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A NOTE ON $(k,\mu)'$-ALMOST KENMOTSU MANIFOLDS

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Abstract. The present paper deals with the study of generalized quasi-conformal curvature tensor inside the setting of $(k,\mu)'$-almost Kenmotsu manifold with respect to η-Ricci soliton. Certain consequences of these curvature tensor on such manifold are likewise displayed. Finally, we illustrate some examples based on this study.

1. Introduction

The idea of k-nullity distribution was started by Gray [15] and Tanno [31] in the study of Riemannian manifolds (M,g), which is defined for any $p \in M$ and $k \in R$ as follows

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}$$

for any X, $Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Blair, Koufogiorgos and Papantoniou [1] introduced a generalized notion of the k-nullity distribution, known as (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p \in M$ and $k,\mu \in R$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ +\mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned}$$

where $h = \frac{1}{2}\mathfrak{L}_\xi\phi$ and $\mathfrak{L}$ denotes the Lie derivative.

In (see, [10], [11], [12]) Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution. Moreover, generalized notion of the k-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, is defined for any $p \in M^{2n+1}$

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and $k, \mu \in R$ as follows:

$$N_p(k,\mu)^{'} = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h^{'}X - g(X,Z)h^{'}Y]\}, \tag{1}$$

where $h^{'}=h \circ \phi$.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), the metric g is called a Ricci soliton if [16]

$$\frac{1}{2}\mathfrak{L}_V g + S + \lambda_1 g = 0,$$

where $\mathfrak{L}$ is the Lie derivative, S the Ricci tensor, V a complete vector field on M and λ_1 is a constant. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g$=-$2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding if λ_1 is negative, zero and positive respectively. A Ricci soliton with V=0 is reduced to Einstein equation. During the last two decades, the geometry of Ricci solitons have been the focus of attention of many mathematicians (see, [5], [6], [7], [8], [9], [13], [17]-[24], [26]). It has became more important after Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904.

The η-Ricci soliton $(\xi, g, \lambda_1, \lambda_2)$ is the generalization of the Ricci soliton (ξ, g, λ_1) and is defined as [6]

$$\mathfrak{L}_\xi g + 2S + 2\lambda_1 g + 2\lambda_2 \eta \otimes \eta = 0, \tag{2}$$

where λ_2 is a real constant.

Thereafter, Ricci solitons and η-Ricci solitons in contact metric manifolds have been studied by various authors such as S. K Yadav et. al (see, [37], [38], [39], [40], [41], [42], [43]) and many others.

Recently, Yano and Sawaki [35], Baishya et al. [4] introduced and studied generalized quasi-conformal curvature tensor C_q in the context of $N(\kappa, \mu)$-manifold. The generalized quasi-conformal curvature tensor C_q is defined for an n-dimensional manifold as

(3)
$$\begin{aligned} C_q(X,Y)Z = \tfrac{n-1}{n}&[\{1+(n-1)a-b\} - \{1+(n-1)(a+b)\}c]C(X,Y)Z \\ &+ [1-b+(n-1)a]\, E(X,Y)Z + (n-1)(b-a)P(X,Y)Z \\ &+ \tfrac{n-1}{n}(c-1)\{1+(n-1)(a+b)\}\hat{C}(X,Y)Z, \end{aligned}$$

for all $X, Y, Z \in \chi(M)$, where the scalars (a, b, c) being real constants and the symbols C, E, P and $\hat{C}$ stand for conformal, concircular, projective and conharmonic curvature tensors respectively. Thus the generalized quasi-conformal curvature tensor C_q can be characterized as, Riemann curvature tensor R if $(a, b, c) \equiv (0, 0, 0)$, conformal curvature C [14] if $(a, b, c) \equiv \left(-\frac{1}{n-2}, -\frac{1}{n-2}, 1\right)$, concircular curvature tensor E [36] if $(a, b, c) \equiv (0, 0, 1)$, projective curvature tensor P [36] if $(a,b,c) \equiv \left(-\frac{1}{n-1}, 0, 0\right)$, conharmonic curvature tensor $\hat{C}$ [25] if $(a,b,c) \equiv \left(-\frac{1}{n-2}, -\frac{1}{n-2}, 0\right)$ and $m-$projective curvature tensor H [30] if $(a,b,c) \equiv \left(-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0\right)$. Thus the equation (3) reduces

$$\begin{aligned} C_q(X,Y)Z &= R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] + b[g(Y,Z)QX \\ &- g(X,Z)QY] - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4}$$

where S, Q and r denotes as usual meaning on M respectively.
The above works motivate us to study generalized quasi-conformal curvature tensor in the domain of $(k,\mu)'$-almost Kenmotsu manifold with respect to η-Ricci soliton.

