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RADIUS ESTIMATES OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Numerical techniques are used to determine the radius of convexity of the starlike functions related to cardioid shaped bounded domain. In addition, radius constants of certain starlikeness associated with right half plane of various starlike functions are computed.

1. Introduction

We first recall the basic definitions in univalent function theory. Let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ denote an open disk. In particular, $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of all analytic functions f in $\mathbb{D}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ containing univalent functions. The classes $\mathcal{S}^*$ and $\mathcal{K}$ have following analytical description:

$$\mathcal{S}^* := \left\{ f \in \mathcal{S} : \Re \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D} \right\},$$

and

$$\mathcal{K} := \left\{ f \in \mathcal{S} : 1 + \Re \frac{zf''(z)}{f'(z)} > 0,\ z \in \mathbb{D} \right\}.$$

These classes contain starlike and convex functions, respectively. Let P be a property and $\mathcal{M}$ be a set of functions. Then, the real number $R_P(\mathcal{M}) = \sup\{r > 0 :\ f \text{ has the property } P \text{ in the disk } \mathbb{D}_r, \forall f \in \mathcal{M}\}$ is radius of property for the set $\mathcal{M}$. If there exists $F_0 \in \mathcal{M}$ such that F_0 has the property P in $\mathbb{D}_{R_P}$, then sharpness follows for the function F_0 (see [6]). Therefore, the radius estimates related to convexity and starlikeness associated with the class $\mathcal{A}$ are given by $R_{\mathcal{K}}(\mathcal{A}) = \sup\{r > 0 :\ f(\mathbb{D}_r) \text{ is convex domain } \forall\ f \in \mathcal{A},\ z \in \mathbb{D}\}$ and $R_{\mathcal{S}^*}(\mathcal{A}) = \sup\{r > 0 :\ f(\mathbb{D}_r) \text{ is starlike domain } \forall\ f \in \mathcal{A},\ z \in \mathbb{D}\}$, respectively. For the class $\mathcal{S}$, the best possible radius of convexity $R_{\mathcal{K}}(\mathcal{S})$ is $2-\sqrt{3}$ due to Nevanlinna [17] and radius of starlikeness $\mathcal{R}_{\mathcal{S}^*}(\mathcal{S}) = \tanh \frac{\pi}{4} \approx 0.65579$ due to Grunsky [8] which are the oldest results in the univalent function theory. In 2009, Sokół [21] determined certain radii results for a class of lemniscate

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Bernoulli functions. Prajapati *et al.* [18] discussed starlikeness, convexity, close-to-convexity of order α for Mittag-Leffler functions. Recently, Bohra and Ravichandran [2] computed some radius constant for certain special functions. Over the years, several authors examined radius estimates for several subclasses of analytic functions. For more details, we refer to [1, 11].

For the functions f, $g \in \mathcal{A}$, the function f is subordinate to the function g, symbolized by $f \prec g$, if there exists a function w satisfying the condition $w(0) = 0$ such that $f = g \circ w$. If the function g is univalent, for subordination of f and g, the equivalent condition is given as: $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Using subordination, Ma and Minda [14] studied the growth, covering and distortion theorem of the unified class $\mathcal{S}^*(\varphi)$ consisting the functions f satisfying $zf'(z)/f(z) \prec \varphi(z)$, $z \in \mathbb{D}$ where φ is the analytic function satisfy inequality $\Re(\varphi(z)) > 0$. This class contains various subclasses of starlike functions. For instance, the class $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ with $-1 \leq B < A \leq 1$ consisting of Janowski starlike functions [9]. The class $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ is initially introduced and studied by Sokół and Stankiewicz [22], and it is a collection of functions $f \in \mathcal{A}$ associated with a special type of Cassinian Curve lemniscate of Bernoulli. In 2015, Mendiratta *et al.* [15, 16] introduced the classes $\mathcal{S}^*_{RL} := \mathcal{S}^*(\varphi_{RL})$, where $\varphi_{RL}(z) = \sqrt{2} - (\sqrt{2} - 1)((1 - z)/(1 + (2\sqrt{2} - 2)z))^{1/2}$ and $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$, and determined the $\mathcal{S}^*_{RL}$ and $\mathcal{S}^*_e$-radii for certain classes. In [12, 20], authors discussed the geometric properties of the classes $\mathcal{S}^*_R := \mathcal{S}^*(\varphi_0(z) = (k^2 + z^2)/k(k - z))$, $k = \sqrt{2} + 1$ and $\mathcal{S}^*_C := \mathcal{S}^*(\varphi_c(z) = (3 + 4z + 2z^2)/3)$, respectively. Note that the functions φ_0 and φ_c map the unit disk $\mathbb{D}$ into cardiod shaped bounded regions in the right half plane which is a type of cycloidal curves. Raina and Sokół [19] discussed the coefficient estimates of the class $\mathcal{S}^*_q := \mathcal{S}^*(\varphi_q(z) = z + \sqrt{1+z^2})$. The class $\mathcal{BS}^*(\alpha) := \mathcal{S}^*(G_\alpha(z) = 1 + z/(1 - \alpha z^2))$, where $\alpha \in [0, 1)$ is related to the Booth lemniscate which is a special type of the Persian curve, introduced by Kargar *et al.* [10]. In 2019, Goel and Kumar [7] obtained several radius estimates, coefficient bounds, structural formula, growth theorem, distortion theorem and inclusion relations for the class $\mathcal{SG}^* := \mathcal{S}^*(G(z))$, where $G(z) = 2/(1 + e^{-z})$ is a modified sigmoid function. Recently, Cho *et al.* [5] introduced $\mathcal{S}^*_s := \mathcal{S}^*(\varphi_s(z) = 1 + \sin z)$ and discussed the sin-starlikeness for the class of several starlike functions.

