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FINITE INTEGRALS ASSOCIATED WITH THE PRODUCT OF ORTHOGONAL POLYNOMIALS AND WRIGHT FUNCTION

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Abstract. Several useful and interesting extensions of the various special functions have been introduced by many authors during the last few decades. Various integral formulas associated with Wright function have been studied and a noteworthy amount of work have found in literature. The principal object of the present paper is to evaluate finite integral formulas containing the product of orthogonal polynomials with generalized Wright function. These integral formulas are expressed in terms of Srivastava and Daoust function. Some interesting particular cases are obtained from the main results by specialising the suitable values of the parameters involved.

1. Introduction and Preliminaries

Integral transforms associated with many special functions, have found remarkable importance and applications in sub-field of physical and mathematical analysis. The most useful significance of integral transforms lies in the fact that they transform a class of differential equations into a class of algebaric equations, so that solution of those differential equations can be obtained by easily algebraic methods and by use of results of integral transforms. Several authors including Choi et al. [2]-[4], Nisar et al. [21], Suthar et al. [30]-[32] and Tadees et al. [33] have been researching integral formulae involving special functions over the past few decades. Further, Shahed and Salem[25] obtained an extension of Wright function and found its essential properties such as integral transforms, integral representation and its representations in terms of other special functions. In a sequel work of Kamarujjama and Khan [6], Khan et al.[7]-[17] and O. Khan et al. [18] developed specific integral formulas that included Bessel functions, generalized Bessel function, generalized Bessel-Maitland function, Struve function, Galue type Struve function, generalized

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Wright function, Mittag-Leffler function, multi-index Mittag-Leffler function and Whittaker function etc. Lately, Khan et al. [11]-[13] obtained intriguing results by applying the integral formulas linked with the product of Jacobi and Laguerre polynomials with generalized Bessel function. Integral transforms involving special functions have gained considerable attention in the bibliography. In the last decade, many papers are investigated to study the integral transforms involving the product of generalized Wright function and other special functions(see above cited work).
Recently, many applications of integrals involving special functions are found in several areas of engineering sciences and mainly in Physics. for e.g., applications of doughnut beams with a dark spot at the centre are investigated in optical tweezers, infar-field fluorescence microscopy of biological samples, which can be expressed as the integrand of an integral transform involving the Bessel functions which is closely related to generalized Wright function. In view of this, it is worth investigating a general integral transform involving generalized wright function and orthogonal polynomials. Motivated by such type of works, in the present paper, we establish two new integrals involving the product of generalized Wright function with orthogonal polynomials, which are expressed in terms of Srivastava and Daoust function. Also, some other integrals whose integrands are associated with ultraspherical polynomials, Gegenbauer polynomials, Tchebicheff polynomials, Legendre polynomials, Hermite polynomials and Whittaker function are derived as particular cases in the form of corollaries of our main results.

The special function called Wright function and its generalizations has gained considerable prominence and significance in the solution of integral and differential equations because of its applications appear naturally. We begin to recall Bessel-Maitland function better-known as the classical Wright function $W_{\nu,\mu}(w)$ [19, 23] named in honor of E. Maitland Wright defined by the follwing series:

$$W_{\nu,\mu}(w) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu n + \mu)} \frac{w^n}{n!}, \quad \mu \in \mathbb{C}, \nu > -1. \tag{1}$$

Moustafa and Salem [25] introduced the generalization of the Wright function as:

$$W_{\nu,\mu}^{\gamma,\delta}(w) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \, \Gamma(\nu n + \mu)} \frac{w^n}{n!}, \tag{2}$$

where ν is real and $\gamma,\ \delta,\ \mu \in \mathbb{C};\ \nu > -1,\ \delta \neq 0, -1, -2, \ldots,$ with $w \in \mathbb{C}$ and $|w| < 1$ with $\nu = -1$, and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$ is a pochhammer symbol and $\Gamma(.)$ is the gamma function [27].

