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THE SINGULARITIES FOR BIHARMONIC PROBLEM WITH CORNER SINGULARITIES

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Abstract. In [8, 9] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities. They consider the Poisson equations with corner singularities, compute the finite element solutions using standard Finite Element Methods and use the extraction formula to compute the stress intensity factor(s), then they posed new PDE with a regular solution by imposing the nonhomogeneous boundary condition using the computed stress intensity factor(s), which converges with optimal speed. From the solution they could get an accurate solution just by adding the singular part. The error analysis was given in [5]. In their approaches, the singular functions and the extraction formula which give the stress intensity factor are the basic elements. In this paper we consider the biharmonic problems with the cramped and/or simply supported boundary conditions and get the singular functions and its duals and find properties of them, which are the cornerstones of the approaches of [8, 9, 10].

1. Introduction

We let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$ and consider the following biharmonic equation

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{1}$$

supplemented by the boundary conditions:

$$\begin{cases} u &= u_n = 0 & \text{on } \partial\Omega_C, \\ u &= M(u) = 0 & \text{on } \partial\Omega_S, \end{cases} \tag{2}$$

where $f \in L^2(\Omega)$ and Δ stands for the Laplacian operator and for $\sigma \in (0, 1)$

$$M(u) = u_{nn} - \sigma u_{tt}. \tag{3}$$

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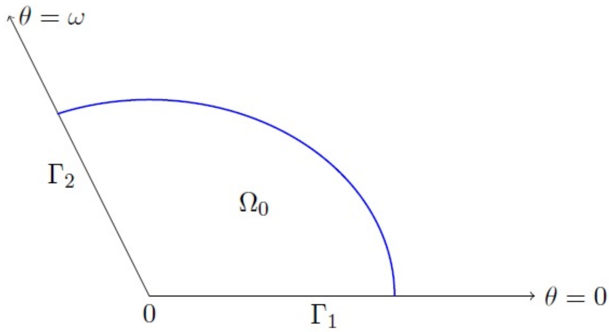

FIGURE 1. $\Omega_0 = \{(r, \theta) : 0 < \theta < \omega,\ 0 < r < 1\}$

Here, σ is the Poisson ratio for the plate and u_n and u_t denote the derivatives of u with respect to the outer normal direction, n, and the tangential direction, t, along $\partial\Omega$. The moments given in the boundary condition are defined in [11].

For simplicity, we assume that there is only one corner with the inner angle $w : \frac{\pi}{2} < \omega \leq 2\pi$ and it satisfies C/S boundary condition as in **Figure 1**.

For the partial differential equation defined on the concave domain and/or with mixed boundary condition, it is well-known that we have domain singularities, which raise the lack of regularity and affect the accuracy of the finite element approximation. Among the many ways to overcome this problem, there have been a way which use the fact that the solution has a singular function representation.([1, 3, 4, 6, 7])

Recently, in [10], they considered the biharmonic equations on cracked domains with clamped (or simply supported or free) boundary conditions along the crack faces and derived formulas extracting stress intensity factors. For this approach the derivation of the singular function(s) and dual singular function(s) is essential steps. In this paper we consider the biharmonic equations on nonconvex domains with mixed boundary condition and give the derivation of the singular function(s) and dual singular function(s) for the several special inner angles.

We identify the singular function(s) and dual singular function(s) and introduce the extraction formula for three cases: $\omega = \pi$, $\frac{3\pi}{2}$, and 2π, respectively.

To isolate the singularity we use cut-off functions which have the following properties; for $0 < a < b < 1$,

$$\eta(a) = 1, \quad \eta'(a) = 0, \quad \eta''(a) = 0, \quad \eta^{(3)}(a) = 0, \quad \eta^{(4)}(a) = 0,$$

and

$$\eta(b) = 0, \quad \eta'(b) = 0, \quad \eta''(b) = 0, \quad \eta^{(3)}(b) = 0, \quad \eta^{(4)}(b) = 0,$$

and $\eta(r) = 1$ for $0 < r < a$, and, $\eta(r) = 0$ for $r > b$, so that η satisfies

$$\int_a^b \eta'(r)dr = -1, \qquad \int_a^b r\eta''(r)dr = 1,$$
$$\int_a^b r^2\eta^{(3)}(r)dr = -2, \qquad \int_a^b r^3\eta^{(4)}(r)dr = 6.$$

