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A Zadeh's max-min composition operator for two triangular fuzzy numbers defined on $\mathbb{R}$

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There are many results for Zadeh's max-min composition operators based on Zadeh's extension principle. We calculate Zadeh's max-min composition operators for two triangular fuzzy numbers defined on $\mathbb{R}$.

Keywords: max-min composition operator, non-positive triangular fuzzy numbers.

MSC: 03E72, 47S40

1 Introduction

In fuzzy set theory, various types of operations between two fuzzy sets have been defined and studied. There are many results for Zadeh's max-min composition operators based on Zadeh's extension principle. We calculated max-min composition operators for two generalized triangular fuzzy sets [7], for two generalized quadratic fuzzy sets [6], and for two generalized trapezoidal fuzzy sets [4]. And we extended the above results to 2-dimensional case. We proved Zadeh's extension principle for 2-dimensional triangular fuzzy numbers [2]. We calculated Zadeh's max-min composition operator for two 2-dimensional quadratic fuzzy numbers [1], parametric operations between 2-dimensional triangular fuzzy number and trapezoidal fuzzy set [3] and algebraic operations for two generalized 2-dimensional quadratic fuzzy sets [5]. The fuzzy sets in the above results [7, 6, 4] were defined on $\mathbb{R}^+$.

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In this paper, we calculate Zadeh's max-min composition operators for two triangular fuzzy numbers defined on $\mathbb{R}$. Precisely, for six positive numbers a, b, c, p, q, r, we calculate the operators between two non-positive triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$ defined in Definition 2.5.

2 Zadeh′s max-min composition operators

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0, 1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0, 1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A *fuzzy set* A on X is a function μ_A from X to the interval $[0, 1]$. The function is called the *membership function* of A.

Definition 2.2. The set $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$ is said to be the *α-cut* of a fuzzy set A.

Definition 2.3. A fuzzy set A on $\mathbb{R}$ is *convex* if

$$\mu_A(\lambda x_1 + (1 - \lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in \mathbb{R}, \quad \forall \lambda \in [0, 1].$$

Definition 2.4. A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if (1) There exists exactly one $x \in \mathbb{R}$ such that $\mu_A(x) = 1$,

(2) $\mu_A(x)$ is piecewise continuous.

Definition 2.5. A *triangular fuzzy number* on $\mathbb{R}$ is a fuzzy number A which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_3 \leq x, \\ \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x < a_2, \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.6. [8] The *extended addition* $A(+)B$, *extended subtraction* $A(-)B$, *extended multiplication* $A(\cdot)B$ and *extended division* $A(/)B$ are fuzzy sets with membership functions defined as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z = x * y} \min\{\mu_A(x), \mu_B(y)\}, \quad (* = +, -, \cdot, /)$$

Remark 2.1. Let A and B be fuzzy sets and $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B, A(-)B$, $A(\cdot)B$ and $A(/)B$ can be calculated as follows.

(1) $(A(+)B)_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$.
(2) $(A(-)B)_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$.
(3) $(A(\cdot)B)_\alpha = [\min(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}),$
$\max(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)})]$.
(4) $(A(/)B)_\alpha = [\min(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}),$
$\max(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)})]$.

3 Main results

In this section, for six positive real numbers a, b, c, p, q, r, we consider two triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$. The other cases can be calculated similarly.

Theorem 3.1. *Let $-a < -p$, $c > r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 > \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$.*

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \quad c < x, \\ \frac{1}{a-b}(x+a), & -a \le x < -b, \\ \frac{1}{c+b}(c-x) & -b \le x < c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \quad r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(r-x) & q \le x < r. \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. From $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b) - a, -\alpha(c+b) + c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p) - p, -\alpha(r-q) + r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p \le x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x < c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r \le x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x < c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), \\ &\qquad (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p)$ and $\mu_{A(\cdot)B}(-bq) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r) \le x < -bq, \\ \frac{bp+cp+bq+cq-\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+bq+cq)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x < (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p). \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), \\ &\qquad (-\alpha(c+b) + c) \cdot (-\alpha(r-q) + r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r)$ and $x = (-\alpha_2(c+b) + c) \cdot (-\alpha_2(r-q) + r)$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot (-\alpha_1(r-q) + r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ (\alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r) \le x \\ \qquad < (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r), \\ \frac{-cq+rb+rc+\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)} \\ (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p) \le x \\ \qquad < (-\alpha_2(c+b) + c) \cdot (-\alpha_2(r-q) + r). \end{cases}$$

(3) $0 < \alpha \le \alpha_2$

There are two cases (i) and (ii).

(i) By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), \\ &\qquad (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ \qquad (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p) \le x < cr. \end{cases}$$

(ii) By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), \\ &\qquad (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $(\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : (1) $\alpha_1 < \alpha \le 1$

By the above facts,
$A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left[\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right].$
Thus $\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $x = \frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}$,
$\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p} \le x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le \frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}. \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$

By the above facts, $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] = \left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{\alpha(q+p)-p}\right].$

$\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $\frac{-\alpha_1(c+b)+c}{\alpha_1(q+p)-p}$. Therefore

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \leq \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)}, \\ \frac{px+c}{(q+p)x+c+b}, & \frac{(-\alpha_1(c+b)+c)}{(\alpha_1(q+p)-p)} \leq x < \infty. \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$. □

Theorem 3.2. *Let $-a < -p$, $c > r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 = \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$.*

Proof. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b) - a, -\alpha(c+b) + c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p) - p, -\alpha(r-q) + r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \ \ -a-p \leq x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \leq x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \leq x < c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \ \ -a-r \leq x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \leq x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \leq x < c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \leq 1$

By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), \\ &\qquad (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)},\\ \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)\leq x<-bq,\\ \frac{bp+cp+be+ce-\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+bq+cq)},\\ -bq\leq x<(-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p).\end{cases}$$

(2) $0<\alpha\leq\alpha_1$

There are two cases (i) and (ii).