In view of above discussed work, this manuscript studies the radius of convexity for the functions belonging to the class $\mathcal{S}^*_R$ by using numerical techniques like bisection, secant method etc. Next result provides $\mathcal{S}^*_B$-radius constants for the functions belonging to classes $\mathcal{S}^*_L$, $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{S}^*_{RL}$, $\mathcal{BS}^*(\alpha)$, $\mathcal{S}^*_C$, $\mathcal{S}^*_R$ and $\mathcal{S}^*_q$. Further, the $\mathcal{S}^*_B$, $\mathcal{S}^*_R$, $\mathcal{S}^*_L$, $\mathcal{S}^*_{RL}$, $\mathcal{BS}^*(\alpha)$, $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{SG}^*$-starlikeness for the class of functions associated with Nephroid are computed. Last result yields $\mathcal{BS}^*(\alpha)$-starlikeness for the classes $\mathcal{S}^*_C$, $\mathcal{S}^*_R$, $\mathcal{S}^*_L$ and $\mathcal{S}^*_q$.

2. Main Results

In the first result, the radius of convexity of the class $\mathcal{S}_R^*$ has been discussed.

Theorem 2.1. *Let $k = \sqrt{2}+1$ and the function $f \in \mathcal{S}_R^*$. Then, the function f maps the disk $|z| < r_R \approx 0.966372$ onto convex region in the right half plane, where r_R is the unique real root of the polynomial*

$$k^3 + (1-k)kr - \left(k^3 + k + 3\right) r^2 + \left(k^2 + 1\right) r^3 + kr^4 - r^5. \tag{1}$$

Proof. We have $f \in \mathcal{S}_R^*$, then there exists a Schwarz function w satisfying $w(0) = 0$, $|w(z)| \leq |z| = r$ such that

$$\frac{zf'(z)}{f(z)} = \frac{k^2 + (w(z))^2}{k(k - w(z))}. \tag{2}$$

On differentiation of equation (2), we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{k^2 + w^2(z)}{k^2 - kw(z)} + \left(\frac{2w(z)}{k^2 + w^2(z)} + \frac{1}{k - w(z)}\right) zw'(z).$$

Therefore,

$$1 + \Re\frac{zf''(z)}{f'(z)} \geq \Re\left(\frac{k^2 + w^2(z)}{k^2 - kw(z)}\right) - \left|\frac{2w(z)}{k^2 + w^2(z)} + \frac{1}{k - w(z)}\right| \times |zw'(z)|.$$