For the approach we are considering, we use a cut-off functions η with $a = 0.75$ and $b = 1$ and another cut-off function $\hat{\eta}$ with $a = 0.5$ and $b = 0.75$ as follows; Let

$$\Phi_5(x) = (1-x)^5(1 + 5x + 15x^2 + 35x^3 + 70x^4)$$

and η and $\hat{\eta}$ be the $\mathcal{C}^4$-continuous cut-off functions defined by

$$\eta(r,\theta) = \begin{cases} 1 & \text{if } r \le 0.75 \\ \Phi_5(4r-3) & \text{if } 0.75 \le r \le 1 \\ 0 & \text{if } 1 \le r, \end{cases} \tag{4}$$

$$\hat{\eta}(r,\theta) = \begin{cases} 1 & \text{if } r \le 0.5 \\ \Phi_5(4r-2) & \text{if } 0.5 \le r \le 0.75 \\ 0 & \text{if } 0.75 \le r. \end{cases} \tag{5}$$

Note that $\eta = 1$ on the support of $\hat{\eta}$.

2. Regularity and singular functions for biharmonic problem

Let $u_s(r,\theta) = r^z\Psi(\theta)$ be one of the singular part of the solution u near the corner such that $z \in (1,2)$. Then it is known that the equation $\Delta^2 u_s = 0$ implies $\Psi(\theta)$ satisfies the following differential equation:

$$\Psi^{(4)}(\theta) + (2z^2 - 4z + 4)\Psi^{(2)}(\theta) + (z^4 - 4z^3 + 4z^2)\Psi(\theta) = 0.$$

In the case $1 < z < 2$, the general solution of this equation can be written as follows:

$$\Psi(\theta) = A\sin z\theta + B\cos z\theta + C\sin(2-z)\theta + D\cos(2-z)\theta.$$

We assume the boundary condition C/S, that is, u_s satisfies the crampted condition on $\Gamma_1(\theta = 0)$ and the simply supported condition on $\Gamma_2(\theta = \omega)$.(see Figure 1.)

Therefore the regularity z can be computed by the following equation(See [2]):

$$\sin(z-1)2\omega - (z-1)\sin 2\omega = 0, \tag{6}$$

If some specific angle $\omega = \omega_0$ is given, then we can find singular function(s) and the corresponding dual singular function(s) by finding $z \in (1,2)$ satisfying (6) with $\omega = \omega_0$.

We consider the three cases and list the singular functions and the corresponding dual singular functions for each case.

2.1. Case 1 : $\omega = \pi$

From (6) we have regularity $z = \frac{3}{2}$ for $\omega = \pi$, and we have one singular function and corresponding dual singular function;

$$s = s_1 = r^{\frac{3}{2}}(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}), \qquad s^* = s_1^* = r^{\frac{1}{2}}(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}). \tag{7}$$

2.2. Case 2 : $\omega = \frac{3\pi}{2}$

From (6) we have regularities $z = \frac{4}{3}$ and $\frac{5}{3}$ for $\omega = \frac{3\pi}{2}$, and we have two singular functions

$$s_1 = r^{\frac{4}{3}}(\sin\frac{4\theta}{3} - 2\sin\frac{2\theta}{3}), \qquad s_2 = r^{\frac{5}{3}}(\cos\frac{5\theta}{3} - \cos\frac{\theta}{3}), \tag{8}$$

and corresponding dual singular functions

$$s_1^* = r^{\frac{2}{3}}(\sin\frac{4\theta}{3} - 2\sin\frac{2\theta}{3}), \qquad s_2^* = r^{\frac{1}{3}}(\cos\frac{5\theta}{3} - \cos\frac{\theta}{3}). \tag{9}$$

2.3. Case 3 : $\omega = 2\pi$

From (6) we have regularities $z = \frac{5}{4}, \frac{3}{2}$, and $\frac{7}{4}$ for $\omega = 2\pi$, and we have three singular functions

$$s_1 = r^{\frac{5}{4}}(\cos\frac{5\theta}{4} - \cos\frac{3\theta}{4}), \quad s_2 = r^{\frac{3}{2}}(\sin\frac{3\theta}{2} - 3\sin\frac{\theta}{2}), \quad s_3 = r^{\frac{7}{4}}(\cos\frac{7\theta}{4} - \cos\frac{\theta}{4}), \tag{10}$$

and corresponding dual singular functions

$$s_1^* = r^{\frac{3}{4}}(\cos\frac{5\theta}{4} - \cos\frac{3\theta}{4}),\ s_2^* = r^{\frac{1}{2}}(\sin\frac{3\theta}{2} - 3\sin\frac{\theta}{2}), s_3^* = r^{\frac{1}{4}}(\cos\frac{7\theta}{4} - \cos\frac{\theta}{4}). \tag{11}$$