There upon, the two auxillary functions for all complex variables $w \neq 0$ and for any order $0 < \nu < 1$ were investigated as:

$$M_\nu^{\gamma,\delta}(w) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \, \Gamma(1-\nu(n+1))} \frac{(-1)^n \, w^n}{n!}, \tag{3}$$

$$F_\nu^{\gamma,\delta}(w) = \sum_{n=1}^{\infty} \frac{(\gamma)_n}{\Gamma(-\nu n) \, (\delta)_n} \frac{(-1)^n \, w^n}{n!}. \tag{4}$$

Also, generalized Wright function $W_{\nu,\mu}^{\gamma,\delta}$ can be easily written in terms of other special functions, for example, two auxilary functions of Wright function, Fox H-function, Mittag-Leffler function and Meijer G-function as mentioned below:

1. On replacing ν with $-\nu$, w with $-w$, setting $\mu = 1-\nu$ in (2) and with the help of (3), we get

$$W_{-\nu,1-\nu}^{\gamma,\delta}(-w) = M_\nu^{\gamma,\delta}(w). \tag{5}$$

2. On replacing ν with $-\nu$, w with $-w$, setting $\mu = 0$ in (2) and with the help of (4), we get

$$W_{-\nu,0}^{\gamma,\delta}(-w) = F_\nu^{\gamma,\delta}(w). \tag{6}$$

3. On replacing w with $-w$ in (2), we can easily relate the generalized Wright function with Fox H-function [5] as:

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)} \, W_{\nu,\mu}^{\gamma,\delta}(-w) = H_{1\ 3}^{1\ 1}\left[w \middle| \begin{matrix} (1-\gamma,1) \\ (0,1), \ \ (1-\mu,\nu), \ \ (1-\delta,1) \end{matrix} \right]. \tag{7}$$

4. On replacing w with $-w$, $\nu = 1$ in (2), we can easily relate the generalized Wright function with Meijer G-function [1] as:

$$W_{\nu,\mu}^{\gamma,\delta}(-w) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \, G_{1\ 3}^{1\ 1}\left[w \middle| \begin{matrix} 1-\gamma \\ 0, 1-\mu, 1-\delta \end{matrix} \right]. \tag{8}$$

5. On replacing $\gamma = 1$ and $\nu = 0$ in (2), we get

$$\frac{\Gamma(\mu)}{\Gamma(\delta)} \, W_{0,\mu}^{1,\delta}(w) = E_{1,\delta}(w), \tag{9}$$

where $E_{1,\delta}(w)$ is the classical Mittag-Leffler function [20, 35].

Several useful and interesting extensions of the various special functions such as Wright, Bessel, Mittag-Leffler, Gauss hypergeometric functions, Fox H-function, Meijer G-function and Jacobi polynomials associated with ultraspherical polynomials, Gegenbauer polynomials, Tchebicheff polynomials, Legender polynomials and Laguerre polynomials have been introduced by various authors during the recent decades. The description of Jacobi and Laguerre polynomials in terms of ultraspherical, Gegenbauer, Tchebicheff, Legender, Hermite polynomials and Whittaker functions are shown below:

The Jacobi polnomial $P_n^{(\lambda,\ \sigma)}(w)$ [24, 26], defined by

$$P_n^{(\lambda,\ \sigma)}(w) = \frac{(1+\lambda)_n}{n!}\ {}_2F_1\left[\begin{array}{cc} -n, \quad 1+\sigma+\lambda+n; & \dfrac{1-w}{2} \\ 1+\lambda; & \end{array}\right], \tag{10}$$

or equivalently

$$P_n^{(\lambda,\ \sigma)}(w) = \sum_{r=0}^{n} \frac{(1+\lambda)_n\ (1+\lambda+\sigma)_{n+r}\ (w-1)^r}{r!\ (n-r)!\ (1+\lambda)_r\ (1+\lambda+\sigma)_n\ 2^r}\ , \tag{11}$$

From equation (10), taking $w = 1$ we have

$$P_n^{(\lambda,\ \sigma)}(1) = \frac{(1+\lambda)_n}{n!} \tag{12}$$

where $P_n^{(\lambda,\sigma)}(w)$ is polynomial of degree n.