Using following well-known inequality related to the Schwarz function,

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2},$$

we have

$$\begin{aligned} 1 + \Re\frac{zf''(z)}{f'(z)} &\geq \frac{k^2 - r^2}{k^2 + kr} - \left(\frac{2r}{k^2 - r^2} - \frac{1}{k - r}\right) \frac{r(1 - |w(z)|^2)}{1 - |z|^2} \\ &\geq \frac{k - r}{k} - \frac{2r^2}{(k^2 - r^2)(1 - r^2)} - \frac{r}{(k - r)(1 - r^2)} \\ &\geq \frac{k - r}{k} - \frac{3r^2 - rk}{(k^2 - r^2)(1 - r^2)} \\ &= \frac{k^3 + (1-k)kr - \left(k^3 + k + 3\right) r^2 + \left(k^2 + 1\right) r^3 + kr^4 - r^5}{k(k^2 - r^2)(1 - r^2)} \\ &\geq 0 \end{aligned}$$

for $k^3 + (1-k)kr - \left(k^3 + k + 3\right) r^2 + \left(k^2 + 1\right) r^3 + kr^4 - r^5 > 0$ with $r < 1$. Next, we determine the smallest real root of the polynomial $k^3 + (1-k)kr - \left(k^3 + k + 3\right) r^2 + \left(k^2 + 1\right) r^3 + kr^4 - r^5$ using some well known numerical techniques. To calculate the approximate value of the smallest real root of the polynomial (1), we define $H(x) = k^3 + (1-k)kx - \left(k^3 + k + 3\right) x^2 + \left(k^2 + 1\right) x^3 + kx^4 - x^5 > 0$ and apply Bisection method, Regula Falsi method, Newton Raphson method and Secant method. We take a and b as the initial approximations and tolerance defined by $|a - b|$ is taken as 10^{-10}. The following table gives the approximate value of the polynomial $H(x)$ at the last iteration, number of iterations required to reach desired accuracy, value of initial approximations and the value of the desired root obtained by the considered methods:

S. No.	Numerical method	root(x)	$H(x)$	Initial approximations	No of iterations
1	Bisection	0.9664	-1.125e-11	$a = 0, b = 1$	34
2	Secant	0.9664	3.079e-11	$a = 0, b = 1$	3
3	Regula falsi	0.9664	-5.427e-12	$a = 0, b = 1$	4
4	Newton Raphson	0.9664	5.329e-15	$a = 1$	4

Table 1. **Calculation of the positive root of the polynomial $H(x)$ using various root finding methods**

It can be seen from the above table that the secant method works best for the given polynomial $H(x)$ as it requires least number of approximations to reach the desired accuracy. Performance of the considered numerical schemes is also demonstrated in figure 1 by plotting the value of function $H(x)$ for various schemes as well as the continuous function.

Let n be a fixed positive integer. The Bell numbers B_n satisfy a relation $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. In [13], authors considered the analytic function

$$Q(z) := e^{e^z - 1} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \ (z \in \mathbb{D})$$

whose coefficients are Bell numbers. The function Q is starlike with respect to 1 and satisfy subordination relation $zf'(z)/f(z) \prec Q(z)$ for $z \in \mathbb{D}$. The class of such functions is denoted by $\mathcal{S}^*_B$. Further, Cho *et.al.* [4, Lemma 3.1, p. 11] obtained the following result related to the class $\mathcal{S}^*_B$: □