Define the index set Λ as follows;

$$\Lambda = \begin{cases} \{1\} & \text{if} \quad \omega = \pi \\ \{1,2\} & \text{if} \quad \omega = \frac{3\pi}{2} \\ \{1,2,3\} & \text{if} \quad \omega = 2\pi. \end{cases} \tag{12}$$

First we have some simple properties of the singular functions s_i and the dual singular functions s_i^*;

Lemma 2.1. *If $u(r,\theta)$ is harmonic, then $r^2 \cdot u(r,\theta)$ is biharmonic.*

Proof. First we recall that

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Let $w = w(r,\theta) = r^2 \cdot u(r,\theta)$, then

$$\begin{aligned}\Delta w = \Delta(r^2 u) &= (r^2u)_{rr} + \frac{1}{r}(r^2u)_r + \frac{1}{r^2}(r^2u)_{\theta\theta} \\ &= r^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) + 4u + 4ru_r = 4u + 4ru_r.\end{aligned}$$

So, we have

$$\frac{1}{4}\Delta^2 w = \frac{1}{4}\Delta^2(r^2 u) = \Delta u + \Delta(r u_r) = 0 + (r u_r)_{rr} + \frac{1}{r}(r u_r)_r + \frac{1}{r^2}(r u_r)_{\theta\theta}$$
$$= (r u_{rr} + u_r + \frac{1}{r} u_{\theta\theta})_r + \Delta u = (r\Delta u)_r + \Delta u = 0,$$

and the lemma since u is harmonic function. □

Lemma 2.2. *The singular functions s_i and the dual singular functions s_i^*, given in* (7) $\sim$ (11), *are biharmonic;*

$$\Delta^2 s_i = \Delta^2 s_i^* = 0, \tag{13}$$

for each $i \in \Lambda$.

Proof. Each of the singular functions s_i and the dual singular functions s_i^* is a combination of the following functions;

$$r^\alpha \sin\alpha\theta, \qquad r^\alpha \cos\alpha\theta, \qquad r^{2-\alpha}\sin\alpha\theta, \qquad r^{2-\alpha}\cos\alpha\theta,$$

with α being a suitable real value.

The first two functions are harmonic because they are the imaginary and real part of the analytic function z^α, so they are biharmonic. Similarly we note that $r^{-\alpha}\sin\alpha\theta$ and $r^{-\alpha}\cos\alpha\theta$ are harmonic, which implies that the last two functions are biharmonic together with Lemma 2.1. □

3. Properties of the singular functions and their duals

In this section we give the cornerstones for the extraction formula for the three cases: $\omega = \pi$, $\frac{3\pi}{2}$, and 2π.

Suppose that the solution u of (1) is given as

$$u = w + \sum_{i\in\Lambda} \lambda_i s_i \hat{\eta},$$

then we have

$$\Delta^2 w + \sum_{i\in\Lambda} \lambda_i \Delta^2(s_i\hat{\eta}) = f.$$

After multiflying $s_j^*\eta$ to both side and integrating, we have

$$< \Delta^2 w, s_j^*\eta > + \sum_{i\in\Lambda} \lambda_i < \Delta^2(s_i\hat{\eta}), s_j^*\eta > = < f, s_j^*\eta > .$$

Note that all the integrands in the following three propositions are non-zero only at $(r,\theta) \in [a,b] \times (0,\omega)$ in polar coordinate and we need to compute $\int_a^b \int_0^\omega \Delta^2(s\hat{\eta})\, s^*\, r d\theta dr$ to get $< \Delta^2(s\hat{\eta}), s^* >$.

Proposition 3.1. *If $\omega = \pi$, then the singular function and their dual singular function in* (7) satisfy the following :

$$< \Delta^2(s\hat{\eta}), s^*\eta > = < \Delta^2(s\hat{\eta}), s^* > = A_1, \tag{14}$$

where $A_1 = 4\pi$.