On taking $\sigma = \lambda$ in equation (10), we have

$$P_n^{(\lambda,\lambda)}(w) = \sum_{r=0}^{n} \frac{(1+\lambda)_n\ (1+2\lambda)_{n+r}\ (w-1)^r}{r!\ (n-r)!\ (1+\lambda)_r(1+2\lambda)_n\ 2^r}, \tag{13}$$

where $P_n^{(\lambda,\lambda)}(w)$ is the ultraspherical polynomial;

For $\lambda = \sigma = \eta - \frac{1}{2}$, (10) reduces to Gegenbauer polynomial $C_n^{\eta}(w)$ [24, 26]:

$$P_n^{\left(\eta-\frac{1}{2},\ \eta-\frac{1}{2}\right)}(w) = \frac{\left(\eta+\frac{1}{2}\right)_n}{(2\eta)_n}\ C_n^{\eta}(w), \tag{14}$$

For $\lambda = \sigma = -\frac{1}{2}$, (10), reduces to Tchebycheff polynomial of first kind $T_n(w)$ as:

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(w) = \frac{\left(\frac{1}{2}\right)_n}{(n)!}\ T_n(w), \tag{15}$$

On taking $\lambda = \sigma = \frac{1}{2}$ in equation (10), we have

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(w) = \frac{\left(\frac{3}{2}\right)_n}{(n+1)!}\ U_n(w), \tag{16}$$

where $U_n(w)$ is the Tchebycheff polynomial of second kind.

On taking $\lambda = \sigma = 0$ in equation (10), we have

$$P_n^{(0,0)}(w) = P_n(w), \tag{17}$$

where $P_n(w)$ is the Legendre polynomial [24, 26].

The generalized Leguerre polynomials $L_n^{\lambda}(w)$ are defined by [24, 26]

$$L_n^{\lambda}(w) = \frac{(1+\lambda)_n}{n!}\, {}_1F_1(-n; 1+\lambda; w). \tag{18}$$

The series representation of (18)

$$L_n^{\lambda}(w) = \sum_{r=0}^{n} \frac{(-1)^r (1+\lambda)_n \; w^r}{r! \; (n-r)! \; (1+\lambda)_r}. \tag{19}$$

The relation of generalized Laguerre polynomials with Hermite polynomials as follows:

$$L_n^{-\frac{1}{2}}(w^2) = \frac{H_{2n}(w)}{(-1)^n \; 2^{2n} \; n!} \tag{20}$$

$$L_n^{\frac{1}{2}}(w^2) = \frac{H_{2n+1}(w)}{(-1)^n \; 2^{2n+1} \; n! \; w} \;, \tag{21}$$

where $H_{2n}(w)$ and $H_{2n+1}(w)$ are known as Hermite polynomials [24, 26]. We also know that

$$L_n^{\lambda}(w) = \frac{(1+\lambda)_n}{n!} \; w^{-\frac{1}{2}-\frac{\lambda}{2}} \; e^{\frac{w}{2}} \; M_{n+\frac{1}{2}+\frac{\lambda}{2},\frac{\lambda}{2}}(w) \tag{22}$$

$$L_n^{\lambda}(w) = \frac{(-1)^n}{n!} \; w^{-\frac{1}{2}-\frac{\lambda}{2}} \; e^{\frac{w}{2}} W_{n+\frac{1}{2}+\frac{\lambda}{2},\frac{\lambda}{2}}(w), \tag{23}$$

where $M_{r,\xi}(w)$ and $W_{r,\xi}(w)$ are the Whittaker functions defined by Whittaker [34]. Whittaker function is expressed in terms of confluent hypergeometric function ${}_1F_1$ (or Kummer's function) [34] as follows:

$$M_{r,\xi}(w) = w^{\xi+\frac{1}{2}} \; e^{-\frac{w}{2}} {}_1F_1\left(\frac{1}{2}+\xi-r, 2\xi+1; w\right), \tag{24}$$

$$W_{r,\xi}(w) = w^{\xi+\frac{1}{2}} \; e^{-\frac{w}{2}} {}_1F_1\left(\frac{1}{2}+\xi-r, 2\xi+1; w\right). \tag{25}$$

