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2-DIMENSIONAL EXPANSION OF QUADRATIC FUZZY NUMBERS THROUGH CALCULATION AND GRAPH

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Abstract. We compute the extended four operations of the 2-dimensional quadratic fuzzy numbers. Then we calculate the intersection between a plane perpendicular to the x-axis, which passes through each vertex, and the resulting 2-dimensional quadratic fuzzy number. We confirm that the equations of the two intersections acquired in this way and the graphs are actually identical, respectively.

1. Introduction

We calculated the operators of various types of fuzzy sets such as triangular, quadratic and trapezoidal (see [1, 3, 8]). We generalized the results of 1-dimensional fuzzy set to 2-dimensional fuzzy set in [2, 4, 5, 6]. We also generalized triangular fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued α-cuts, we obtained parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$ in [1]. We also demonstrated that 2-dimensional Zadeh's max-min operator constitutes the generalization of 1-dimensional Zadeh's max-min operator in [2, 6].

In this paper, with the result of the extended computation of a 2-dimensional quadratic fuzzy number being a 1-dimensional extension, we try to visually confirm it using a graph. Taking the examples of two 2-dimensional quadratic fuzzy sets, we obtain the equations of the the intersections between planes perpendicular to the x-axis, which passes through each vertex, and two 2-dimensional quadratic fuzzy numbers. Then, the extended four operations of the two 1-dimensional quadratic fuzzy sets are calculated and graphed. Meanwhile, we compute the extended four operations of the 2-dimensional quadratic fuzzy numbers, which are the two examples above. Then we calculate the intersection between a plane perpendicular to the x-axis, which passes through each

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vertex, and the resulting 2-dimensional quadratic fuzzy number. We confirm that the equations of the two intersections acquired in this way and the graphs are actually identical, respectively.

2. Preliminaries

Let X be a set. We define α-cut and α-set of the fuzzy set A with the membership function $\mu_A(x)$.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_\alpha = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ if $\alpha = 0$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Definition 2.2. ([9]) The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \ * = +, -, \cdot, /$$

We defined the parametric operations for two fuzzy numbers defined on $\mathbb{R}$ and showed that the results for parametric operations are the same as those for the extended operations. For this, we proved that for all fuzzy numbers A and all $\alpha \in [0,1]$, there exists a piecewise continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $A_\alpha = \{f_\alpha(t) | t \in [0,1]\}$. If A is continuous, then the corresponding function $f_\alpha(t)$ is also continuous. The corresponding function $f_\alpha(t)$ is said to be the *parametric α-function* of A. The parametric α-function of A is denoted by $f_\alpha(t)$ or $f_A(t)$.

Theorem 2.3. ([1]) *Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and $f_A(t), f_B(t)$ be the parametric α-functions of A and B, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-cuts as follows.*
(1) parametric addition $A(+)_pB : (A(+)_pB)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0,1]\}$
(2) parametric subtraction $A(-)_pB : (A(-)_pB)_\alpha = \{f_A(t) - f_B(1-t) \mid t \in [0,1]\}$
(3) parametric multiplication $A(\cdot)_pB : (A(\cdot)_pB)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0,1]\}$
(4) parametric division $A(/)_pB : (A(/)_pB)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}$

Theorem 2.4. ([1]) *Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_pB = A(+)B, A(-)_pB = A(-)B, A(\cdot)_pB = A(\cdot)B$ and $A(/)_pB = A(/)B$.*

Definition 2.5. ([7]) A *quadratic fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \beta \leq x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2 + 1, & \alpha \leq x < \beta, \end{cases}$$

where $a > 0$.

The above quadratic fuzzy number is denoted by $A = [\alpha, k, \beta]$.

Theorem 2.6. ([5]) *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Definition 2.7. ([5]) Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$
$$B^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric operations defined by parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-sets as the followings.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi. \end{cases}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, define

$$(A(*)_pB)^0 = \lim_{\alpha \to 0^+} (A(*)_pB)^\alpha \quad \text{and} \quad (A(*)_pB)^1 = \lim_{\alpha \to 1^-} (A(*)_pB)^\alpha,$$

where $* = +,\ -,\ \cdot,\ /$.

Definition 2.8. ([2]) A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b > 0$ is called the *2-dimensional quadratic fuzzy number* and denoted by $[a,\ x_1,\ b,\ y_1]^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1)$ $(k \in \mathbb{R})$ are symmetric quadratic fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(1-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Theorem 2.9. ([2]) *Let $A = [a_1,\ x_1,\ b_1,\ y_1]^2$ and $B = [a_2,\ x_2,\ b_2,\ y_2]^2$ be two 2-dimensional quadratic fuzzy numbers. Then we have the following.*

(1) $A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2.$

(2) $A(-)_pB = \Big[a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big]^2.$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 2-dimensional quadratic fuzzy numbers.