2. Almost Kenmotsu manifolds

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure, if it admits a $(1,1)$-tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying (see, [2], [3]):

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{5}$$

where I denote the identity endomorphism. Also $\phi\xi = 0$ and $\eta \circ \phi = 0$ both can be derived from (5) easily. If a manifold M with a (ϕ,ξ,η)-structure admits a Reimannian metric g such that

$$(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of T_pM^{2n+1}, then M is said to have an almost contact structure (ϕ,ξ,η,g). The fundamental 2-form θ on an almost contact metric manifold is defined by $\theta(X,Y)=g(X,\phi Y)$ for any X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by

$$N_\phi = [\phi,\phi] + 2d\eta \otimes \xi,$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [2]. Recently in (see, [10], [11], [12], [28]) almost contact metric manifold with the closed η and $d\theta = 2\eta \wedge \theta$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for any vector fields X, Y. It is well known [27] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by D and defined by $D = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, D is an integrable distribution. Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h{=}\frac{1}{2}\mathfrak{L}_\xi\phi$ and $l{=}R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [29]:

$$\begin{gathered} h\xi = 0, l\xi = 0,\ \mathrm{tr}(h) = 0,\ \mathrm{tr}(h\phi) = 0,\ h\phi + \phi h = 0, \\ \nabla_X\xi = -\phi^2X - \phi hX\ (\Rightarrow \nabla_\xi\xi = 0), \\ \phi l\phi - l = 2(h^2 - \phi^2), \\ R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) \\ +(\nabla_Y\phi h)X - (\nabla_X\phi h)Y, \end{gathered} \tag{6}$$

for any vector fields X, Y . The $(1,1)$-type symmetric tensor field $h^{'}{=}h\circ\phi$ is anticommuting with ϕ and $h^{'}\xi{=}0$. Also it is clear that (see, [32], [33], [34]):

$$h = 0 \Leftrightarrow h^{'} = 0,\ h^{'2} = (k+1)\phi^2\ (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{7}$$

3. ξ belongs to the $(k,\mu)^{'}$-nullity distribution

Let $X \in D$ be the eigenvector of $h^{'}$ corresponding to the eigenvalue λ. Then from (7) it is clear that λ^2=-$(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $\lambda^{'}$ and - $\lambda^{'}$ the corresponding eigenspaces related to the non-zero eigenvalue λ and - λ of $h^{'}$, respectively. Before going to our main work, we recall theorem which will be used later on:

Theorem 3.1. *([10]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)^{'}$-nullity distribution and $h^{'} \neq 0$. Then $k < -1$, μ=-2 and Spec $(h^{'})$=$\{0,\lambda,-\lambda\}$, with 0 as simple eigenvalue and λ=$\pm\sqrt{-k-1}$. The distributions $[\xi]\oplus[\lambda]^{'}$ and $[\xi]\oplus[-\lambda]^{'}$ are integrable with totally geodesic leaves. The distributions $[\lambda]^{'}$ and $[-\lambda]^{'}$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as:*

$$\begin{gathered} i)\ K(X,\xi) = k - 2\lambda\ \text{ if } X \in [\lambda]^{'} \text{and } K(X,\xi) = k + 2\lambda \text{ if } X \in [-\lambda]^{'}, \\ ii)\ K(X,Y) = k - 2\lambda,\ \text{if } X,Y \in [\lambda]^{'}, \\ iii)\ K(X,Y) = k + 2\lambda,\ \text{if } X,Y \in [-\lambda]^{'}, \\ iv)\ K(X,Y) = -(k+2),\ \text{if } X \in [\lambda]^{'}, Y \in [-\lambda]^{'}. \end{gathered}$$

Theorem 3.2. *([10]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,-2)^{'}$-nullity distribution and $h^{'} \neq 0$. Then for any X_λ, Y_λ,$Z_\lambda \in [\lambda]^{'}$ and $X_{-\lambda}$,$Y_{-\lambda}$,$Z_{-\lambda} \in [-\lambda]^{'}$, the Riemannian curvature tensor satisfies*

$$\begin{gathered} i) R(X_\lambda,Y_\lambda)Z_{-\lambda} = 0, \\ ii) R(X_{-\lambda},Y_{-\lambda})Z_\lambda = 0, \\ iii) R(X_\lambda,Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda,Z_\lambda)Y_{-\lambda}, \end{gathered}$$

$$iv) R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$v) R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$vi) R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].$$

Theorem 3.3. *([33]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh' \tag{8}$$

Moreover, the scalar curvature of M^{2n+1} is 2n(k-2n).