In 1969, the multivariable hypergeometric function introduced and studied by Srivastava and Daoust, generally known as Srivastava-Daoust function [28] which is given as:

$$F^{p:m_1;.....;m_s}_{\ell:q_1;.....;q_s}\left[\begin{matrix} (a_j:\alpha_j^1,....\alpha_j^s)_{1,p}: & (c_j^1,r_j^1)_{1,q_1}; & ....; & (c_j^s,r_j^s)_{1,q_s}; \\ (b_j:\beta_j^1,....\beta_j^s)_{1,\ell}: & (d_j^1,\delta_j^1)_{1,m_1}; & ....; & (d_j^s,\delta_j^s)_{1,m_s}; \end{matrix} \; w_1, w_2, ..., w_s\right]$$

(26)

$$= \sum_{n_1,n_2,...,n_s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha_j^1+...+n_s\alpha_j^s} \prod_{j=1}^{q_1}(c_j^1)_{n_1 r_j^1} \cdots \prod_{j=1}^{q_s}(c_j^s)_{n_s r_j^s}}{\prod_{j=1}^{l}(b_j)_{n_1\beta_j^1+...+n_s\beta_j^s} \prod_{j=1}^{m_1}(d_j^1)_{n_1\delta_j^1} \cdots \prod_{j=1}^{m_s}(d_j^s)_{n_s\delta_j^s}} \frac{w_1^{n_1}}{(n_1)!} ..... \frac{w_s^{n_s}}{(n_s)!}.$$

The multiple series (26) converges absolutely under the precise conditions (see[29]).

We are required the most useful result due to Prudnikov et al. [22] by means of which we have established our main result in the present article:

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}\, d\theta$$

$$= (\zeta-\xi)^{-1}(1+t_1)^{-\alpha}(1+t_2)^{-\beta}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{27}$$

provided that $\mathbb{R}(\alpha)>0$, $\mathbb{R}(\beta)>0$, $\zeta\neq\xi$ and the constants t_1 and t_2 are such that none of the expression $1+t_1$, $1+t_2$, $[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]$, (where $\xi\leq\theta\leq\zeta$) is non zero.

2. Main Results

We have establish an intriguing integral, which is explicity written in terms of Srivastava and Daoust functions by introducing the product of orthogonal polynomials with generalized Wright function with proper arguments in the integrand of (27).

Theorem 2.1. *Let α and β such that $\mathbb{R}(\alpha)>0$, $\mathbb{R}(\beta)>0$, $\zeta\neq\xi$ and the constants t_1 and t_2 are such that none of the expression $1+t_1$, $1+t_2$, $[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]$, (where $\xi\leq\theta\leq\zeta$) is non zero, then the underlying integral holds:*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}\, W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times P_n^{(\lambda,\,\sigma)}\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$= \frac{\Gamma(\alpha)\;\Gamma(\beta)\;\Gamma(\mu)}{\Gamma(\alpha+\beta)\;(\zeta-\xi)\;(1+t_1)^{\alpha}\;(1+t_2)^{\beta}}\, F_{5:0;1}^{5:0;0}\left[\begin{matrix}(\gamma:1,1), & (1+\lambda:1,1), & (1+\lambda+\sigma:1,2), \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1),\end{matrix}\right.$$
$$\left.\begin{matrix}(\alpha:1,2), & (\beta:1,2), & \_; & \_; \\ (1+\lambda+\sigma:1,1), & (\alpha+\beta:2,4), & \_; & (1+\lambda:1);\end{matrix}\;\frac{w}{1+t_1+t_2+t_1t_2},\frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{28}$$

where $W_{\nu,\mu}^{\gamma,\delta}(w)$ and $P_n^{(\lambda,\,\sigma)}(w)$ are known as generalized Wright function and Jacobi polynomial defined in (2) and (11) respectively, and $\Delta(p;\lambda)$ abbreviates the arrangement of p parameters $\frac{\lambda}{p}.\frac{\lambda+1}{p}...\frac{\lambda+p-1}{p}$ and $p\geq 1$.