3. 2-dimensional quadratic fuzzy set

In this section, taking the examples of two 2-dimensional quadratic fuzzy sets, we obtain the equations of the the intersections between planes perpendicular to the x-axis, which passes through each vertex, and two 2-dimensional quadratic fuzzy numbers. Then, the extended four operations of the two 1-dimensional quadratic fuzzy sets are calculated and graphed. Meanwhile, we compute the extended four operations of the 2-dimensional quadratic fuzzy numbers, which are the two examples above. Then we calculate the intersection between a plane perpendicular to the x-axis, which passes through each vertex, and the resulting

2-dimensional quadratic fuzzy number. We confirm that the equations of the two intersections acquired in this way and the graphs are actually identical, respectively.

Let $A^2 = [6,\ 10,\ 8,\ 16]^2$ and $B^2 = [4,\ 8,\ 5,\ 12]^2$. The graphs of μ_{A^2} and μ_{B^2} are as follows:

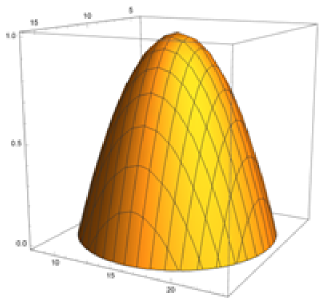

FIGURE 1. $\mu_{A^2}(x, y)$

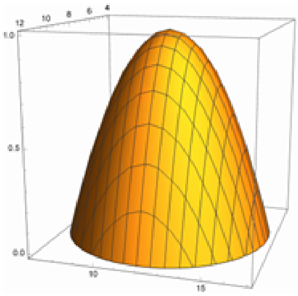

FIGURE 2. $\mu_{B^2}(x, y)$

The intersections of A^2, B^2 and planes perpendicular to the x-axis, which passes through each vertex are called A and B, respectively. The membership functions of A and B are $\mu_A = \frac{-1}{64}(y-16)^2 + 1$ and $\mu_B = \frac{-1}{25}(y-12)^2 + 1$.

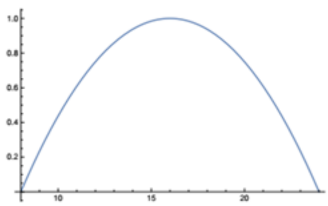

FIGURE 3. $\mu_A(y)$

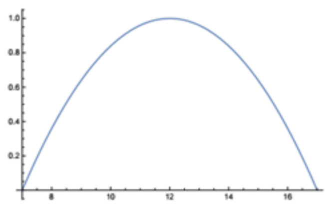

FIGURE 4. $\mu_B(y)$

We calculate exactly the above four operations using α-cuts. Let $A_\alpha = \left[{a_1}^{(\alpha)}, {a_2}^{(\alpha)}\right]$ and $B_\alpha = \left[{b_1}^{(\alpha)}, {b_2}^{(\alpha)}\right]$ be the α-cuts of A and B, respectively. Then we have

$$A_\alpha = \left[16 - 8\sqrt{1-\alpha}, 16 + 8\sqrt{1-\alpha}\right], B_\alpha = \left[12 - 5\sqrt{1-\alpha}, 12 + 5\sqrt{1-\alpha}\right]$$

(1) Addition: Since $A_\alpha\,(+)\,B_\alpha = \left[28 - 13\sqrt{1-\alpha}, 28 + 13\sqrt{1-\alpha}\right]$, we have

$$\mu_{A(+)B}\,(y) = \begin{cases} 0, & y \leq 15, 41 \leq y \\ \frac{1}{169}\left(-615 + 56y - y^2\right), & 15 \leq y \leq 41 \end{cases}$$

By Theorem 2.9, $A(+)_p B = [10, 18, 13, 28]^2$. Thus

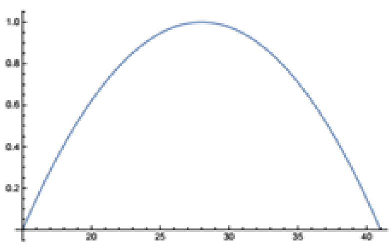

FIGURE 5. $\mu_{A(+)B}(y)$

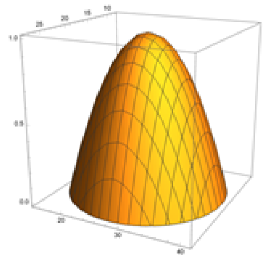

FIGURE 6. $\mu_{A(+)B}(x,y)$

$$\mu_{A(+)B}(x,y)=\begin{cases}1-\left(\frac{(x-18)^2}{10^2}+\frac{(y-28)^2}{13^2}\right), & \\ \qquad\qquad\text{if}\quad 13^2(x-18)^2+10^2(y-28)^2\le 10^2 13^2 \\ 0, \qquad\quad\text{if}\quad 13^2(x-18)^2+10^2(y-28)^2> 10^2 13^2\end{cases}$$