4. η-Ricci soliton on almost Kenmotsu manifolds

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution. Then from (6) we write $\mathfrak{L}_\xi g$ in term of the Levi-Civita connection ∇, as

$$\begin{aligned} (\mathfrak{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 2[g(X, Y) - \eta(X)\eta(Y) - g(\phi h X, Y)]. \end{aligned} \tag{9}$$

From (2) and (9), we obtain

$$S(X, Y) = -(1+\lambda_1)g(X, Y) - g(h'X, Y) + (1-\lambda_2)\eta(X)\eta(Y), \tag{10}$$

$$QX = -(1+\lambda_1)X + (1-\lambda_2)\eta(X)\xi - h'X, \tag{11}$$

$$S(X, \xi) = S(\xi, X) = -(\lambda_1 + \lambda_2)\eta(X), \tag{12}$$

$$S(\xi, \xi) = -(\lambda_1 + \lambda_2). \tag{13}$$

From (8) and (13), we get

$$\lambda_1 + \lambda_2 = -2nk, \tag{14}$$

for any $X, Y \in \chi(M)$.
This leads to the following:

Theorem 4.1. *In an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n > 1$ with ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$ admitting η-Ricci soliton $(g, \xi, \lambda_1, \lambda_2)$ then $\lambda_1 + \lambda_2$=-2nk.*

With the help of the theorem 4.1, we have the following corollary

Corollary 4.2. *An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$ admitting Ricci soliton is always expanding.*

The generalized quasi-conformal curvature C_q tensor in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with ξ belongs to the $(k, \mu)^{'}$-nullity distribution bearing η-Ricci soliton (ξ, g, λ, μ), reduces to

$$\begin{aligned} C_q(X,Y)Z &= R(X,Y)Z + \left\{(a+b)(1+\lambda_1) + \frac{cr}{2n+1}(\frac{1}{2n} + a + b)\right\} \\ &\quad \{g(Y,Z)X - g(X,Z)Y\} + a\{g(h^{'}X,Z)Y - g(h^{'}Y,Z)X\} \\ &\quad +b\{g(X,Z)h^{'}Y - g(Y,Z)h^{'}X\} \\ &\quad +a(1-\lambda_2)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad +b(1-\lambda_2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \end{aligned} \tag{15}$$

where equations (4), (10) and (11) are used.

5. ξ-Generalized quasi-conformally flat almost Kenmotsu manifold

In this section we discuss ξ-generalized quasi-conformally flat on $(M^{2n+1}, \phi, \xi, \eta, g)$ with ξ belongs to the $(k, \mu)^{'}$-nullity distribution bearing η-Ricci soliton. Now, we recall the following definition:

Definition 5.1. *An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with ξ belongs to the $(k, \mu)^{'}$-nullity distribution is said to be ξ-generalized quasi-conformally flat if $C_q(X,Y)\xi$=0 on M^{2n+1}.*

In view of (1) and (15), we have

$$\begin{aligned} C_q(X,Y)\xi &= \{k + a(2-\lambda_2) + \frac{cr}{2n+1}(\frac{1}{2n} + a + b) + b + (a+b)\lambda_1\} \\ &\quad [\eta(Y)X - \eta(X)Y] + (\mu - b)[\eta(Y)h^{'}X - \eta(X)h^{'}Y]. \end{aligned} \tag{16}$$

With reference to the definition 5.1 and putting $h^{'}X$=X and $h^{'}Y$=Y in (16), we obtain

$$\begin{aligned} &\{k + a(2-\lambda_2) + \tfrac{cr}{2n+1}(\tfrac{1}{2n} + a + b) + b + (a+b)\lambda_1 \\ &\quad +(\mu - b)\}[\eta(Y)X - \eta(X)Y] = 0. \end{aligned} \tag{17}$$

Again substituting X=$h^{'}X$ in (17) and use of (7), we get

$$\begin{aligned} \pm\sqrt{k+1}\{k + a(2-\lambda_2) + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn) \\ +b + (a+b)\lambda_1 + (\mu - b)\}\eta(Y)\phi X = 0, \end{aligned}$$

for any $X, Y \in M^{2n+1}$. It is obvious that
Case (i) $\sqrt{k+1}$=0, that is, k=-1. Dileo and Pastore [10] proved that in almost Kenmotsu manifold with ξ belongs to the $(k, \mu)^{'}$-nullity distribution if k=-1, then $h^{'}$=0 and the manifold is locally a wrapped product of an almost Kähler manifold and an open interval. Thus k=-1, contradicts our hypothesis $h^{'} \neq 0$.

