = 3,\ \ *(\mathbf{w}) = 4,\ \ *_1(\mathbf{v}) = 3,\ \ *_2(\mathbf{v}) = 7,\\
&(\mathbf{vw})_{47} = \begin{vmatrix} v_4 & w_4 \\ v_7 & w_7 \end{vmatrix} = \begin{vmatrix} 0 & \frac{7}{8} \\ \frac{3}{5} & 1 \end{vmatrix} = -\frac{21}{40}.
\end{aligned}
$$

We need more definitions for the formula (1) and the good clipping condition in the next section. Let us denote the notation $\bullet_{\mathbf{v}}$ and $*_{\mathbf{v}}$ by ordered sets of indices satisfying the following

$$
*_{\mathbf{v}} := \{i \in [n] \mid v_i \neq 0,1\} = \{*_1(\mathbf{v}), *_2(\mathbf{v}), \ldots\},\ \ \bullet_{\mathbf{v}} := [n] \setminus *_{\mathbf{v}}.
$$

Let A_I^J denote a minor with indices I and J of a matrix $A = (a_{i,j}) = (\mathbf{abc})$, where $\mathbf{a},\mathbf{b},\mathbf{c}$ are column vectors. For example, let $I = \{1,4\}$ and $J = \{2,3\}$.

Then

$$A_I^J = \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{4,2} & a_{4,3} \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_4 & c_4 \end{vmatrix}.$$

We call an index I *well-ordered* if $i_1 < i_2 < \cdots < i_s$. We consider two different notions of union operation for ordered sets. One is the *ordered union* $\cup$ respecting the order between two well-ordered indices, for instance, for $t \notin I$,

$$I \cup \{t\} \;:=\; \{i_1, i_2, \dots, t, \dots, i_s\} \quad \text{when } i_1 < i_2 < \cdots < t < \cdots < i_s.$$

The other is the *joining union* $\vee$ as concatenation as follows,

$$I \vee \{t\} \;:=\; \{i_1, i_2, \dots, i_s, t\}.$$

We abbreviate a set of one element $\{x\}$ to x omitting the brace symbols, for example, $I \vee \{t\} =: I \vee t$ and $I \setminus \{t\} =: I \setminus t$.

Let H^+ denote the intersection of all the half spaces H_i^+,

$$H^+ = \bigcap_{i \in [m]} H_i^+,$$

where m is the total number of hyperplanes and $m = 3$ in this paper. Let I be a set of indices for several hyperplanes usually not including the m-th auxiliary plane, i.e. $I \subset [m-1]$ and let H_I denote the intersection of $H^+ \setminus H_m$ and the hyperplanes H_i for $i \in I$, i.e.,

$$H_I := \bigcap_{i \in I} H_i \cap H^+ \setminus H_m.$$

3. Volume formula of the hypercubes clipped by three hyperplanes and good clipping conditions

From Theorem 4.6 in [2], we can see the following complicated volume formula of a hypercube clipped by m hyperplanes $H_1, H_2, \dots, H_m$ with good clipping conditions:

$$\begin{aligned} &\mathrm{vol}\ ([0,1]^n \cap \bigcap_{i=1}^{m} H_i^+) \\ &= \sum_{I \subset \{1,2,\dots,m-1\}} \ \sum_{\mathbf{v} \in F^{|I|} \cap H_I} \frac{(-1)^{|0_{\mathbf{v}}| + \frac{|I|(|I|+1)}{2}} (g_m(\mathbf{v}) A^I_{*_{\mathbf{v}}})^n}{n! |A^I_{*_{\mathbf{v}}}| \prod_{t \in I} A^{I \vee m \setminus t}_{*_{\mathbf{v}}} \prod_{t \in \bullet_{\mathbf{v}}} A^{I \vee m}_{*_{\mathbf{v}} \vee t}}, \end{aligned} \tag{1}$$

where the good clipping conditions show concrete expressions:

(A) For any $I \subset \{1, 2, \dots, m-1\}$, $F^{|I|-1} \cap H_I = \varnothing$,

(B) For any $I \subset \{1, 2, \dots, m-1\}$ and $\mathbf{v} \in F^{|I|} \cap H_I$, $\prod_{t \in I} A^{I \cup m \setminus t}_{*_{\mathbf{v}}} \prod_{t \in \bullet_{\mathbf{v}}} A^{I \cup m}_{*_{\mathbf{v}} \cup t} \neq 0$.

In order to understand the formula and the good clipping conditions, we have to see the Notation section in [2]. However, we will use and consider only $k = 3$ case, so we need not so many notations in (1). Now we introduce $k = 3$ version of the formula (1) which is our starting point (see Corollary 5.1 in [2]).