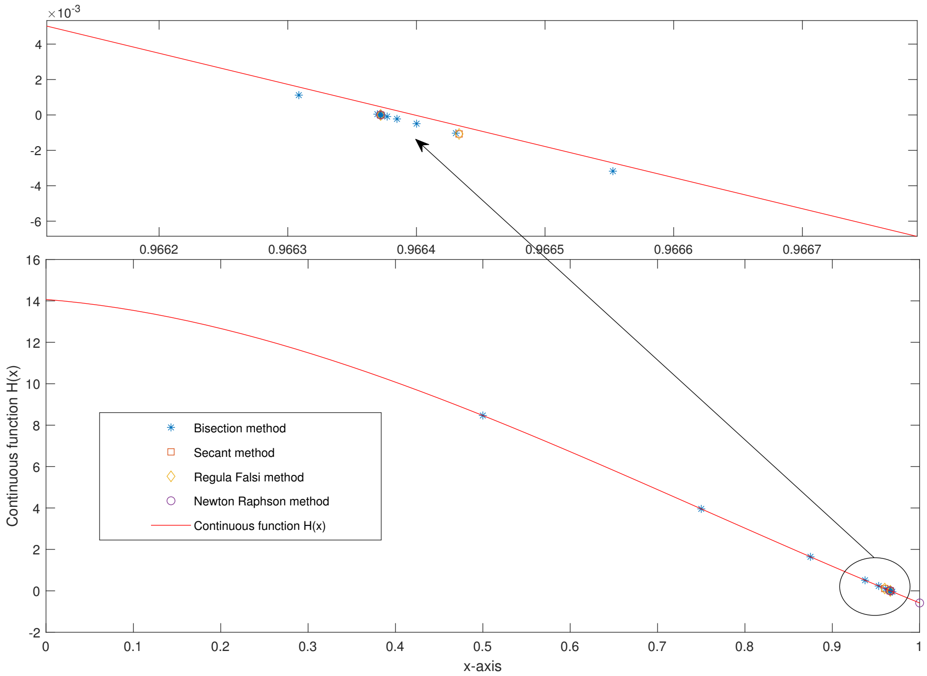

FIGURE 1. **Plot of the continuous function $H(x)$ and its value for various numerical methods**

Lemma 2.2. *Let $Q(z) := e^{e^z-1}$, $z \in \mathbb{D}$ and $\rho : [e^{1/e-1}, e^{e-1}] \to \mathbb{R}^+$ be the function*

$$\rho(a) := \begin{cases} \frac{ea-e^{1/e}}{e}, & e^{\frac{1}{e}-1} \leq a \leq \frac{e^{1/e}+e^e}{2e}; \\ \frac{e^e-ea}{e}, & \frac{e^{1/e}+e^e}{2e} \leq a \leq e^{e-1}. \end{cases}$$

Then, for $w \in \mathbb{C}$, the following inclusions hold:

$$|w-a| < \rho(a) \subset Q(\mathbb{D}) \subset |w-1| < \frac{e^e-e}{e}.$$

Next, we compute the $\mathcal{S}^*_B$-radius estimate for the various known subclasses of starlike functions. The main technique involved in tackling the $\mathcal{S}^*_B$-starlikeness for the classes of functions f is to determine the disk which contain the image of the open unit disk under the quantity $zf'(z)/f(z)$.

Theorem 2.3. *The $\mathcal{S}^*_B$-radius estimate for the various subclasses $\mathcal{S}^*_L$, $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{S}^*_{RL}$, $\mathcal{BS}^*(\alpha)$, $\mathcal{S}^*_C$, $\mathcal{S}^*_R$ and $\mathcal{S}^*_q$ are given:*

(i) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_L) = (e^2 - e^{(2/e)})/e^2 \approx 0.717546.$
(ii) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_e) = (e-1)/e \approx 0.632121.$
(iii) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_s) = \log\left((\sqrt{2e^2 - 2e^{1+\frac{1}{e}} + e^{2/e}} - e^{1/e} + e)/e\right) \approx 0.452894.$

(iv) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_{RL}) = \dfrac{-e^2+2\sqrt{2}e^2-2\sqrt{2}e^{1+\frac{1}{e}}+e^{2/e}}{-e^2+2\sqrt{2}e^2-8e^{1+\frac{1}{e}}+4\sqrt{2}e^{1+\frac{1}{e}}-2e^{2/e}+2\sqrt{2}e^{2/e}} \approx 0.743676.$

(v) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{BS}^*(\alpha)) = \dfrac{e-\sqrt{-2e^{1+\frac{1}{e}}\alpha+e^{2/e}\alpha+e^2\alpha+e^2}}{\alpha(e^{1/e}-e)}$, *where* $0 \leq \alpha < 1$.

(vi) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_C) = \frac{\sqrt{2\left(5e^2-3e^{1+\frac{1}{e}}\right)-2e}}{2e} \approx 0.304916.$

(vii) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_R) = r^*_R$, *where*

$$r^*_R = \frac{e^{1/e}\left(1+\sqrt{2}\right)-2e\left(1+\sqrt{2}\right)+\sqrt{\left(3+2\sqrt{2}\right)\left(8e^2-8e^{1+\frac{1}{e}}+e^{2/e}\right)}}{2e} \approx 0.650794.$$

(viii) $\mathcal{R}_{\mathcal{S}^*_B}(\mathcal{S}^*_q) = ((\sqrt{2e^4-2e^{2+\frac{2}{e}}+e^{4/e}}+e^2)/2e^2)^{1/2} \approx 0.389544.$

The first five radius estimates are sharp.

Proof. Consider $|z| = r$.