Case (ii) $k \neq 1$, than we have

$$\{k + a(2-\lambda_2) + \tfrac{c(k-2n)}{2n+1}(1+2an+2bn) \\ +b+(a+b)\lambda_1) + (\mu - b)\} = 0.$$

Thus we can state the following theorem:

Theorem 5.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belongs to the $(k,\mu)'$-nullity distribution admitting η-Ricci soliton satisfies $C_q(X,Y)\xi = 0$. Then*

Curvature condition	*Remarks on λ_1, λ_2*
$R(X,Y)\xi$=0	$k = -\mu$
$P(X,Y)\xi$=0	$\lambda_1 + \lambda_2 = -(1+n+n\mu)$
$C(X,Y)\xi$=0	$2\lambda_1 + \lambda_2 = -2n(k+\mu-1) - \mu(2n-1) - 3$
$E(X,Y)\xi$=0	$k = \frac{-2n(1+\mu)}{2n-1}$
$\hat{C}(X,Y)\xi$=0	$2\lambda_1 + \lambda_2 = (k+\mu)[1-4n] - 4$
$H(X,Y)\xi$=0	$2n\lambda_1 + \lambda_2 = -[4n(k+\mu) + 2(1+n)]$

6. ϕ-Generalized quasi-conformally semi-symmetric almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with $(k,\mu)'$-nullity distribution bearing η-Ricci soliton

We consider ϕ-generalized quasi-conformally semi-symmetric η-Ricci soliton on $(M^{2n+1}, \phi, \xi, \eta, g)$ with ξ belongs to the $(k,\mu)'$-nullity distribution. Then

$$C_q \cdot \phi = 0.$$

Which is equivalent to

$$C_q(X,Y)\phi Z - \phi(C_q(X,Y)Z) = 0. \tag{18}$$

Fix Z=ξ in (18), we obtain

$$\phi(C_q(X,Y)\xi) = 0. \tag{19}$$

From (16) and (19), we have

$$\begin{aligned}&\left\{k + a(2-\lambda_2) + \tfrac{cr}{2n+1}(\tfrac{1}{2n} + a + b) + b + (a+b)\lambda_1\right\} \\ &(\eta(Y)\phi X - \eta(X)\phi Y) \\ &+(\mu - b)\{\eta(Y)\phi(h'X) - \eta(X)\phi(h'Y)\} = 0.\end{aligned} \tag{20}$$

Again letting X=$h'X$ in (20) and using (7), we get

$$\pm\sqrt{k+1}\ [(k + a(2-\lambda_2) + b + (a+b)\lambda_1) \\ +\tfrac{c(k-2n)}{2n+1}(1+2an+2bn))\phi^2 X\eta(Y) + (\mu - b)\sqrt{k+1}\eta(Y)\phi X] = 0.$$

for any vector fields X, Y on M^{2n+1}. Now, at this stage we have two cases
Case (i) $\sqrt{k+1}$=0, that is, k=-1, it contradicts our hypothesis $h' \neq 0$.

Case (ii) $k \neq 1$, then we get

$$[(k + a(2 - \lambda_2) + b + (a + b)\lambda_1) + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn)) \\ \phi^2 X\eta(Y) + (\mu - b)\sqrt{k+1}\eta(Y)\phi X] = 0.$$

This leads to the following:

Theorem 6.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belongs to the $(k, \mu)'$-nullity distribution admitting η-Ricci soliton satisfies $C_q \cdot \phi = 0$. Then*

Curvature condition	*Remarks on λ_1, λ_2*
$R \cdot \phi$=0	$k = \mu = 0$
$P \cdot \phi$=0	$\lambda_1 + \lambda_2 = -2(1 + nk)$, $\mu = 0$
$C \cdot \phi$=0	$2\lambda_1+\lambda_2 = -[\frac{(k-2n)}{2n+1}(1+2n)+k(2n-1)+3]$, $\mu = -\frac{1}{2n-1}$
$E \cdot \phi$=0	$k = -1$, $\mu = 0$
$\hat{C}(X, Y)\xi$=0	$2\lambda_1 + \lambda_2 = (k + \mu)[1 - 4n] - 4$
$H(X, Y)\xi$=0	$2n\lambda_1 + \lambda_2 = -[4n(k + \mu) + 2(1 + n)]$

7. An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with $(k, \mu)'$-nullity distribution bearing η-Ricci soliton satisfying $C_q \cdot S$=0

We consider the condition $C_q \cdot S$=0, in an almost Kenmotsu manifold with ξ belongs to the $(k, \mu)'$-nullity distribution admitting η-Ricci soliton. Precisely, we prove the following results:

Theorem 7.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belongs to the $(k, \mu)'$-nullity distribution bearing bearing η-Ricci soliton under the restriction $C_q \cdot S = 0$. Then M is*
i) locally isometric to the Riemannian product of an $(n + 1)$-dimensional manifold with constant sectional curvature -4 and a flat n-dimensional manifold.
ii) locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold with constant sectional curvature -9 and n-dimensional manifold with constant sectional curvature -1
iii) an η-Einstein manifold.