Substituting $x=18$ into $\mu_{A(+)B}(x,y)$,

$$\mu_{A(+)B}(18,y)=\begin{cases}1-\left(\frac{(y-28)^2}{13^2}\right), & (y-28)^2\le 13^2 \\ 0, & (y-28)^2>13^2\end{cases}$$

This result indicates that $\mu_{A(+)B}(y)$ and $\mu_{A(+)B}(18,y)$ match. The section cut parallel to the x-axis at the vertex of Figure 6 is shown in Figure 7, and this section is shown in Figure 5.

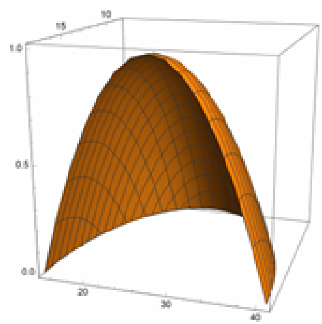

FIGURE 7. $\mu_{A(+)B}(18,y)$

(2) Subtraction: Since $A_\alpha\,(-)\,B_\alpha=\left[4-13\sqrt{1-\alpha},4+13\sqrt{1-\alpha}\right]$, we have

$$\mu_{A(-)B}(y)=\begin{cases}0, & y\le -19, 17\le y \\ \frac{1}{169}\left(153+8y-y^2\right), & -9\le y\le 17\end{cases}$$

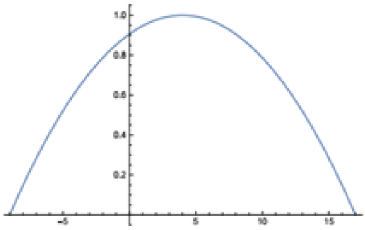

FIGURE 8. $\mu_{A(-)B}(y)$

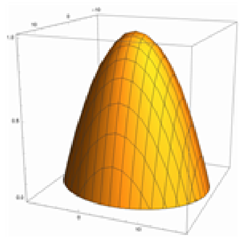

FIGURE 9. $\mu_{A(-)B}(x,y)$

By theorem 2.9, $A(-)_pB = [10, 2, 13, 4]^2$. Thus

$$\mu_{A(-)B}\,(x,y) = \begin{cases} 1 - \left(\frac{(x-2)^2}{10^2} + \frac{(y-4)^2}{13^2}\right), & \\ & \text{if} \quad 13^2(x-2)^2 + 10^2(y-4)^2 \leq 10^2 13^2 \\ 0, & \text{if} \quad 13^2(x-2)^2 + 10^2(y-4)^2 > 10^2 13^2 \end{cases}$$

Substituting $x = 2$ into $\mu_{A(-)B}(x,y)$,

$$\mu_{A(-)B}\,(2,y) = \begin{cases} 1 - \left(\frac{(y-4)^2}{13^2}\right), & (y-4)^2 \leq 13^2 \\ 0, & (y-4)^2 > 13^2 \end{cases}$$

This result indicates that $\mu_{A(-)B}\,(y)$ and $\mu_{A(-)B}\,(2,y)$ match. The section cut parallel to the x-axis at the vertex of Figure 9 is shown in Figure 10, and this section is shown in Figure 8.

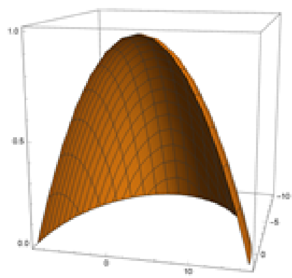

FIGURE 10. $\mu_{A(-)B}(2,y)$

(3) Multiplication: Since $A_\alpha\,(\times)\,B_\alpha = [(16 - 8\sqrt{1-\alpha})\,(12 - 5\sqrt{1-\alpha})\,,$ $(16 + 8\sqrt{1-\alpha})\,(12 + 5\sqrt{1-\alpha})]$, we have

$$\mu_{(A\times B)}\,(y) = \begin{cases} 0, & y \leq 56, 408 \leq y \\ \frac{1}{200}\left(-776 - 5y + 44\sqrt{2}\sqrt{8+5y}\right), & 56 \leq y \leq 408 \end{cases}$$