Proof. In order to obtain our main result of Theorem 2.1, we indicate the left hand side of Theorem 2.1 by I, writing $W^{\gamma;\delta}_{\nu,\mu}(w)$ and $P_n^{(\lambda,\ \sigma)}(w)$ in their summation formula in the integrand with the help of equation (2) and (11) respectively, we have

$$I = \int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}$$

$$\times \sum_{n=0}^{\infty} \frac{(\theta-\xi)^n(\zeta-\theta)^n\ (\gamma)_n\ (w)^n}{\Gamma(\nu n+\mu)\ [(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{2n}\ (\delta)_n\ n!}$$

(29)
$$\times \sum_{r=0}^{n} \frac{(1+\lambda+\sigma)_{n+r}\ (1+\lambda)_n\ (\theta-\xi)^r\ (\zeta-\theta)^r\ (-1)^r}{(n-r)!\ r!\ (1+\lambda)_r\ (1+\lambda+\sigma)_n\ [(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{2r}}\ d\theta.$$

The above equation (29) directly follows from the fact using the following Lemma (see, [24])

(30)
$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} C(r,n) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} C(r,n+r),$$

and exchanging the order of integration and summation, we get

$$I = \sum_{r,n=0}^{n} \frac{(\gamma)_{n+r}\ (1+\lambda)_{n+r}\ (1+\lambda+\sigma)_{n+2r}\ (w)^{n+r}\ (-1)^r}{(\delta)_{n+r}\ \Gamma(\nu(n+r)+\mu)\ (n+r)!\ n!\ r!\ (1+\lambda)_r\ (1+\lambda+\sigma)_{n+r}}$$

$$\times \int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha+n+2r-1}(\zeta-\theta)^{\beta+n+2r-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta+2n+2r}}\ d\theta$$

applying the result(27) and after some simplification , we get

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\mu)}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}\ \Gamma(\alpha+\beta)}$$

(31)
$$\times \sum_{r,n=0}^{n} \frac{(\gamma)_{n+r}\ (1+\lambda)_{n+r}\ (1+\lambda+\sigma)_{n+2r}\ (\alpha)_{n+2r}\ (\beta)_{n+2r}\ \left(\frac{w}{1+t_1+t_2+t_1t_2}\right)^n \left(\frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right)^r}{\Gamma(\nu(n+r)+\mu)\ (\delta)_{n+r}\ (1)_{n+r}\ (1+\lambda+\sigma)_{n+r}\ (\alpha+\beta)_{2n+4r}\ (1+\lambda)_r\ n!\ r!}$$

Finally, after summing up and some simplification, the above series with the help of (26) can be easily obtained our main result (28). This completes the proof of Theorem 2.1.

Theorem 2.2. *Let α and β such that $\mathbb{R}(\alpha)>0$, $\mathbb{R}(\beta)>0$, $\zeta\neq\xi$ and the constants t_1 and t_2 are such that none of the expression $1+t_1$, $1+t_2$, $[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]$, (where $\xi\leq\theta\leq\zeta$) is non zero, then the underlying integral holds:*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times L_n^{(\lambda)}\left[\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$= \frac{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(\mu)}{\Gamma(\alpha+\beta)\ (\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}} F_{4:0;1}^{4:0;0}\left[\begin{matrix} (\gamma:1,1), & (1+\lambda:1,1), & (\alpha:1,2), \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), \end{matrix}\right.$$
$$\left.\begin{matrix} (\beta:1,2), & \_; & \_; \\ (\alpha+\beta:2,4), & \_; & (1+\lambda:1); \end{matrix} \ \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{32}$$

Proof. In order to obtain above Theorem 2.2 , we indicate the left hand side of (32) by I^* and using (2) and (19), we have

$$I^* = \int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}$$
$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_n\ (\theta-\xi)^n(\zeta-\theta)^n\ (w)^n}{(\delta)_n\ \Gamma(\nu n+\mu)\ [(\zeta-\xi)+t_1(\theta-\xi)+t_1(\zeta-\theta)]^{2n}\ n!}$$
$$\times \sum_{r=0}^{n} \frac{(1+\lambda)_n\ (\theta-\xi)^r\ (\zeta-\theta)^r\ (-1)^r}{(n-r)!\ r!\ (1+\lambda)_r\ [(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{2r}}\ d\theta. \tag{33}$$