(i) By the above fact,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha&=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_2^{(\alpha)}\cdot b_2^{(\alpha)}]\\&=[\alpha(a-b)-a)\cdot(-\alpha(r-q)+r),\\&\qquad(-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=0$ on the interval $[-ar,cr]^c$ and $\mu_{A(\cdot)B}(x)=\alpha_1$ at $x=(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x=(-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)},\\ -ar\leq x<(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r),\\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)},\\ (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)\leq x<cr.\end{cases}$$

(ii) By the above fact,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha&=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_1^{(\alpha)}\cdot b_1^{(\alpha)}]\\&=[(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r),\\&\qquad(\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=0$ on the interval $[-ar,ap]^c$ and $\mu_{A(\cdot)B}(x)=\alpha_1$ at $x=(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x=(\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)},\\ -ar\leq x<(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r),\\ \frac{2ap+aq-bp-\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)},\\ (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)\leq x<ap.\end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : If $\alpha_1<\alpha\leq 1$, by the above fact,
$A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p},\frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$. $\mu_{A(/)B}(\frac{-b}{q})=1$. Therefore,

$$\mu_{A(/)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & -\infty<x\leq\frac{-b}{q},\\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q}\leq x\leq 0.\end{cases}$$

Hence $A(/)B$ has values in $(\alpha_2,1]$ on $\mathbb{R}$. □

Theorem 3.3. *Let $-a < -p$, $c > r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 < \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.*

Proof. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively, Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b) - a, -\alpha(c+b) + c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p) - p, -\alpha(r-q) + r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p \le x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x < c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r \le x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x < c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$

By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), \\ &\qquad (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = \alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r)$ and $x = (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+be+ce)}, \\ -bq \le x < (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p). \end{cases}$$

(2) $\alpha_1 < \alpha \le \alpha_2$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), \\ &\qquad (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x \\ \qquad < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp+\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x \\ \qquad < (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(3) $0 < \alpha \le \alpha_1$

There are two cases (i) and (ii).

(i) By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), \\ &\qquad (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \le x < cr. \end{cases}$$

(ii) By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), \\ &\qquad (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)},\\ -ar\le x<(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r),\\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)},\\ (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)\le x<ap.\end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : If $\alpha_1<\alpha\le 1$, by the above fact,
$A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p},\frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$. $\mu_{A(/)B}(\frac{-b}{q})=1$. Therefore,

$$\mu_{A(/)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & -\infty<x\le\frac{-b}{q},\\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q}\le x\le 0.\end{cases}$$

Hence $A(/)B$ has values in $(\alpha_1,1]$ on $\mathbb{R}$. □

4 Conclusion

We have computed Zadeh's max-min composition operator for two non-positive triangular fuzzy numbers A and B. In the case of $B_0\subset A_0$, we got three kinds of conclusions according to the three magnitude relationship between $\mu_A(0)$ and $\mu_B(0)$, i.e., $\mu_A(0)>\mu_B(0)$, $\mu_A(0)=\mu_B(0)$ and $\mu_A(0)<\mu_B(0)$. For each case, $A(+)B$ and $A(-)B$ were triangular fuzzy numbers, and $A(\cdot)B$ was a slightly distorted triangular fuzzy number, but $A(/)B$ was a different type of fuzzy number. In conclusion, $A(+)B$, $A(-)B$, $A(\cdot)B$ can be applied where the shape of the triangular fuzzy number comes out, and $A(/)B$ can be applied where appropriate.

Firstly, in the case of $\mu_A(0)>\mu_B(0)$, the fuzzy number defined on the whole real number can be applied to a place with the value of the closed interval $[\mu_B(0),1]$. In other words, it can be applied to the case where the membership function of a fuzzy number F is $\mu_F(\cdot):\mathbb{R}\to(\mu_B(0),1]$.

Secondly, in the case of $\mu_A(0)=\mu_B(0)$, the fuzzy number defined on the interval $(-\infty,0]$ can be applied to the place with the value of the closed interval $[\mu_B(0),1]$. In other words, it can be applied to the case where the membership function of a fuzzy number F is $\mu_F(\cdot):(-\infty,0]\to[\mu_B(0),1]$.

Finally, in the case of $\mu_A(0)<\mu_B(0)$, the fuzzy number defined on the interval $(-\infty,0]$ can be applied to the place with the value of the closed interval $[\mu_A(0),1]$. In other words, it can be applied to the case where the membership function of a fuzzy number F is $\mu_F(\cdot):(-\infty,0]\to[\mu_A(0),1]$.

Remark 4.1. We calculated Zadeh's max-min composition operator for two non-positive triangular fuzzy numbers $A=(-a,-b,c)$ and $B=(-q,p,r)$ for six positive numbers a,b,c,p,q,r. Our results were obtained for the case of $-a<-p$ and $r<c$.

Similar results can be obtained when $-a < -p$ and $c < r$, although the calculation is complex.

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