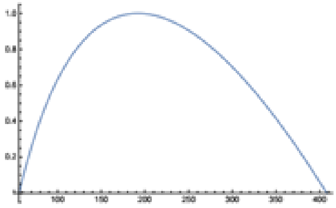

FIGURE 11. $\mu_{A(\times)B}(y)$

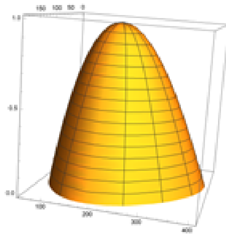

FIGURE 12. $\mu_{A(\times)B}(x,y)$

By theorem 2.9, $\left(A\left(\times\right)_p B\right)^\alpha = \{(x_a\left(t\right), y_a\left(t\right)) \mid 0 \leq t \leq 2\pi\}$, where

$$\begin{cases} x_\alpha\left(t\right) = 80 + 88\sqrt{1-\alpha}\cos t + 24\left(1-\alpha\right)\cos^2 t \\ y_\alpha\left(t\right) = 192 + 176\sqrt{1-\alpha}\sin t + 40\left(1-\alpha\right)\sin^2 t \end{cases}$$

Since $x_\alpha\left(\frac{3\pi}{2}\right) = 80$ and $y_\alpha\left(\frac{3\pi}{2}\right) = 192 - 176\sqrt{1-\alpha} + 40\left(1-\alpha\right)$, we have

$$\alpha = \frac{1}{200}\left(-776 - 5\left(y_\alpha\left(\frac{3\pi}{2}\right)\right) + 44\sqrt{2}\sqrt{8 + 5\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)}\right)$$

This result indicates that $\mu_{A(\times)B}\left(y\right)$ and $\mu_{A(\times)B}\left(80, y\right)$ match. The section cut parallel to the x-axis at the vertex of Figure 12 is shown in Figure 13, and this section is shown in Figure 11.

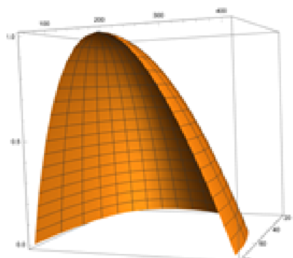

FIGURE 13. $\mu_{A(\times)B}(80,y)$

(4) Division: Since $A_\alpha\left(/\right)B_\alpha = \left[\frac{16-8\sqrt{1-\alpha}}{12+5\sqrt{1-\alpha}}, \frac{16+8\sqrt{1-\alpha}}{12-5\sqrt{1-\alpha}}\right]$, we have

$$\mu_{A(/)B}\left(y\right) = \begin{cases} 0, & y \leq \frac{8}{17}, \frac{24}{7} \leq y \\ \frac{-192+464y-119y^2}{(8+5y)^2}, & \frac{8}{17} \leq y \leq \frac{24}{7} \end{cases}$$

By Theorem 2.9, $\left(A\left(/\right)_p B\right)^\alpha = \{(x_a\left(t\right), y_a\left(t\right)) \mid 0 \leq t \leq 2\pi\}$, where

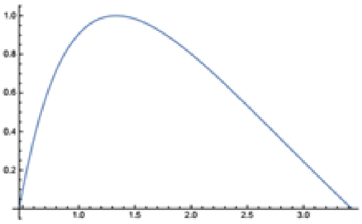

FIGURE 14. $\mu_{A(/)B}(y)$

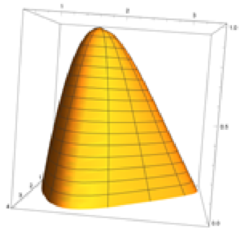

FIGURE 15. $\mu_{A(/)B}(x,y)$

$$x_\alpha\left(t\right)=\frac{10+6\sqrt{1-\alpha}\cos t}{8-4\sqrt{1-\alpha}\cos t},\quad y_\alpha\left(t\right)=\frac{16+8\sqrt{1-\alpha}sint}{12-5\sqrt{1-\alpha}sint}$$

Since $x_\alpha\left(\frac{3\pi}{2}\right)=\frac{10}{8}$ and $y_\alpha\left(\frac{3\pi}{2}\right)=\frac{16-8\sqrt{1-\alpha}}{12+5\sqrt{1-\alpha}}$, we have

$$\alpha=\frac{-192+464\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)-119\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)^2}{\left(8+5\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)\right)^2}$$

This result indicates that $\mu_{A(/)B}\left(y\right)$ and $\mu_{A(/)B}\left(\frac{10}{8},y\right)$ match. The section cut parallel to the x-axis at the vertex of Figure 15 is shown in Figure 16, and this section is shown in Figure 14.

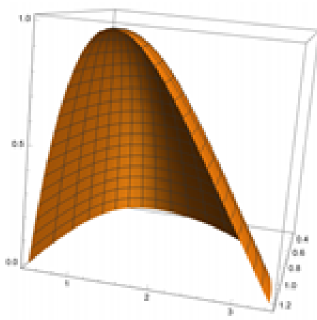

FIGURE 16. $\mu_{A(/)B}(\frac{10}{8},y)$

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