Now applying the result (30) in (33) and then by exchanging the order of integration and summation , we get

$$I^* = \sum_{r,n=0}^{n} \frac{(\gamma)_{n+r}\ (1+\lambda)_{n+r}\ (w)^{n+r}(-1)^r}{(\delta)_{n+r}\ \Gamma(\nu(n+r)+\mu)\ (1+\lambda)_r\ (n+r)!\ r!\ n!}$$
$$\times \int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha+n+2r-1}(\zeta-\theta)^{\beta+n+2r-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta+2n+2r}}\ d\theta$$

using (27) , we have

$$I^* = \frac{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(\mu)}{\Gamma(\alpha+\beta)\ (\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}}$$
$$\times \sum_{r,\ n=0}^{n} \frac{(\gamma)_{n+r}\ (1+\lambda)_{n+r}\ (\alpha)_{n+2r}\ (\beta)_{n+2r}\ \left(\frac{w}{1+t_1+t_2+t_1t_2}\right)^n \left(\frac{-w}{(1+t_1+t_1+t_1t_1)^2}\right)^r}{\Gamma(\nu(n+r)+\mu)\ (\delta)_{n+r}\ (1)_{n+r}\ (\alpha+\beta)_{2n+4r}\ (1+\lambda)_r\ n!\ r!}$$

Finally, with the help of (26), summing up the above series, we can obtain our main result (32). This completes the proof of Theorem 2.2.

3. Particular Cases

By making use of Theorem 2.1 and Theorem 2.2, we established some fascinating integral formulas associated with the product of Ultraspherical, Gegenbauer, Tchebicheff, Legendre, Hermite polynomials and Whittaker function with generalized Wright function as particular cases of our main results.

Corollary 3.1. *On taking σ=λ in Theorem 2.1 and with the help of* (13), *the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times P_n^{(\lambda,\ \lambda)}\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$=\frac{\Gamma(\mu)}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}}\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,F_{5:0;1}^{5:0;0}\left[\begin{matrix}(\gamma:1,1), & (1+\lambda:1,1), & (1+2\lambda:1,2),\\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1),\end{matrix}\right.$$

(34)

$$\left.\begin{matrix}(\alpha:1,2) & (\beta:1,2), & \_; & \_;\\ (1+2\lambda:1,1), & (\alpha+\beta:2,4), & \_; & (1+\lambda:1);\end{matrix}\ \frac{w}{1+t_1+t_2+t_1t_2},\frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right].$$

Corollary 3.2. *On taking $\lambda=\sigma=\eta-\frac{1}{2}$ in Theorem 2.1 with the help of* (14), *the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times C_n^{\eta}\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$=\frac{\Gamma(\mu)\ (2\eta)_n}{(\eta+\frac{1}{2})_n\ (\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}}\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,F_{5:0;1}^{5:0;0}\left[\begin{matrix}(\gamma:1,1), & (\eta+\frac{1}{2}:1,1), & (2\eta:1,2),\\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1),\end{matrix}\right.$$

(35)

$$\left.\begin{matrix}(\alpha:1,2) & (\beta:1,2), & \_; & \_;\\ (2\eta:1,1), & (\alpha+\beta:2,4), & \_; & (\eta+\frac{1}{2}:1);\end{matrix}\ \frac{w}{1+t_1+t_2+t_1t_2},\frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right]$$

Corollary 3.3. *On replacing* $\lambda = \sigma = -\frac{1}{2}$ *in Theorem 2.1 and with the help of* (15), *the underlying integral holds:*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right] \times T_n\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$

$$= \frac{\Gamma(\mu)\ n!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}\ (\frac{1}{2})_n} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F_{5:0;1}^{5:0;0}\left[\begin{array}{llllllll} (\gamma:1,1), & (\frac{1}{2}:1,1), & (0:1,2), & (\alpha:1,2) & (\beta:1,2), & \_; & \_; & \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (0:1,1), & (\alpha+\beta:2,4), & \_; & (\frac{1}{2}:1); & \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{36}$$

Corollary 3.4. *On replacing* $\lambda = \sigma = \frac{1}{2}$ *in Theorem 2.1 with the help of* (16), *the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right] \times U_n\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$

$$= \frac{\Gamma(\mu)\ (n+1)!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}\ (\frac{3}{2})_n} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F_{5:0;1}^{5:0;0}\left[\begin{array}{llllllll} (\gamma:1,1), & (\frac{3}{2}:1,1), & (2:1,2), & (\alpha:1,2) & (\beta:1,2), & \_; & \_; & \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (2:1,1), & (\alpha+\beta:2,4), & \_; & (\frac{3}{2}:1); & \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{37}$$