(i) Let the function $f \in \mathcal{S}^*_L$. Then

$$\left|\frac{zf'(z)-f(z)}{f(z)}\right| \leq 1-\sqrt{1-r}. \tag{3}$$

From Lemma 2.2 and the inequality (3), it is noted that the function $f \in \mathcal{S}^*_B$ if

$$\sqrt{1-r} \geq e^{(1/e)-1},$$

which gives the desired result. The radius estimate is best possible for the function

$$f_l(z) = \frac{4ze^{(2\sqrt{1+z}-2)}}{(1+\sqrt{1+z})^2}.$$

(ii) Since the function $f \in \mathcal{S}^*_e$, then

$$\left|\frac{zf'(z)-f(z)}{f(z)}\right| \leq 1-e^{-r}. \tag{4}$$

Using Lemma 2.2 and the inequality (4), it is observed that the function $f \in \mathcal{S}^*_B$ if $1-e^{-r} \leq 1-e^{(1/e)-1}$. The sharpness follows for the function f_e given as

$$\log\frac{f_e(z)}{z} = \int_0^z \frac{e^u-1}{u}du.$$

(iii) Let $f \in \mathcal{S}^*_s$. By a simple calculation, we have

$$\left|\frac{zf'(z)-f(z)}{f(z)}\right| \leq \sinh r. \tag{5}$$

By Lemma 2.2 and the inequality (5), the function f belongs to the class $\mathcal{S}^*_B$ provided

$$\sin hr \leq 1-e^{(1/e)-1},$$

equivalently

$$e^{2r+1} + 2e^{(er+1)/e} - 2e^{r+1} - e \leq 0$$

or

$$e^{2r} + 2(e^{(1/e)-1} - 1)e^{r} - 1 \leq 0.$$

The function

$$f_s(z) = z \exp\left(\int_0^z \frac{\sin z}{z}\, dz\right)$$

shows the sharpness.

(iv) Let the function $f \in \mathcal{S}^*_{RL}$. Then it is computed that

$$\begin{aligned} \left|\frac{zf'(z) - f(z)}{f(z)}\right| &\leq \left|\sqrt{2} - (\sqrt{2}-1)\left(\frac{1-z}{1+2(\sqrt{2}-1)z}\right)^{1/2} - 1\right| \\ &\leq 1 - \left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right). \end{aligned} \tag{6}$$

By using Lemma 2.2, the disk defined by (6) lies in the domain $Q(\mathbb{D})$ if the following inequality hold:

$$\sqrt{2} - (\sqrt{2}-1)\left(\frac{1+r}{1-2(\sqrt{2}-1)r}\right)^{1/2} \geq e^{(1/e)-1},$$

equivalently

$$\frac{1+r}{1-2(\sqrt{2}-1)r} \leq \frac{(\sqrt{2} - e^{(1/e)-1})^2}{(\sqrt{2}-1)^2}.$$

From the last inequality, we get the best possible radius estimate which is verified by the function

$$\log\frac{f_{RL}(z)}{z} = \int_0^z \frac{1}{z}\left(\left(\sqrt{2}-1\right)\left(-\sqrt{\frac{1-z}{2\left(\sqrt{2}-1\right)z+1}}\right) + \sqrt{2} - 1\right) dz.$$

(v) For $0 \leq \alpha < 1$, if $f \in \mathcal{BS}^*(\alpha)$, then

$$\left|\frac{zf'(z) - f(z)}{f(z)}\right| \leq \frac{r}{1-\alpha r^2}. \tag{7}$$

From Lemma 2.2, the disk defined by (7) is contained in the domain $Q(\mathbb{D})$ if

$$\frac{r}{1-\alpha r^2} \leq 1 - e^{(1/e)-1},$$

and it is equivalent to

$$\alpha r^2 + \left(\frac{e}{e - e^{1/e}}\right) r \leq 1,$$

which gives the required radius estimate. The obtained estimate is best possible for the function $f_{B\alpha}$ defined as

$$\log \frac{f_{B\alpha}(z)}{z} = z\frac{\tanh^{-1}(\sqrt{\alpha}z)}{\sqrt{\alpha}}.$$

(vi) Let the function $f \in \mathcal{S}^*_C$. We have

$$\left|\frac{zf'(z) - f(z)}{f(z)}\right| \leq \frac{1}{3}(4r + 2r^2). \tag{8}$$

From the Lemma 2.2 and the disk defined by (8), the function f belongs to $\mathcal{S}^*_B$ if

$$1 - \frac{1}{3}(4r + 2r^2) \geq \frac{e^{(1/e)}}{e}.$$

Therefore, we get the required estimate.