Proof. The first equality comes from the fact that $\eta = 1$ on the support of $\hat{\eta}$. Note that $s = r^{3/2}(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2})$ and $s^* = r^{1/2}(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2})$. Now, using the fact that $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$, we have, for $(r,\theta) \in [a,b] \times (0,\omega)$

$$\Delta(s\hat{\eta}) = -2r^{-\frac{1}{2}}\cos\frac{\theta}{2}\ \hat{\eta}(r) - 4r^{\frac{1}{2}}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}(4\hat{\eta}'(r) + r\hat{\eta}''(r)),$$

and

$$\begin{aligned}\Delta^2(s\hat{\eta}) &= -2\Delta(r^{-\frac{1}{2}}\cos\frac{\theta}{2}\ \hat{\eta}(r)) - 4\Delta(r^{\frac{1}{2}}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}(4\hat{\eta}'(r) + r(\hat{\eta}''(r)))\\ &= -4r^{-\frac{3}{2}}\cos\frac{\theta}{2}(-2+4\cos\theta)\ \hat{\eta}'(r) - 4r^{-\frac{1}{2}}\cos\frac{\theta}{2}((5-4\cos\theta)\,\hat{\eta}''(r))\\ &- 32r^{\frac{1}{2}}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\ \hat{\eta}^{(3)}(r) - 4r^{\frac{3}{2}}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\ \hat{\eta}^{(4)}(r).\end{aligned}$$

Now we can conclude $< \Delta^2(s\hat{\eta}), s^* >= 4\pi$ by multiflying $s^* = r^{1/2}(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2})$ to the above equation and using the properties of $\hat{\eta}$ and the follow integrals;

$$\int_0^\pi \cos\frac{\theta}{2}\cos\frac{3\theta}{2}d\theta = 0, \qquad \int_0^\pi \cos\frac{\theta}{2}\cos\theta\cos\frac{3\theta}{2}\,d\theta = \frac{\pi}{4},$$
$$\int_0^\pi \cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\cos\frac{3\theta}{2}\,d\theta = -\frac{\pi}{8}, \qquad \int_0^\pi \cos^2\frac{\theta}{2}d\theta = \frac{\pi}{2},$$
$$\int_0^\pi \cos^2\frac{\theta}{2}\cos\theta d\theta = \frac{\pi}{4}, \qquad \int_0^\pi \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\,d\theta = \frac{\pi}{8}.$$

□

Proposition 3.2. *If $\omega = 3\pi/2$, then the singular functions in* (8) and their dual singular functions in (9) satisfy the following :

$$< \Delta^2(s_i\hat{\eta}), s_j^*\eta >=< \Delta^2(s_i\hat{\eta}), s_j^* >= A_i\delta_{ij}, \tag{15}$$

where $A_1 = A_2 = 8\pi$.

Proof. Note that we have two singular functions $s_1 = r^{4/3}\left(\sin\frac{4\theta}{3} - 2\sin\frac{2\theta}{3}\right)$ and $s_2 = r^{5/3}\left(\cos\frac{5\theta}{3} - \cos\frac{\theta}{3}\right)$ and by the similar methods in the proof of Proposition 3.1 we have

$$\begin{aligned}\Delta(s_1\hat{\eta}) &= -\frac{4}{3}r^{-\frac{2}{3}}\sin\frac{2\theta}{3}\left(r\sin^2\frac{\theta}{3}\ (3r\hat{\eta}''(r) + 11\hat{\eta}'(r)) + 2\hat{\eta}(r)\right)\\ &= -\frac{8}{3}r^{-\frac{2}{3}}\sin\frac{2\theta}{3}\ \hat{\eta}(r) - \frac{44}{3}r^{-\frac{2}{3}}\sin\frac{2\theta}{3}\sin^2\frac{\theta}{3}\ r\hat{\eta}'(r)\\ &- 4r^{-\frac{2}{3}}\sin\frac{2\theta}{3}\sin^2\frac{\theta}{3}\ r^2\hat{\eta}''(r)\end{aligned}$$

and

$$\begin{aligned}\Delta(s_2\hat{\eta}) &= \frac{2}{3}r^{-\frac{1}{3}}\cos\frac{\theta}{3}\left(-2r\sin^2\frac{\theta}{3}\left(2\cos\frac{2\theta}{3}+1\right)(3r\hat{\eta}''(r)+13\hat{\eta}'(r))-4\hat{\eta}(r)\right)\\ &= -\frac{8}{3}r^{-\frac{1}{3}}\cos\frac{\theta}{3}\,\hat{\eta}(r)-\frac{52}{3}r^{-\frac{1}{3}}\cos\frac{\theta}{3}\sin^2\frac{\theta}{3}\left(2\cos\frac{2\theta}{3}+1\right)r\hat{\eta}'(r)\\ &\quad -4r^{-\frac{1}{3}}\cos\frac{\theta}{3}\sin^2\frac{\theta}{3}\left(2\cos\frac{2\theta}{3}+1\right)r^2\hat{\eta}''(r).\end{aligned}$$