Corollary 3.5. *On taking* $\sigma = \lambda = 0$ *in Theorem 2.1 and with the help of* (17), *the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right] \times P_n\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$

$$= \frac{\Gamma(\mu)}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ F^{5:0;0}_{5:0;1} \left[\begin{array}{llllll} (\gamma:1,1), & (1:1,1), & (1:1,2), & (\alpha:1,2) & (\beta:1,2), & \_; \ \ \_; \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (1:1,1), & (\alpha+\beta:2,4), & \_; \ (1:1); \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2} \right] \tag{38}$$

Corollary 3.6. *On taking $\lambda = -\frac{1}{2}$ in Theorem 2.2 and using the relation (20), the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}\ W^{\gamma,\delta}_{\nu,\mu}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right] \times H_{2n}\left[\frac{\sqrt{(\theta-\xi)(\zeta-\theta)}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]}\right] d\theta$$

$$= \frac{\Gamma(\mu)\ 2^{2n}(-1)^n\ n!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ F^{4:0;0}_{4:0;1} \left[\begin{array}{lllll} (\gamma:1,1), & (\frac{1}{2}:1,1), & (\alpha:1,2), & (\beta:1,2), & \_; \ \ \_; \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (\alpha+\beta:2,4), & \_; \ (\frac{1}{2}:1); \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2} \right] \tag{39}$$

Corollary 3.7. *On taking $\lambda = \frac{1}{2}$ in Theorem 2.2 and using the relation (21), the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-\frac{3}{2}}(\zeta-\theta)^{\beta-\frac{3}{2}}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta-1}}\ W^{\gamma,\delta}_{\nu,\mu}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right] \times H_{2n+1}\left[\frac{\sqrt{(\theta-\xi)(\zeta-\theta)}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]}\right] d\theta$$

$$= \frac{\Gamma(\mu)\ 2^{2n+1}(-1)^n\ n!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ F^{4:0;0}_{4:0;1} \left[\begin{array}{lllll} (\gamma:1,1), & (\frac{3}{2}:1,1), & (\alpha:1,2), & (\beta:1,2), & \_; \ \ \_; \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (\alpha+\beta:2,4), & \_; \ (\frac{3}{2}:1); \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2} \right] \tag{40}$$

Corollary 3.8. *On applying the relation* (22) *in Theorem 2.2, then the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-\frac{\lambda}{2}-\frac{3}{2}}(\zeta-\theta)^{\beta-\frac{\lambda}{2}-\frac{3}{2}}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta-\lambda-1}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$

$$\times e^{\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]}} M_{n+\frac{1}{2}+\frac{\lambda}{2},\frac{\lambda}{2}}\left[\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$

$$= \frac{\Gamma(\mu)\ 2^{\lambda+1}\ n!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}(1+\lambda)_n} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F_{4:0;1}^{4:0;0}\left[\begin{array}{llllll} (\gamma:1,1), & (1+\lambda:1,1), & (\alpha:1,2), & (\beta:1,2), & \_; & \_; \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (\alpha+\beta:2,4), & \_; & (1+\lambda:1); \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{41}$$

Corollary 3.9. *On applying the relation* (23) *in Theorem 2.2, the underlying integral holds :*

$$\int_{\xi}^{\zeta} \frac{(\theta-\xi)^{\alpha-\frac{\lambda}{2}-\frac{3}{2}}(\zeta-\theta)^{\beta-\frac{\lambda}{2}-\frac{3}{2}}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta-\lambda-1}} W_{\nu,\mu}^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$

$$\times e^{\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]}} W_{n+\frac{1}{2}+\frac{\lambda}{2},\frac{\lambda}{2}}\left[\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$

$$= \frac{\Gamma(\mu)\ 2^{\lambda+1}\ n!}{(\zeta-\xi)\ (1+t_1)^{\alpha}\ (1+t_2)^{\beta}(-1)^n} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F_{4:0;1}^{4:0;0}\left[\begin{array}{llllll} (\gamma:1,1), & (1+\lambda:1,1), & (\alpha:1,2), & (\beta:1,2), & \_; & \_; \\ (\Delta(\nu;\mu):1,1), & (\delta:1,1), & (1:1,1), & (\alpha+\beta:2,4), & \_; & (1+\lambda:1); \end{array} \frac{w}{1+t_1+t_2+t_1t_2}, \frac{-w}{(1+t_1+t_2+t_1t_2)^2}\right] \tag{42}$$