(vii) Since $f \in \mathcal{S}^*_R$, then a computation yields

$$\left|\frac{zf'(z) - f(z)}{f(z)}\right| \leq \frac{kr + r^2}{k^2 - kr}. \tag{9}$$

Using Lemma 2.2 and the inequality (9), the function f belongs to the class $\mathcal{S}^*_B$ provided

$$\frac{kr + r^2}{k^2 - kr} - 1 \leq -e^{(1/e)-1},$$

equivalently

$$r^2 + (ek + k(e - e^{1/e}))r - k^2(e - e^{1/e}) \leq 0,$$

which yields the required result.

(viii) Since the function $f \in \mathcal{S}^*_q$, then

$$\left|\frac{zf'(z) - f(z)}{f(z)}\right| \leq 1 - r - \sqrt{1 - r^2}. \tag{10}$$

From Lemma 2.2, it is noted that the disk defined by (10) is contained in the domain $Q(\mathbb{D})$ if

$$r + \sqrt{1 - r^2} \geq \frac{e^{(1/e)}}{e}$$

or

$$4r^4 - 4r^2 + (e^{2((1/e)-1)} - 1)^2 \leq 0,$$

that yields the radius constant.

$\square$

Recently, Wani and Swaminathan [23] introduced a class $\mathcal{S}^*_{Ne}$ which is associated to the Caratheodory function $\varphi_{Ne}(z) = 1 + z - z^3/3$. This function starlike with respect to 1 and maps the unit disk onto the nephroid shaped region. By technique used in Theorem **??**, next result provides the $\mathcal{S}^*_B$, $\mathcal{S}^*_R$, $\mathcal{S}^*_L$, $\mathcal{S}^*_{RL}$, $\mathcal{BS}^*(\alpha)$, $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{SG}^*$-radii for the class $\mathcal{S}^*_{Ne}$.

Theorem 2.4. *For the class $\mathcal{S}^*_{Ne}$, the following radius estimates are obtained:*

(i) *The $\mathcal{S}^*_B$-radius estimate=0.514559.*
(ii) *The $\mathcal{S}^*_R$-radius estimate=0.169937.*
(iii) *The $\mathcal{S}^*_L$-radius estimate=0.393849.*
(iv) *The $\mathcal{S}^*_{RL}$-radius estimate=0.278708.*
(v) *The $\mathcal{BS}^*(\alpha)$-radius estimate is a real root of the equation $(1+\alpha)r^3 + 3(1+\alpha)r - 3 = 0$, where $0 \le \alpha < 1$.*
(vi) *The $\mathcal{S}^*_e$-radius estimate=0.570294.*
(vii) *The $\mathcal{S}^*_s$-radius estimate=0.718059.*
(viii) *The $\mathcal{SG}^*$-radius estimate=0.43473.*

The first five results are sharp.

Proof. Let $|z| = r$ and the function $f \in \mathcal{S}^*_{Ne}$. Then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le r + \frac{r^3}{3}. \tag{11}$$

(i) In view of Lemma 2.2, it is easy to check that the disk in (11) lies inside the domain $Q(\mathbb{D})$ if
$$r + \frac{r^3}{3} \le 1 - e^{(1/e)-1},$$
equivalently
$$r^3 + 3r - 3 + 3e^{(1/e)-1} \le 0.$$
The last inequality gives the required radius estimate.

(ii) As an application of [12, Lemma 2.2, p.202], it is evident that the disk given by (11) lies in the domain $\varphi_0(\mathbb{D}) := \{w \in \mathbb{C} \colon |w + (w^2+4w-4)^{1/2}| < 2(\sqrt{2}-1)\}$ if
$$r + \frac{r^3}{3} \le 3 - 2\sqrt{2}$$
or
$$r^3 + 3r - 9 + 6\sqrt{2} \le 0,$$
which gives the required estimate.