Theorem 3.1. *The volume of the unit hypercube $[0,1]^n$ intersecting three halfspaces H_1^+, H_2^+, and H_3^+ under seven conditions (see Remark 1) is*

$$\begin{aligned}\mathrm{vol}\,([0,1]^n \cap H_1^+ \cap H_2^+ \cap H_3^+) &= \sum_{\mathbf{v}\in F^0\cap H_1^+\cap H_2^+\cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|} g_3(\mathbf{v})^n}{n!\prod_{k=1}^n c_k} \\ &- \sum_{\mathbf{v}\in F^1\cap H_1\cap H_2^+\cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|}\ sgn(a_{*(\mathbf{v})})\ a_{*(\mathbf{v})}^{n-1}\ g_3(\mathbf{v})^n}{n!\ c_{*(\mathbf{v})}\prod_{k=1,k\neq *(\mathbf{v})}^n (\mathbf{ac})_{*(\mathbf{v})k}} \\ &- \sum_{\mathbf{v}\in F^1\cap H_1^+\cap H_2\cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|}\ sgn(b_{*(\mathbf{v})})\ b_{*(\mathbf{v})}^{n-1}\ g_3(\mathbf{v})^n}{n!\ c_{*(\mathbf{v})}\prod_{k=1,k\neq *(\mathbf{v})}^n (\mathbf{bc})_{*(\mathbf{v})k}} \\ &- \sum_{\mathbf{v}\in F^2\cap H_1\cap H_2\cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|}\ sgn((\mathbf{ab})_{*_1(\mathbf{v})*_2(\mathbf{v})})\ (\mathbf{ab})_{*_1(\mathbf{v})*_2(\mathbf{v})}^{n-1}\ g_3(\mathbf{v})^n}{n!\ (\mathbf{ac})_{*_1(\mathbf{v})*_2(\mathbf{v})}(\mathbf{bc})_{*_1(\mathbf{v})*_2(\mathbf{v})}\prod_{k=1,k\neq *_1(\mathbf{v}),*_2(\mathbf{v})}^n (\mathbf{abc})_{*_1(\mathbf{v})*_2(\mathbf{v})k}}.\end{aligned} \tag{2}$$

Remark 1. *The good clipping conditions change to the following seven conditions ($(A) \Rightarrow (1),(2),(3)$; $(B) \Rightarrow (4),(5),(6),(7)$):*

(1) $F^0 \cap H_1 \cap H_2^+ \cap H_3^+ = \emptyset$,
(2) $F^0 \cap H_1^+ \cap H_2 \cap H_3^+ = \emptyset$,
(3) $F^1 \cap H_1 \cap H_2 \cap H_3^+ = \emptyset$,
(4) $\prod_{k=1}^n c_k \neq 0$,
(5) $\prod_{k=1,k\neq *(\mathbf{v})}^n (\mathbf{ac})_{*(\mathbf{v})k} \neq 0$,
(6) $\prod_{k=1,k\neq *(\mathbf{v})}^n (\mathbf{bc})_{*(\mathbf{v})k} \neq 0$,
(7) $(\mathbf{ac})_{*_1(\mathbf{v})*_2(\mathbf{v})} \neq 0$, $(\mathbf{bc})_{*_1(\mathbf{v})*_2(\mathbf{v})} \neq 0$, $\prod_{k=1,k\neq *_1(\mathbf{v}),*_2(\mathbf{v})}^n (\mathbf{abc})_{*_1(\mathbf{v})*_2(\mathbf{v})k} \neq 0$.

Because the good clipping condition (A) with $m = 3$ implies three cases $F^0 \cap H_1 \cap H_2^+ \cap H_3^+ \setminus H_3 = \emptyset$ or $F^0 \cap H_1^+ \cap H_2 \cap H_3^+ \setminus H_3 = \emptyset$ or $F^1 \cap H_1 \cap H_2 \cap H_3^+ \setminus H_3 = \emptyset$, and above each term in the formula (2) with vertices in the auxiliary hyperplane H_3 gives 0 value from $g_3(\mathbf{v}) = 0$, so we could not consider the case $\mathbf{v} \in H_3$. Hence we get above three cases $(A) \Rightarrow (1),(2),(3)$.

The good clipping condition (B) means that the denominator of each term in the formula (2) is not 0, so $(B) \Rightarrow (4),(5),(6),(7)$ is trivial.

Even though the seven conditions which were originated in the good clipping conditions are complicated, they are all measure zero conditions which can be avoided by applying an appropriate perturbation on the hyperplanes. Therefore we can obtain the volume of the clipped hypercube by applying a small ϵ-perturbation of the hyperplanes and taking $\lim_{\epsilon\to 0}$. However, we should be cautious about the perturbing way; for example, the simpler changes $a_i + i\epsilon$ instead of $a_i + \epsilon^2 + i\epsilon$ or $b_i + \epsilon^i$ instead of $b_i + \epsilon^{2i-1}$ in Theorem 3.2 may not satisfy the seven conditions all together. Our main result Theorem 3.2 gives a concrete method that guarantees safe use of ϵ-perturbation that does not violate any of the seven conditions.