Proof. The condition $(C_q(X, Y) \cdot S)(Z, V) = 0$ is equivalent to

$$S(C_q(\xi, Y)\xi, V) + S(\xi, (C_q(\xi, Y)V) = 0. \tag{21}$$

Also from (8), (13), (14) and (16) we have

$$S(C_q(\xi, Y)\xi, V) = \{k + a(2 - \lambda_2) + b + (a + b)\lambda_1 \\ + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn)\}[2nk\eta(Y)\eta(V) - S(Y, V)] \\ +(\mu - b)S(V, h'Y). \tag{22}$$

Also,

$$
\begin{aligned}
S(\xi,(C_q(\xi,Y)V) &= k[2nkg(Y,V)-2nk\eta(Y)\eta(V)] \\
&+\left\{(a+b)(1+\lambda_1)+\tfrac{c(k-2n)}{2n+1}(1+2an+2bn)\right\} \\
&[2nk\eta(Y)\eta(V)-2nkg(Y,V)]-2ang(h^{'}Y,V) \\
&-b(1-\lambda_2)[2nk\eta(Y)\eta(V)-2nkg(Y,V)].
\end{aligned} \tag{23}
$$

Using (22) and (23) in (21), we get

$$
\begin{aligned}
&\left\{k+a(2-\lambda_2)+b+(a+b)\lambda_1+\tfrac{c(k-2n)}{2n+1}(1+2an+2bn)\right\} \\
&[2nk\eta(Y)\eta(V)-S(Y,V)]+(\mu-b)S(V,h^{'}Y) \\
&+2nk^2[g(Y,V)-\eta(Y)\eta(V)]+2nk\mu g(h^{'}Y,V) \\
&+2nk\left\{(a+b)(1+\lambda_1)+\tfrac{c(k-2n)}{2n+1}(1+2an+2bn))\right\} \\
&[\eta(Y)\eta(V)-g(Y,V)]-2ang(h^{'}Y,V) \\
&-2nkb(1-\lambda_2)[\eta(Y)\eta(V)-g(Y,V)]=0.
\end{aligned} \tag{24}
$$

On substituting $Y{=}h^{'}Y$ in (24) and using (8), we obtain

$$
\begin{aligned}
&(k+2)(k+5)[\{-k+a(1+\lambda_2)-(\mu-b)+\{(a+b)(1+\lambda_1)\} \\
&S(h^{'2}Y,V)+\{2nk(k-b(1+\lambda_2)+2n(k\mu-a)\}g(h^{'2}Y,V)]=0.
\end{aligned} \tag{25}
$$

With the help of (7), equation (25) reduces to

$$
\begin{aligned}
&(k+1)(k+2)(k+5)[-pS(Y,V)-qg(Y,V) \\
&+(2nkp+q)\eta(Y)\eta(V)]=0,
\end{aligned}
$$

where $p=-k+a(1+\lambda_2)-(\mu-b)+(a+b)(1+\lambda_1)$, $q=2nk(k-b(1+\lambda_2)+2n(k\mu-a)$, for any vector fields Y, V on M^{2n+1}.

Now, we discuss the following cases. $[-\lambda]^{'}$

Case (i) $(k+1)$=0, that is, k=-1. Then according to Dileo and Pastore [10], it contradicts our hypothesis $h^{'}\neq 0$.

Case (ii) $k\neq -1$, $(k+2)$=0, that is, k=-2 then λ=1. So from Theorem 3.2, we get

$$R(X_\lambda,Y_\lambda)Z_\lambda=-4[g(Y_\lambda,Z_\lambda)X_\lambda-g(X_\lambda,Z_\lambda)Y_\lambda],$$

$$R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda}=0,$$

for any vector field

$$X_\lambda,Y_\lambda,Z_\lambda\in[\lambda]^{'}$$

and

$$X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in[-\lambda]^{'}.$$

Also $\mu=-2$, thus from Theorem 3.1 we get $K(X,\xi)=-4$ for any $X\in[\lambda]^{'}$ and $K(X,\xi)=0$ for any $X\in-\lambda'$. Again from Theorem 7.1, we find $K(X,Y)=-4$ for any $X,Y\in[\lambda]^{'}$, $K(X,Y)=0$ for any $X,Y\in[-\lambda]^{'}$. As is shown [10] that the distribution $[\xi]\oplus[\lambda]^{'}$ is integrable with totally geodesic leaves and the distribution $[-\lambda]^{'}$ is integrable with totally umbilical leaves by $H=-(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Thus $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Therefore the manifold M^{2n+1} is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$.