4. Conclusion

The formulas derived in this paper confirm the usefulness of the integral transform techniques used to deal with the application of special functions in different domains. The paper aims at presenting the study of finite integrals associated with Jacobi and Laguerre polynomials with generalized Wright function. Also, it is shown that the generalized Wright function, Jacobi and Laguerre polynomials can be expressed in terms of Fox-H function, Meijer G-function, Mittag-Leffler function, Ultraspherical polynomials, Gegenbauer polynomials, Tchebicheff polynomials of first and second kind, Legendre

polynomials, Hermite polynomials and Whittaker function after some suitable parametric replacement. It is pointed out that with the help of Theorem 2.1 and Theorem 2.2, we can also establish some results involving the product of auxiliary functions $M_\nu^{\gamma,\delta}(w)$, and $F_\nu^{\gamma,\delta}(x)$ with other special functions and orthogonal polynomials.
To give an example, we derive the following integrals:

(i) On replacing ν with $-\nu$, w with $-w$ and $\mu = -(1-\nu)$ in (28) and with the help of (5), we get an interesting integral involving the Wright auxiliary function $M_\nu^{\gamma,\delta}(w)$ and Jacobi polynomials:

$$\int_\xi^\zeta \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}\, M_\nu^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times\, P_n^{(\lambda,\,\sigma)}\left[1-\frac{2(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(1-\nu)}{\Gamma(\alpha+\beta)\,(\zeta-\xi)\,(1+t_1)^\alpha\,(1+t_2)^\beta}\, F^{5:0;0}_{5:0;1}\left[\begin{matrix}(\gamma:1,1), & (1+\lambda:1,1), & (1+\lambda+\sigma:1,2),\\ (\Delta(-\nu;(1-\nu)):1,1), & (\delta:1,1), & (1:1,1),\end{matrix}\right.$$

(43)

$$\left.\begin{matrix}(\alpha:1,2), & (\beta:1,2), & \_; & \_; \\ (1+\lambda+\sigma:1,1), & (\alpha+\beta:2,4), & \_; & (1+\lambda:1);\end{matrix}\ \frac{-w}{1+t_1+t_2+t_1t_2}, \frac{w}{(1+t_1+t_2+t_1t_2)^2}\right]$$

(ii) Similarly, on replacing ν with $-\nu$, w with $-w$ and $\mu = 0$ in (32) and with the help of (6), we get an interesting integral involving the Wright auxillary function $F_\nu^{\gamma,\delta}(w)$ and Laguerre polynomials:

$$\int_\xi^\zeta \frac{(\theta-\xi)^{\alpha-1}(\zeta-\theta)^{\beta-1}}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^{\alpha+\beta}}\, F_\nu^{\gamma,\delta}\left[\frac{w(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]$$
$$\times\, L_n^{(\lambda)}\left[\frac{(\theta-\xi)(\zeta-\theta)}{[(\zeta-\xi)+t_1(\theta-\xi)+t_2(\zeta-\theta)]^2}\right]d\theta$$
$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)\,(\zeta-\xi)\,(1+t_1)^\alpha\,(1+t_2)^\beta}\, F^{4:0;0}_{4:0;1}\left[\begin{matrix}(\gamma:1,1), & (1+\lambda:1,1), & (\alpha:1,2),\\ (\Delta(-\nu;0):1,1), & (\delta:1,1), & (1:1,1),\end{matrix}\right.$$

(44)

$$\left.\begin{matrix}(\beta:1,2), & \_; & \_; \\ (\alpha+\beta:2,4), & \_; & (1+\lambda:1);\end{matrix}\ \frac{-w}{1+t_1+t_2+t_1t_2}, \frac{w}{(1+t_1+t_2+t_1t_2)^2}\right]$$

This shows that the findings presented here, being general, can be utilized to yield several known and new integrals which may be potentially useful in various fields of Applied Mathematics, Mathematical Physics, Engineering Sciences, etc.

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