Theorem 3.2. *From the ϵ changes of coefficients of $g_i(\mathbf{x})$ in H_i^+:*

$$\begin{aligned} g_1(\mathbf{x}) \to g_{1,\epsilon}(\mathbf{x}) &= a_{1,\epsilon}x_1 + a_{2,\epsilon}x_2 + \cdots + a_{n,\epsilon}x_n + r_{1,\epsilon} \\ &= (a_1 + \epsilon^2 + \epsilon)x_1 + (a_2 + \epsilon^2 + 2\epsilon)x_2 + \cdots + (a_n + \epsilon^2 + n\epsilon)x_n \\ &\quad + (r_1 + \epsilon), \\ g_2(\mathbf{x}) \to g_{2,\epsilon}(\mathbf{x}) &= b_{1,\epsilon}x_1 + b_{2,\epsilon}x_2 + \cdots + b_{n,\epsilon}x_n + r_{2,\epsilon} \\ &= (b_1 + \epsilon)x_1 + (b_2 + \epsilon^3)x_2 + \cdots + (b_n + \epsilon^{2n-1})x_n + (r_2 + \epsilon^2), \\ g_3(\mathbf{x}) \to g_{3,\epsilon}(\mathbf{x}) &= c_{1,\epsilon}x_1 + c_{2,\epsilon}x_2 + \cdots + c_{n,\epsilon}x_n + r_3, \end{aligned}$$

where $c_{i,\epsilon} = \begin{cases} c_i, & \text{if } c_i \neq 0 \\ \epsilon, & \text{if } c_i = 0 \end{cases}$,

the volume of the unit hypercube $[0,1]^n$ that intersects three halfspaces H_1^+, H_2^+, and H_3^+ is obtained from $\lim_{\epsilon\to 0} \mathrm{vol}\,([0,1]^n \cap H_{1,\epsilon}^+ \cap H_{2,\epsilon}^+ \cap H_{3,\epsilon}^+)$, *where* $H_{i,\epsilon}^+ = \{\mathbf{x} | g_{i,\epsilon}(\mathbf{x}) \geq 0\}$, $H_{i,\epsilon} = \{\mathbf{x} | g_{i,\epsilon}(\mathbf{x}) = 0\}$.

In fact, we need not change all coefficients of $g_i(\mathbf{x})$ to satisfy the seven conditions. More precisely, we can only change coefficients whose indices are related to the seven conditions.

If the hyperplane H_2 does not intersect $[0,1]^n$, then the following corollary holds only with three conditions (1), (4), (5) in Remark 1. Because $H_2 \cap [0,1]^n = \emptyset$ implies (2) $F^0 \cap H_1^+ \cap H_2 \cap H_3^+ = \emptyset$, (3) $F^1 \cap H_1 \cap H_2 \cap H_3^+ = \emptyset$ and $(6)^*$ $F^1 \cap H_1^+ \cap H_2 \cap H_3^+ = \emptyset$, $(7)^*$ $F^2 \cap H_1 \cap H_2 \cap H_3^+ = \emptyset$, and $(6)^*$, $(7)^*$ imply (6), (7) trivially.

If two hyperplanes H_1, H_2 do not intersect $[0,1]^n$, then the formula (2) becomes that of Barrow and Smith [1] with only one condition (4) in Remark 1. Because $H_1 \cap [0,1]^n = \emptyset$ implies (1) $F^0 \cap H_1 \cap H_2^+ \cap H_3^+ = \emptyset$ and $(5)^*$ $F^1 \cap H_1 \cap H_2^+ \cap H_3^+ = \emptyset$, and $(5)^*$ implies (5) trivially.

Corollary 1. *The volume of the standard unit hypercube $[0,1]^n$ intersecting the two halfspaces H_1^+ and H_3^+ under three conditions (see Remark 2) is*

$$\begin{aligned} \mathrm{vol}\,([0,1]^n \cap H_1^+ \cap H_3^+) = &\sum_{\mathbf{v} \in F^0 \cap H_1^+ \cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|} g_3(\mathbf{v})^n}{n! \prod_{k=1}^n c_k} \\ &- \sum_{\mathbf{v} \in F^1 \cap H_1 \cap H_3^+} \frac{(-1)^{|0_\mathbf{v}|}\, sgn(a_{*(\mathbf{v})})\, a_{*(\mathbf{v})}^{n-1}\, g_3(\mathbf{v})^n}{n!\, c_{*(\mathbf{v})} \prod_{k=1, k\neq *(\mathbf{v})}^n (\mathbf{ac})_{*(\mathbf{v})k}}. \end{aligned} \tag{3}$$

Remark 2. *The two halfspaces in the above general position should satisfy the following three conditions:*
(1) $F^0 \cap H_1 \cap H_3^+ = \emptyset$,
(2) $\prod_{k=1}^n c_k \neq 0$,
(3) $\prod_{k=1, k\neq *(\mathbf{v})}^n (\mathbf{ac})_{*(\mathbf{v})k} \neq 0$.