Case (iii) $k \neq -1$, $k \neq -2$ and $(k+5)$=0, that is, k=-5 then λ=2. Thus from Theorem 3.2, we get

$$R(X_\lambda, Y_\lambda)Z_\lambda = -9[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = [g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also we conclude that $\mu = -2$, thus in view of Theorem 3.1 that $K(X, \xi) = -9$ for any $X \in [\lambda]'$ and $K(X, \xi) = -1$ for any $X \in [-\lambda]'$. Again from Theorem 3.1, we have $K(X, Y) = -9$ for any $X, Y \in [\lambda]'$, $K(X, Y) = 2$ for any $X, Y \in [-\lambda]'$. As is shown [10] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is intregrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. So $\lambda = 2$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Therefore we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{2n+1}(-9) \times \mathbb{R}^n$.

Case (iv) $k \neq -1$, $k \neq -2$ and $k \neq -5$ then we have

$$S(Y, V) = -\frac{q}{p} g(Y, V) + \frac{(2nkp + q)}{q} \eta(Y)\eta(V),$$

which means that the manifold is an η-Einstein manifold. This leads the proof of the Theorem 7.1. □

8. An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with $(k, \mu)'$-nullity distribution bearing η-Ricci soliton satisfying $((\xi \wedge_S X) \cdot C_q)$=0

In this section we discuss the condition $((\xi \wedge_S X) \cdot C_q)$=0 on almost Kenmotsu manifolds with the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution admitting η-Ricci soliton. First we prove the following theorem.

Theorem 8.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution bearing η-Ricci soliton under the restriction $((\xi \wedge_S X) \cdot C_q)$=0. Then M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$.*

Proof. The condition $((\xi \wedge_S X) \cdot C_q) = 0$ holds on M^{2n+1}. Then we get

$$((\xi \wedge_S X) \cdot C_q)(Y, Z)U = 0, \tag{26}$$

for any $X, Y, Z, U \in \chi(M)$. The equation (26) equivalent to

$$\begin{aligned} S(X, C_q(Y,Z)U)\xi - S(\xi, C_q(Y,Z)U)X - S(X,Y)C_q(\xi,Z)U \\ +S(\xi,Y)C_q(X,Z)U - S(X,Z)C_q(\xi,Z)U + S(\xi,Y)C_q(X,Z)U \\ -S(X,Z)C_q(Y,\xi)U + S(\xi,Z)C_q(Y,X)U - S(X,U)C_q(Y,Z)\xi \\ +S(\xi,U)C_q(Y,Z)X = 0. \end{aligned} \tag{27}$$

Taking the inner product of (27) with ξ, we obtain

$$\begin{aligned} S(X, C_q(Y,Z)U) - S(\xi, C_q(Y,Z)U)\eta(X) - S(X,Y)\eta(C_q(\xi,Z)U) \\ +S(\xi,Y)\eta(C_q(X,Z)U) - S(X,Z)\eta(C_q(\xi,Z)U) + S(\xi,Y)\eta(C_q(X,Z)U) \\ -S(X,Z)\eta(C_q(Y,\xi)U) + S(\xi,Z)\eta(C_q(Y,X)U) - S(X,U)\eta(C_q(Y,Z)\xi) \\ +S(\xi,U)\eta(C_q(Y,Z)X) = 0. \end{aligned} \tag{28}$$

Using (8), (15) and (16), for U=ξ, equation (28) reduces to

$$\begin{aligned} \Big\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn))\Big\} \\ [S(X,Y)\eta(Z) - S(X,Z)\eta(Y)] + (\mu - b)[S(X,h'Y)\eta(Z) - S(X,h'Z)\eta(Y)] \\ +S(\xi,\xi)[\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1 + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn))\} \\ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + a\{g(h'X,Z)\eta(Y) - g(h'Y,Z)\eta(X)\}] = 0. \end{aligned} \tag{29}$$

For fix Z=ξ in (29), using (12) and (13), we obtain

$$\begin{aligned} \Big\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) + \tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn)\Big\} \\ [S(X,Y) - 2nk\eta(X)\eta(Y)] + (\mu - b)[S(X,h'Y] = 0. \end{aligned} \tag{30}$$