(iii) Using Lemma [1, Lemma 2.2, p.6559], we observe that the disk defined by (11) lies in the domain $\sqrt{1+z}(\mathbb{D})$ if
$$r + \frac{r^3}{3} \le \sqrt{2} - 1,$$
equivalently
$$r^3 + 3r + 3 - 3\sqrt{2} \le 0,$$
which yields the desired estimate.

(iv) By [15, Lemma 3.2, p.9], it is noted that the disk defined by (11) lies inside the domain $\varphi_{RL}(z)(\mathbb{D})$ if

$$r+\frac{r^3}{3}\leq\sqrt{\sqrt{2\sqrt{2}-2}-(2\sqrt{2}-2)}$$

or

$$r^3+3r-3\sqrt{\sqrt{2\sqrt{2}-2}-(2\sqrt{2}-2)}\leq 0.$$

The real root of this inequality is the desired radius estimate.

(v) Using [3, Lemma 3.4, p. 1392] and the inequality (11), the function f is to be in the class $\mathcal{BS}^*(\alpha)$ if

$$r+\frac{r^3}{3}\leq\frac{1}{1+\alpha},$$

which gives the radius estimate.

(vi) Using [16, Lemma 2.2, p. 368], it is noted that the disk defined by (11) lies inside the domain $e^z(\mathbb{D})$ if

$$r+\frac{r^3}{3}\leq\frac{e-1}{e},$$

which gives the radius estimate.

(vii) By an application of [5, Lemma 3.3, p. 219] and the inequality (11), the function $f\in\mathcal{S}^*_s$ if

$$r+\frac{r^3}{3}\leq\sin 1,$$

which gives the radius estimate.

(viii) Let $G(\mathbb{D})=\{w\in\mathbb{C};|\log(w/(2-w))|<1\}$ where $G(z)=2/(1+e^{-z})$ is the modified Sigmoid function which is convex and symmetric with respect to the real axis. By an application of [7, Lemma 2.2, p. 961], it is noted that the disk defined by (11) lies inside the domain $G(\mathbb{D})$ *i.e.* the function $f\in\mathcal{SG}^*$ if

$$r^3+3r-3\frac{e-1}{e+1}\leq 0,$$

which gives the radius estimate.

□

Next theorem provides the $\mathcal{BS}^*(\alpha)$-radius estimate for the classes $\mathcal{S}^*_C$, $\mathcal{S}^*_R$, $\mathcal{S}^*_L$ and $\mathcal{S}^*_q$.

Theorem 2.5. *The $\mathcal{BS}^*(\alpha)$-radius estimate for*

(i) *the class $\mathcal{S}^*_C$ is given as:*

$$\mathcal{R}_{\mathcal{BS}^*(\alpha)}(\mathcal{S}^*_C)=\sqrt{\frac{2\alpha+5}{2+2\alpha}}-1.$$

In particular, $\mathcal{R}_{\mathcal{BS}^*(0.9)}(\mathcal{S}^*_C)=0.337712$.

(ii) *the class $\mathcal{S}^*_R$ is given as:*

$$\mathcal{R}_{\mathcal{BS}^*(\alpha)}(\mathcal{S}^*_R) = \frac{-k(2+\alpha) + k\sqrt{\alpha^2 + 8\alpha + 8}}{2(1+\alpha)}.$$

In particular, $\mathcal{R}_{\mathcal{BS}^*(0.9)}(\mathcal{S}^*_R) = 0.699645$.

(iii) *the class $\mathcal{S}^*_L$ is r_l which is a positive real root of the equation*

$$(1+\alpha)(1-\sqrt{1-r}) - 1 = 0.$$

In particular, $\mathcal{R}_{\mathcal{BS}^*(0.9)}(\mathcal{S}^*_L) = 0.775623$.

(iv) *the class $\mathcal{S}^*_q$ is r_q which is a positive real root of the equation*

$$(1+\alpha)(1-r-\sqrt{1-r^2}) - 1 = 0.$$

In particular, $\mathcal{R}_{\mathcal{BS}^*(0.9)}(\mathcal{S}^*_q) = 0.42942046$.

The first estimate is sharp.

Proof. (i) Using [3, Lemma 3.4, p. 1392] and the disk defined by (8), the function f belongs to the class $\mathcal{BS}^*(\alpha)$ provided

$$2(1+\alpha)r^2 + 4(1+\alpha)r - 3 \leq 0,$$

which yields the required estimate.
As part (i), we get the proof of the part (ii), (iii) and (vi).

□

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