Corollary 2. *(Barrow and Smith) The volume of the standard unit hypercube* $[0,1]^n$ *intersecting one halfspace* H_3^+ *under the condition* $\prod_{k=1}^n c_k \neq 0$ *is*

$$\text{vol}\,([0,1]^n \cap H_3^+) = \sum_{\mathbf{v}\in F^0\cap H_3^+} \frac{(-1)^{|0_{\mathbf{v}}|} g_3(\mathbf{v})^n}{n! \prod_{k=1}^n c_k} \tag{4}$$

4. ϵ-perturbation of Hyperplanes

Now we explain an ϵ-perturbation ($\epsilon > 0$) of hyperplanes. If the hyperplanes do not satisfy all of the conditions, we change each hyperplane H_i to $H_{i,\epsilon}$ and take a limit $\lim_{\epsilon\to 0} \text{vol}\,([0,1]^n \cap H_{1,\epsilon}^+ \cap H_{2,\epsilon}^+ \cap H_{3,\epsilon}^+)$, then we can get the volume $\text{vol}\,([0,1]^n \cap H_1^+ \cap H_2^+ \cap H_3^+)$.

Now we prove Theorem 3.2.

Proof of Theorem 3.2. If the first condition is violated, then we change $r_1 \to r_{1,\epsilon} = r_1 + \epsilon$, and if the second condition is violated, then we change $r_2 \to r_{2,\epsilon} = r_2 + \epsilon^2$, and if the third condition is violated, then we change $r_1 \to r_1 + \epsilon, r_2 \to r_2 + \epsilon^2$. Geometrically, the condition $F^0 \cap H_1 = \emptyset$ (first condition is exactly $F^0 \cap H_1 \cap H_2^+ \cap H_3^+ = \emptyset$) means that all vertices of hypercube do not touch the hyperplane H_1. So if some vertex of hypercube meets hyperplane H_1, then the small parallel movement of H_1 implies the property $F^0 \cap H_1 = \emptyset$ trivially, and the second case similarly. For the third condition $F^1 \cap H_1 \cap H_2 \cap H_3^+ = \emptyset$, the condition $F^1 \cap H_1 \cap H_2 \neq \emptyset$ means that there exists some value x_i ($0 < x_i < 1$) such that $a_i x_i + M_1 + r_1 = 0$ and $b_i x_i + M_2 + r_2 = 0$ are satisfied, where $M_1 = a_1\delta_1 + \cdots + a_{i-1}\delta_{i-1} + a_{i+1}\delta_{i+1} + \cdots + a_n\delta_n$ and $M_2 = b_1\delta_1 + \cdots + b_{i-1}\delta_{i-1} + b_{i+1}\delta_{i+1} + \cdots + b_n\delta_n$ ($\delta_k = 0$ or 1 for all $k = 1, \ldots, n$). So we get $x_i = -\frac{M_1+r_1}{a_i} = -\frac{M_2+r_2}{b_i}$, however the new relation $-\frac{M_1+r_1+\epsilon}{a_i} = -\frac{M_2+r_2+\epsilon^2}{b_i}$ is impossible for all sufficiently small ϵ, i.e., $a_i\epsilon^2 - b_i\epsilon \neq 0$ for any $\epsilon \in (0, \epsilon_0)$ with sufficiently small positive ϵ_0.

If the fourth condition is violated, then we change $c_i = 0 \to c_{i,\epsilon} = \epsilon$ for all zero coefficient c_i's. This changing trivially satisfies the condition 4).

If the fifth condition is violated, then we change

$$(a_1, a_2, \ldots, a_n) \to (a_{1,\epsilon}, a_{2,\epsilon}, \ldots, a_{n,\epsilon}) = (a_1+\epsilon^2+\epsilon, a_2+\epsilon^2+2\epsilon, \ldots, a_n+\epsilon^2+n\epsilon).$$

In fact, we need not change all a_i, specifically we only need to change the smallest index set $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ such that $(\mathbf{ac})_{ij} = 0$ for any distinct indices $i, j \in \{i_1, i_2, \ldots, i_k\}$ and $(\mathbf{ac})_{ji_1}, \ldots, (\mathbf{ac})_{ji_k} \neq 0$ for arbitrary index $j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$. This change shows that $(\mathbf{ac})_{ij} = 0$ implies $(\mathbf{a}_\epsilon \mathbf{c}_\epsilon)_{ij} \neq 0$ for $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is a sufficiently small positive number. Three kind of $(\mathbf{ac})_{ij}$ must be considered: $\begin{vmatrix} a_i & c_i \\ a_j & c_j \end{vmatrix}$, $\begin{vmatrix} a_i & 0 \\ a_j & c_j \end{vmatrix}$, and $\begin{vmatrix} a_i & 0 \\ a_j & 0 \end{vmatrix}$ (the case $\begin{vmatrix} a_i & c_i \\ a_j & 0 \end{vmatrix}$ is almost same as the second one, and all $c_i, c_j \neq 0$).

The first case of $\begin{vmatrix} a_i & c_i \\ a_j & c_j \end{vmatrix} = 0$ implies

$$\begin{vmatrix} a_{i,\epsilon} & c_{i,\epsilon} \\ a_{j,\epsilon} & c_{j,\epsilon} \end{vmatrix} = \begin{vmatrix} a_i + \epsilon^2 + n_1\epsilon & c_i \\ a_j + \epsilon^2 + n_2\epsilon & c_j \end{vmatrix} = (c_j - c_i)\epsilon^2 + (n_1 c_j - n_2 c_i)\epsilon \neq 0$$

for $\epsilon \in (0, \epsilon_0)$ and two different positive integers n_1, n_2.