Let $X, Y \in [\lambda]'$ and keeping in mind (8), we get from (30) that

$$\begin{aligned} 2n(1+\lambda)\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) \\ +\tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn)\} + 2n\lambda(\mu - b)(1+\lambda) = 0. \end{aligned} \tag{31}$$

Next, for $X, Y \in [-\lambda]'$ in (30) and using (8) we obtain

$$\begin{aligned} -2n(1-\lambda)\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) \\ +\tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn)\} + 2n\lambda(\mu - b)(1-\lambda) = 0. \end{aligned} \tag{32}$$

With the help of (31) and (32), we have

$$\begin{aligned} 4n(\lambda - 1)\{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) \\ +\tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn) + \lambda(\mu - b)\} = 0. \end{aligned}$$

Now, there are following case arises

Case(i) If λ=1, then k=-2. So by the Theorem 3.1 and Theorem 3.2, it is clear that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case(ii) If $\lambda \neq 1$ then we get

$$\begin{aligned} \{(k + a(2-\lambda_2) + b + (a+b)\lambda_1) \\ +\tfrac{c(k-2n)}{2n+1}(1 + 2an + 2bn) + \lambda(\mu - b)\} = 0. \end{aligned}$$

This leads the proof of the Theorem 8.1 □

Precisely, one can also prove the following results:

Corollary 8.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belongs to the $(k,\mu)^{'}$-nullity distribution bearing η-Ricci soliton under the restriction $((\xi \wedge_S X) \cdot C_q)$=0. Then*

Curvature condition	*Remarks on λ_1, λ_2*
$((\xi \wedge_S X) \cdot R)$=0	$k = -\mu$
$((\xi \wedge_S X) \cdot P)$=0	$\lambda_1 + \lambda_2 = -2n(k+\mu) - 2$
$((\xi \wedge_S X) \cdot C)$=0	$\lambda_1 + \lambda_2 = [\frac{1}{n} - n]\{2n(k+\mu+1) + \mu + \frac{2n+1}{2n-1}\} - 2$
$((\xi \wedge_S X) \cdot E)$=0	$k = -[1 + \mu\{1 + \frac{1}{2n}\}]$
$((\xi \wedge_S X) \cdot \widetilde{C})$=0	$-2(\lambda_1 + \lambda_2) = (2n-1)(k+\mu) + 5$
$((\xi \wedge_S X) \cdot H)$=0	$-(2\lambda_1 + \lambda_2) = -2[(nk + n\mu) - 2]$

9. An example of almost Kenmotsu manifold with $(k,\mu)^{'}$-nullity distributions admitting an expanding η-Ricci soliton

We consider a 5-dimensional differentiable manifold

$$M^5 = \{(x,y,z,u,v) \in \mathbb{R}^5 \mid (x,y,z,u,v) \neq (0,0,0)\},$$

where (x,y,z,u,v) denote the standard coordinate in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4, e_5 are the vector fields in $\mathbb{R}^5$ which satisfies [10]

$$[e_1, e_2] = -2e_2, \ \ [e_1, e_3] = -2e_3, \ \ [e_1, e_4] = 0, \ \ [e_1, e_5] = 0,$$

$$[e_i, e_j] = 0, \ \text{ where } i, j = 2, 3, 4, 5.$$

We define the Riemannian metric g by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

$$g(e_1, e_i) = g(e_i, e_j) = 0, \ for \ \ i \neq j; i, j = 2, 3, 4, 5.$$

Let the 1-form η be $\eta(Z) = g(Z, e_1)$ for any $Z \in \chi(M^5)$. Let ϕ be the $(1,1)$-tensor field given by

$$\phi(e_1) = 0, \ \phi(e_2) = e_4, \ \ \phi(e_3) = e_5, \ \ \phi(e_4) = -e_2, \ \ \phi(e_5) = -e_3.$$

In view of linearity properties of ϕ and g, we have

$$\phi^2 X = -X + \eta(X)e_1, \eta(e_1) = 1, g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for arbitrary vector fields $X, Y \in \chi(M^5)$. Moreover,

$$h^{'} e_1 = 0, \ h^{'} e_2 = e_4, \ h^{'} e_3 = e_3, \ h^{'} e_4 = -e_4, \ \ h^{'} e_5 = e_5.$$

We recall the Koszul's formula as

$$\begin{aligned} 2g(\nabla_X Y, Z) \ &= \ Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]) \end{aligned}$$

for arbitrary vector fields $X, Y, Z \in \chi(M^5)$. With the help of Koszul's formula we have