By applying the Laplace operator again we get

$$\begin{aligned}\Delta^2(s_1\hat{\eta}) &= -\frac{10}{9}r^{-\frac{5}{3}}\sin\frac{2\theta}{3}(-3+11\cos\frac{2\theta}{3})\hat{\eta}'(r)-\frac{2}{9}r^{-\frac{5}{3}}\sin\frac{2\theta}{3}(79-55\cos\frac{2\theta}{3})r\hat{\eta}''(r)\\ &\quad -\frac{264}{9}r^{-\frac{5}{3}}\sin\frac{2\theta}{3}\sin^2\frac{\theta}{3}r^2\hat{\eta}^{(3)}(r)-4r^{-\frac{5}{3}}\sin\frac{2\theta}{3}\sin^2\frac{\theta}{3}r^3\hat{\eta}^{(4)}(r)\end{aligned}$$

and

$$\begin{aligned}\Delta^2(s_2\hat{\eta}) &= -\frac{14}{9}r^{-\frac{4}{3}}\cos\frac{\theta}{3}(-13\cos\frac{2\theta}{3}+13\cos\frac{4\theta}{3}+8)\hat{\eta}'(r)\\ &\quad -\frac{2}{9}r^{-\frac{4}{3}}\cos\frac{\theta}{3}(91\cos\frac{2\theta}{3}-91\cos\frac{4\theta}{3}+24)r\hat{\eta}''(r)\\ &\quad -\frac{104}{3}r^{-\frac{4}{3}}\cos\frac{\theta}{3}\sin^2\frac{\theta}{3}\left(2\cos\frac{2\theta}{3}+1\right)r^2\hat{\eta}^{(3)}(r)\\ &\quad -4r^{-\frac{4}{3}}\cos\frac{\theta}{3}\sin^2\frac{\theta}{3}\left(2\cos\frac{2\theta}{3}+1\right)r^3\hat{\eta}^{(4)}(r).\end{aligned}$$

Now after a series of computations similar to those in the proof of Proposition 3.1, we may conclude $<\Delta^2(s_i\hat{\eta}), s_j^*>= A_i\delta_{ij}$ where $A_1 = A_2 = 8\pi$. □

Proposition 3.3. *If* $\omega = 2\pi$, *then the singular functions in* (10) and their dual singular functions in (11) satisfy the following :

$$<\Delta^2(s_i\hat{\eta}), s_j^*\eta>=<\Delta^2(s_i\hat{\eta}), s_j^*>= A_i\delta_{ij}, \tag{16}$$

where $A_1 = 4\pi$, $A_2 = 24\pi$, $A_3 = 12\pi$.

Proof. We may have the proof by the similar way to those of Proposition 3.1 and Proposition 3.2. □

4. Stress intensity factors and their extraction formula

Now we may introduce the extraction formula which gives the stress intensity factor using the three propositions given in the previous section. We only need the following lemma, which can be obtained by a similar method to that of Lemma 3.5 in paper [10].

Lemma 4.1. *For* $w \in H^3(\Omega)$, *we have*

$$<\Delta^2 w, s_j^*\eta>=<w, \Delta^2(s_j^*\eta)> . \tag{17}$$

Proof. See Lemma 3.5 in paper [10]. □

Theorem 4.2. *The stress intensity factors λ_j can be expressed in terms of u and f by the following extraction formula:*

$$\lambda_j = \frac{1}{A_j}\int_\Omega f\eta s_j^* dx + \frac{1}{A_j}\int_\Omega u\Delta^2(\eta s_j^*)dx, \quad j \in \Lambda. \tag{18}$$

Proof. The proof is a direct consequence of the Proposition 3.1, Proposition 3.2, Proposition 3.3 and Lemma 4.1. □

For example, if we consider the biharmonic problem (1) defined on a domain with a corner point with inner angle $\omega = \frac{3\pi}{2}$ together with the boundary condition (2), then we have singularity there, with two singular functions given in (8). The solution u of the problem (1) has the following form

$$u = w + \lambda_1 s_1 \hat{\eta} + \lambda_2 s_2 \hat{\eta},$$

and the coefficient λ_i, called by 'the stress intensity factor', can be obtained by

$$\lambda_j = \frac{1}{8\pi}\int_\Omega f\eta s_j^* dx + \frac{1}{8\pi}\int_\Omega u\Delta^2(\eta s_j^*)dx, \quad (j = 1, 2),$$

where the dual singular functions s_j^* are given in (9).

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