The second case of $\begin{vmatrix} a_i & 0 \\ a_j & c_j \end{vmatrix} = 0$ implies $a_i = 0$ and $\begin{vmatrix} a_{i,\epsilon} & c_{i,\epsilon} \\ a_{j,\epsilon} & c_{j,\epsilon} \end{vmatrix}$ $= \begin{vmatrix} \epsilon^2 + n_1\epsilon & \epsilon \\ a_j + \epsilon^2 + n_2\epsilon & c_j \end{vmatrix} = -\epsilon^3 + (c_j - n_2)\epsilon^2 + (n_1 c_j - a_j)\epsilon \neq 0$ for $\epsilon \in (0, \epsilon_0)$ and two different positive integers n_1, n_2.

The third case of $\begin{vmatrix} a_i & 0 \\ a_j & 0 \end{vmatrix} = 0$ implies $\begin{vmatrix} a_i + \epsilon^2 + n_1\epsilon & \epsilon \\ a_j + \epsilon^2 + n_2\epsilon & \epsilon \end{vmatrix} = (n_1 - n_2)\epsilon^2 + (a_i - a_j)\epsilon \neq 0$ for $\epsilon \in (0, \epsilon_0)$ and two different positive integers n_1, n_2.

If the sixth condition is violated in the fifth condition violation, then we change

$$(b_1, b_2, \ldots, b_n) \to (b_{1,\epsilon}, b_{2,\epsilon}, \ldots, b_{n,\epsilon}) = (b_1 + \epsilon, b_2 + \epsilon^3, \ldots, b_n + \epsilon^{2n-1})$$

using the same method of choosing the smallest index set as the fifth case. Also the property $\begin{vmatrix} b_{i,\epsilon} & c_{i,\epsilon} \\ b_{j,\epsilon} & c_{j,\epsilon} \end{vmatrix} \neq 0$ is easily obtained in a similar way to those used in the calculations of the three cases in the violation of the fifth condition. If we use the changing rule $(b_1+\epsilon, b_2+\epsilon^2, \ldots, b_n+\epsilon^n)$ instead of $(b_1+\epsilon, b_2+\epsilon^3, \ldots, b_n+\epsilon^{2n-1})$, then we can find an unavoidable example $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \to \begin{vmatrix} \epsilon & 1 \\ \epsilon^2 & \epsilon \end{vmatrix} \equiv 0$.

If the seventh condition is violated, then we change $(\mathbf{abc})_{ijk} \to (\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$ for the corresponding indices i, j, k, where the $\mathbf{a}_\epsilon, \mathbf{b}_\epsilon, \mathbf{c}_\epsilon$ are already defined in the above conditions, i.e., $a_{m,\epsilon} = \epsilon^2 + m\epsilon + a_m, b_{m,\epsilon} = \epsilon^{2m-1} + b_m, c_{m,\epsilon} = \epsilon$ or $c_m (\neq 0)$ for any index $m \in \{i, j, k\}$ $(i < j < k)$. There are eight cases which we have to consider depending on $c_i, c_j, c_k = 0$ or not.

In the first $(c_{i,\epsilon}, c_{j,\epsilon}, c_{k,\epsilon}) = (\epsilon, \epsilon, \epsilon)$ and second $(c_{i,\epsilon}, c_{j,\epsilon}, c_{k,\epsilon}) = (\epsilon, \epsilon, c_k)$ cases, we know that the highest degree term of $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$ (as a polynomial of ϵ) becomes $(j - i)\epsilon^{2k+1}$. Hence this implies $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk} \neq 0$ for $\epsilon \in (0, \epsilon_0)$.

In the third $(c_{i,\epsilon}, c_{j,\epsilon}, c_{k,\epsilon}) = (\epsilon, c_j, \epsilon)$ and fourth $(c_{i,\epsilon}, c_{j,\epsilon}, c_{k,\epsilon}) = (\epsilon, c_j, c_k)$ cases, we know that the highest degree term of $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$ becomes ϵ^{2k+2}. Hence this implies $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk} \neq 0$ for $\epsilon \in (0, \epsilon_0)$.

In the fifth $(c_i, \epsilon, \epsilon)$ and sixth (c_i, ϵ, c_k) cases, we know that the highest degree term of $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$ becomes $-\epsilon^{2k+2}$. Hence this implies $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk} \neq 0$ for $\epsilon \in (0, \epsilon_0)$.