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_1}e_4 = 0, \quad \nabla_{e_1}e_5 = e_1,\\
&\nabla_{e_2}e_1 = 2e_2, \quad \nabla_{e_2}e_2 = -2e_1, \quad \nabla_{e_2}e_3 = 0, \quad \nabla_{e_2}e_4 = 0, \quad \nabla_{e_2}e_5 = 0,\\
&\nabla_{e_3}e_1 = 2e_3, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -2e_1, \quad \nabla_{e_3}e_4 = 0, \quad \nabla_{e_3}e_5 = 0,\\
&\nabla_{e_4}e_1 = 0, \quad \nabla_{e_4}e_2 = 0, \quad \nabla_{e_4}e_3 = 0, \quad \nabla_{e_4}e_4 = 0, \quad \nabla_{e_4}e_5 = 0,\\
&\nabla_{e_5}e_1 = 0, \quad \nabla_{e_5}e_2 = 0, \quad \nabla_{e_5}e_3 = 0, \quad \nabla_{e_5}e_4 = 0, \quad \nabla_{e_5}e_5 = 0.
\end{aligned}$$

It is notice that $\nabla_X\xi = -\phi^2 X + h'X$ for ξ=e_1. Thus the manifold is an almost contact metric manifold with the almost contact structure (ϕ, η, ξ, g) such that $d\eta$=0 and dθ=2$\eta \wedge \theta$, so that the manifold is an almost Kenmotsu manifold. Also, the curvature tensors

$$\begin{aligned}
&R(e_1,e_2)e_1 = 4e_2, R(e_1,e_2)e_2 = -4e_1 = R(e_1,e_3)e_3, R(e_1,e_3)e_1 = 4e_3,\\
&R(e_1,e_4)e_1 = 0, R(e_1,e_4)e_4 = 0, R(e_1,e_5)e_1 = 0, R(e_1,e_5)e_5 = 0,\\
&R(e_2,e_3)e_2 = -4e_3, R(e_2,e_3)e_3 = -4e_2, R(e_2,e_4)e_2 = 0 = R(e_2,e_4)e_4,\\
&R(e_2,e_5)e_2 = 0, R(e_2,e_5)e_5 = 0, R(e_3,e_4)e_3 = 0, R(e_3,e_4)e_4 = 0,\\
&R(e_3,e_5)e_3 = 0, R(e_3,e_5)e_5 = 0, R(e_4,e_5)e_4 = 0, R(e_4,e_5)e_5 = 0.
\end{aligned}$$

It is clear that the characteristic vector field ξ-belongs to the $(k,\mu)'$-nullity distribution with k=-2 and μ=-2. The Ricci tensors S is given by

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = -8, \;\; S(e_4,e_4) = S(e_5,e_5) = 0. \tag{33}$$

In case of η-Ricci soliton given by (10), it is sufficient to verify that

$$S(e_i,e_i) = -(1+\lambda_1)g(e_i,e_i) - g(h'e_i,e_i) + (1-\lambda_2)\eta(e_i)\eta(e_i), \tag{34}$$

for all $i = 1,2,3,4,5$. From (34), we can easily find that

$$S(e_2,e_2) = -(1+\lambda_1)g(e_2,e_2). \tag{35}$$

In view of (33) and (35), we get λ_1=7. Also, from (34) we have

$$S(e_1,e_1) = -(1+\lambda_1)g(e_1,e_1) - g(h'e_1,e_1) + (1-\lambda_2)\eta(e_1)\eta(e_1), \tag{36}$$

Keeping in mind λ_1=7, from (36), we obtain λ_2=1. Thus, the structure $(g,\xi,7,1)$ is an η-Ricci soliton in an almost Kenmotsu manifold with $(k,\mu)'$-nullity distributions. At this stage λ_1=7, i.e., $\lambda_1 > 0$ it means η-Ricci soliton is an expanding in nature. This verifies our Theorem 4.1.

With reference to this example and the Theorem (see, Dileo, and Pastore [10]), we conclude that

Theorem 9.1. *There exist a* 5*-dimensional almost Kenmotsu manifold with* $(k,-2)'$*-nullity distribution with* $h' \neq 0$ *which is locally isometric to the warped product* $\mathbb{H}^{n+1}(k-2\lambda) \times_f \mathbb{R}^n$ *or* $B^{n+1}(k+2\lambda) \times_{f'} \mathbb{R}^n$*, where,* f=$ce^{(1-\lambda)t}$ *and* f'=$\acute{c}e^{(1+\lambda)t}$*, with* c*,* $\acute{c}$ *positive constants.*

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