In the seventh (c_i, c_j, ϵ) and eighth (c_i, c_j, c_k) cases, we know that the highest and second large degree term of $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$ becomes $(c_i - c_j)\epsilon^{2k+1} + (jc_i - ic_j)\epsilon^{2k}$ (in the seventh case with $k \neq j+1$ and eighth case) or $(c_i - c_j)\epsilon^{2k+1} + (jc_i - ic_j + 1)\epsilon^{2k}$ (in the seventh case with $k = j+1$). Hence this implies $(\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk} \neq 0$ for $\epsilon \in (0, \epsilon_0)$. ■

From the above proof, we can summarize the following changing rules for each violation of the seven conditions:

Violation of restriction 1) then $r_1 \to r_{1,\epsilon} = r_1 + \epsilon$,

Violation of restriction 2) then $r_2 \to r_{2,\epsilon} = r_2 + \epsilon^2$,

Violation of restriction 3) then $r_1 \to r_{1,\epsilon} = r_1 + \epsilon$ and $r_2 \to r_{2,\epsilon} = r_2 + \epsilon^2$,

Violation of restriction 4) then $c_i \to c_{i,\epsilon} = \begin{cases} c_i, & \text{if } c_i \neq 0 \\ \epsilon, & \text{if } c_i = 0 \end{cases}$

Violation of restriction 5) then $(\mathbf{ac})_{ij} \to (\mathbf{a}_\epsilon \mathbf{c}_\epsilon)_{ij}$, $(a_1, a_2, \ldots, a_n) \to (a_1 + \epsilon^2 + \epsilon, a_2 + \epsilon^2 + 2\epsilon, \ldots, a_n + \epsilon^2 + n\epsilon)$ with the same $c_{i,\epsilon}$ as in the violation of 4)

Violation of restriction 6) then $(\mathbf{bc})_{ij} \to (\mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ij}$, $(b_1, b_2, \ldots, b_n) \to (b_1 + \epsilon, b_2 + \epsilon^3, \ldots, b_n + \epsilon^{2n-1})$ with the same $c_{i,\epsilon}$ as in the violation of 4)

Violation of restriction 7) then $(\mathbf{abc})_{ijk} \to (\mathbf{a}_\epsilon \mathbf{b}_\epsilon \mathbf{c}_\epsilon)_{ijk}$, with the same $a_{i,\epsilon}, b_{i,\epsilon}, c_{i,\epsilon}$ in the violation 4), 5), and 6).

As an additional remark, we don't need to change all values of a_i, b_i for the violation of 5) or 6); we only need to change the coefficients a_i, b_i at the maximal indices set coming from the $\{ij\}, \{ijk\}$ in the violation of 5), 6), and 7).

Remark 3. *The case clipped by two half spaces encounters three conditions. We can avoid each condition by changing : 1)* $r_1 \to r_1 + \epsilon$*; 2)* $c_i = 0 \to \epsilon$*; 3)* $(a_1, a_2, \ldots, a_n) \to (a_1 + \epsilon^2 + \epsilon, a_2 + \epsilon^2 + 2\epsilon, \ldots, a_n + \epsilon^2 + n\epsilon)$ *or* $\to (a_1 + \epsilon, a_2 + \epsilon^3, \ldots, a_n + \epsilon^{2n-1})$*. For the case clipped by one halfspace, only one change rule 2) is necessary.*

5. Two ϵ-perturbation examples

Now we show two examples of the ϵ-perturbation Method.

Example 1. Let us calculate the volume of the region of $[0,1]^3$ that intersects three halfspaces $H_1^+ = \{\mathbf{x} | -x_1 + x_2 \geq 0\}$, $H_2^+ = \{\mathbf{x} | -2x_1 + x_2 \geq 0\}$, and $H_3^+ = \{\mathbf{x} | -x_1 + 2x_2 \geq 0\}$.

These three halfspaces do not satisfy the seven conditions, so we must apply the ϵ-perturbation method. Restrictions 1), 2), 3), 4) are not satisfied. Hence we have to change $g_i(\mathbf{x}) \to g_{i,\epsilon}(\mathbf{x})$ while only changing $r_1 = 0 \to \epsilon$, $r_2 = 0 \to \epsilon^2$, and $c_3 = 0 \to \epsilon$, i.e., $H^+_{1,\epsilon} = \{\mathbf{x}|\ -x_1+x_2+\epsilon \geq 0\}$, $H^+_{2,\epsilon} = \{\mathbf{x}|\ -2x_1+x_2+\epsilon^2 \geq 0\}$, and $H^+_{3,\epsilon} = \{\mathbf{x}|\ -x_1+2x_2+\epsilon x_3 \geq 0\}$.

There are four vertices satisfying $\mathbf{v}_\epsilon \in F^0 \cap H^+_{1,\epsilon} \cap H^+_{2,\epsilon} \cap H^+_{3,\epsilon}$: those are $\mathbf{v}_{1,\epsilon} = (0,0,0), \mathbf{v}_{2,\epsilon} = (0,0,1), \mathbf{v}_{3,\epsilon} = (0,1,0), \mathbf{v}_{4,\epsilon} = (0,1,1)$, and no vertices satisfy $\mathbf{v}_\epsilon \in F^1 \cap H_{1,\epsilon} \cap H^+_{2,\epsilon} \cap H^+_{3,\epsilon}$, and there are three vertices satisfying $\mathbf{v}_\epsilon \in F^1 \cap H^+_{1,\epsilon} \cap H_{2,\epsilon} \cap H^+_{3,\epsilon}$: those are $\mathbf{v}_{5,\epsilon} = (\frac{1+\epsilon^2}{2},1,0), \mathbf{v}_{6,\epsilon} = (\frac{1+\epsilon^2}{2},1,1), \mathbf{v}_{7,\epsilon} = (\frac{\epsilon^2}{2},0,1)$, and no vertices satisfy $\mathbf{v}_\epsilon \in F^2 \cap H_{1,\epsilon} \cap H_{2,\epsilon} \cap H^+_{3,\epsilon}$.

Then $\mathbf{v}_{1,\epsilon}$ lays on H_3, so $N_{\mathbf{v}_{1,\epsilon}} = 0$, where $N_\mathbf{v}$ denotes each sigma term given by a vertex $\mathbf{v}$ in the formula (2). Therefore we get the final clipped volume $\frac{1}{4}$ from the following calculation:

$$\begin{aligned}
&\mathrm{vol}\,([0,1]^3 \cap H^+_{1,\epsilon} \cap H^+_{2,\epsilon} \cap H^+_{3,\epsilon})\\
=&\sum_{\mathbf{v}\in\{\mathbf{v}_{2,\epsilon},\mathbf{v}_{3,\epsilon},\mathbf{v}_{4,\epsilon}\}} N_\mathbf{v} + \sum_{\mathbf{v}\in\emptyset} N_\mathbf{v} + \sum_{\mathbf{v}\in\{\mathbf{v}_{5,\epsilon},\mathbf{v}_{6,\epsilon},\mathbf{v}_{7,\epsilon}\}} N_\mathbf{v} + \sum_{\mathbf{v}\in\emptyset} N_\mathbf{v},\\
=&-\frac{\epsilon^2}{12} - \frac{2}{3\epsilon} + \frac{(2+\epsilon)^3}{12\epsilon} + \frac{(3-\epsilon^2)^3}{72\epsilon} - \frac{(3+2\epsilon-\epsilon^2)^3}{72\epsilon} + \frac{\epsilon^2(2-\epsilon)^3}{72},\\
=&\frac{1}{4} + \frac{\epsilon^2}{2} - \frac{\epsilon^5}{72} \to \frac{1}{4}\ (\text{when } \epsilon \to 0).
\end{aligned}$$

In fact, conditions 5), 6), 7) are also violated. However the small change of $H_{3,\epsilon}$ implies the satisfaction of the remaining conditions, i.e., we can use $(\mathbf{a}\mathbf{c}_\epsilon)_{ij}, (\mathbf{b}\mathbf{c}_\epsilon)_{ij}, (\mathbf{a}\mathbf{b}\mathbf{c}_\epsilon)_{123}$ instead of $(\mathbf{a}_\epsilon\mathbf{c}_\epsilon)_{ij}, (\mathbf{b}_\epsilon\mathbf{c}_\epsilon)_{ij}, (\mathbf{a}_\epsilon\mathbf{b}_\epsilon\mathbf{c}_\epsilon)_{123}$ for easy calculation in this example.

Note that the limit point of $\mathbf{v}_{2,\epsilon}, \mathbf{v}_{7,\epsilon}$ lays on H_3, so we can naturally predict that $\lim_{\epsilon\to 0} N_{\mathbf{v}_{2,\epsilon}}, N_{\mathbf{v}_{7,\epsilon}} = 0$. In general, whether the good clipping conditions are satisfied or not, if the limit point of $\mathbf{v}_\epsilon$ lays on the last auxiliary hyperplane, we can see that $\lim_{\epsilon\to 0} N_{\mathbf{v}_\epsilon} = 0$ is true by Section 3.3 in [2].

Remark 4. *If we choose three halfspaces in* $[0,1]^3$ *as* $H^+_1 = \{\mathbf{x}|\ 3x_1+4x_2+5x_3-1 \geq 0\}$, $H^+_2 = \{\mathbf{x}|\ -6x_1-8x_2-10x_3+7 \geq 0\}$, *and* $H^+_3 = \{\mathbf{x}|\ x_3 - \frac{1}{5} \geq 0\}$, *then condition 7) is violated, i.e.,*

$$(\mathbf{abc})_{123} = \begin{vmatrix} 3 & -6 & 0 \\ 4 & -8 & 0 \\ 5 & -10 & 1 \end{vmatrix} = 0.$$

Additionally, we know

$$(\mathbf{abc}_\epsilon)_{123} = \begin{vmatrix} 3 & -6 & \epsilon \\ 4 & -8 & \epsilon \\ 5 & -10 & 1 \end{vmatrix} \equiv 0, \ \textit{and} \ \begin{vmatrix} \epsilon^2+\epsilon+3 & \epsilon-6 & \epsilon \\ \epsilon^2+2\epsilon+4 & \epsilon^2-8 & \epsilon \\ \epsilon^2+3\epsilon+5 & \epsilon^3-10 & 1 \end{vmatrix} \equiv 0.$$

Hence this case needs the perturbation rule of Theorem 3.2:

$$(\mathbf{a}_\epsilon\mathbf{b}_\epsilon\mathbf{c}_\epsilon)_{123} = \begin{vmatrix} \epsilon^2+\epsilon+3 & \epsilon-6 & \epsilon \\ \epsilon^2+2\epsilon+4 & \epsilon^3-8 & \epsilon \\ \epsilon^2+3\epsilon+5 & \epsilon^5-10 & 1 \end{vmatrix} \neq 0.$$

Example 2. Let us calculate the volume of the region of $[0,1]^3$ that intersects three halfspaces $H_1^+ = \{\mathbf{x} |\ 3x_1 + 4x_2 + 5x_3 - 1 \geq 0\}$, $H_2^+ = \{\mathbf{x} |\ -6x_1 - 8x_2 - 10x_3 + 7 \geq 0\}$, and $H_3^+ = \{\mathbf{x} |\ x_3 - \frac{1}{5} \geq 0\}$.

These three halfspaces violate conditions 4), 5), 6), and 7). More exactly $c_i = 0$ for $i = 1, 2$, and $(\mathbf{ac})_{ij}, (\mathbf{bc})_{ij} = 0$ for $i = 1, j = 2$, and $(\mathbf{abc})_{123} = 0$. Hence we must change $g_i(\mathbf{x}) \to g_{i,\epsilon}(\mathbf{x})$ with changing of $g_1(\mathbf{x}) \to g_{1,\epsilon}(\mathbf{x}) = (\epsilon^2 + \epsilon + 3)x_1 + (\epsilon^2 + 2\epsilon + 4)x_2 + (\epsilon^2 + 3\epsilon + 5)x_3 - 1$, $g_2(\mathbf{x}) \to g_{2,\epsilon}(\mathbf{x}) = (\epsilon - 6)x_1 + (\epsilon^3 - 8)x_2 + (\epsilon^5 - 10)x_3 + 7$, and $g_3(\mathbf{x}) \to g_{3,\epsilon}(\mathbf{x}) = \epsilon x_1 + \epsilon x_2 + x_3 - \frac{1}{5}$.

No vertices satisfy $\mathbf{v}_\epsilon \in F^0 \cap H_{1,\epsilon}^+ \cap H_{2,\epsilon}^+ \cap H_{3,\epsilon}^+$, and no vertices satisfy $\mathbf{v}_\epsilon \in F^1 \cap H_{1,\epsilon} \cap H_{2,\epsilon}^+ \cap H_{3,\epsilon}^+$, and one vertex satisfies $\mathbf{v}_\epsilon \in F^1 \cap H_{1,\epsilon}^+ \cap H_{2,\epsilon} \cap H_{3,\epsilon}^+$, that is $\mathbf{v}_{1,\epsilon} = (0, 0, \frac{7}{10-\epsilon^5})$, and no vertices satisfy $\mathbf{v}_\epsilon \in F^2 \cap H_{1,\epsilon} \cap H_{2,\epsilon} \cap H_{3,\epsilon}^+$.

Therefore we get the final clipped volume $\frac{25}{576}$ from the following calculation:

$$
\begin{aligned}
&\text{vol } ([0,1]^3 \cap H_{1,\epsilon}^+ \cap H_{2,\epsilon}^+ \cap H_{3,\epsilon}^+) \\
=&\sum_{\mathbf{v} \in \emptyset} N_{\mathbf{v}} + \sum_{\mathbf{v} \in \emptyset} N_{\mathbf{v}} + \sum_{\mathbf{v} = \mathbf{v}_{1,\epsilon}} N_{\mathbf{v}} + \sum_{\mathbf{v} \in \emptyset} N_{\mathbf{v}}, \\
=&\frac{(25 + \epsilon^5)^3}{750(10 - \epsilon^5)(6 - 11\epsilon + \epsilon^6)(8 - 10\epsilon - \epsilon^3 + \epsilon^6)} \to \frac{25}{576}.
\end{aligned}
$$

We have considered three dimensional cubes in these examples. However the volume formula applies to any dimension. Our ϵ-perturbation method is just one way of overcoming the conditions. We leave the problems open to find other efficient ways to overcome these conditions with three or more half spaces.

References

[1] D. L. Barrow and P. W. Smith, *Spline Notation Applied to a Volume Problem*, The American Mathematical Monthly, vol. **86(1)** (1979), 50-51.

[2] Yunhi Cho and Seonhwa Kim, *Volume of Hypercubes Clipped by Hyperplanes and Combinatorial Identities*, Electronic Journal of Linear Algebra, vol. **36** (2020), 228-255.

[3] Jim Lawrence, *Polytope volume computation*, Mathematics of Computation, vol. **57(195)** (1991), 259-271.

[4] A.D. Logvinenko, B. Funt and C. Godau, Metamer Mismatching, IEEE Transactions on Image Processing, vol. **23(1)** (2014), 34-43.

[5] A.D. Logvinenko and E. Demidenko, *On Counting Metamers*, IEEE Transactions on Image Processing, vol. **25(2)** (2016), 770-775.

[6] K. P. Murphy, *Machine Learning: A Probabilistic Perspective*, MIT Press, (2012).

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