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GRADED w-NOETHERIAN MODULES OVER GRADED RINGS

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Abstract. In this paper, we study the basic theory of the category of graded w-Noetherian modules over a graded ring R. Some elementary concepts, such as w-envelope of graded modules, graded w-Noetherian rings and so on, are introduced. It is shown that: (1) A graded domain R is graded w-Noetherian if and only if $R^g_{\mathfrak{m}}$ is a graded Noetherian ring for any gr-maximal w-ideal $\mathfrak{m}$ of R, and there are only finite numbers of gr-maximal w-ideals including a for any nonzero homogeneous element a. (2) Let R be a strongly graded ring. Then R is a graded w-Noetherian ring if and only if R_e is a w-Noetherian ring. (3) Let R be a graded w-Noetherian domain and let $a \in R$ be a homogeneous element. Suppose $\mathfrak{p}$ is a minimal graded prime ideal of (a). Then the graded height of the graded prime ideal $\mathfrak{p}$ is at most 1.

1. Introduction

In 1978, HOM and EXT are introduced for the category of graded modules by Goto and Watanabe in [6] and [7]. Since then the study of homological methods on graded rings have received a good deal of attention in the literature (refer to [4, 17, 24]). Multiplicative ideal theory plays an important role in characterizations of the ring structure.

In recent years, there are many references studying multiplicative ideal theory over graded rings (refer to [1–3, 16]). Traditionally, in these references, the assumption is required generally that R is a commutative Γ-graded ring, while Γ is a torsion-free cancellative commutative monoid. Sometimes it is required in most of these references that Γ-graded domains are integral domains. In 1979, D. F. Anderson in [1] proved that if G is a torsion-free group and every nonzero homogeneous element is a unit, the graded domain $R = \bigoplus_{\sigma\in G} R_\sigma$ is completely integrally closed. In 1982, D. D. Anderson and D. F. Anderson introduced the notions of graded Krull domains, graded GCD domains, and graded UFDs over a torsion-free monoid Γ (refer to [2]). In [16], M. H. Park

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proved the graded version of Mori-Nagata theorem: If R is a Γ-graded Noetherian domain and Γ is a torsion-free monoid, then the integral closure of R is a graded Krull domain. But it is worth noting that a torsion-free cancellative commutative monoid can embed into a commutative abelian group.

A few literatures deal with the star operation theory on graded domains. In 1980, the graded v-ideal I is introduced by D. D. Anderson and D. F. Anderson [3] for the case that Γ is a torsion-free cancellative monoid and R is a domain graded by Γ. In the paper, a graded v-ideal I means that I is both a graded ideal and a v-ideal. In [16], M. H. Park call a graded domain R (graded by a torsion-free cancellative monoid Γ) a graded SM domain if R satisfies the ascending chain condition on graded w-ideals, where a graded w-ideal I means that I is both a graded ideal and a w-ideal. In that paper, it is proved that the complete integral closure of a graded SM domain R is a graded Krull domain. Star operation theory provides many methods and techniques for multiplicative ideal theory on integral domains. Then the problem we face is that how to study star operations on graded rings. Especially, it is about the w-operation. Then let us trace the development of the w-operation in the ungraded case.

In [5], F. G. Wang and R. McCasland called a nonzero finitely generated ideal J a GV-ideal if $J^{-1} = R$ and a torsion-free module M a w-module if $Jx \subseteq M$, for $J \in \mathrm{GV}(R)$ and $x \in K \otimes_R M$, implies $x \in M$, where $\mathrm{GV}(R)$ is the set of GV-ideals of R, and K is the quotient field of R. The w-envelope of a torsion-free module M is defined as

$$M_w = \{x \in K \otimes_R M \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}.$$

In 2011, Yin *et al.* [23] generalized these notions to the case that R is a commutative ring with the help of functors Hom and Ext. Let R be a commutative ring. A finitely generated ideal J of R is called a GV-ideal if $\mathrm{Hom}_R(R/J, R) = 0$ and $\mathrm{Ext}^1_R(R/J, R) = 0$ for all $J \in \mathrm{GV}(R)$. An R-module M is called GV-torsion-free if $\mathrm{Hom}_R(R/J, M) = 0$ for all $J \in \mathrm{GV}(R)$. M is called a w-module if G is GV-torsion-free and $\mathrm{Ext}^1_R(R/J, M) = 0$ for all $J \in \mathrm{GV}(R)$. The w-envelope of a GV-torsion-free module M is defined as

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. Applying the w-operation, we can characterize many integral domains such as Krull domains and PVMDs and their categories of modules.

In [21], a graded commutative ring R is called a *graded domain* if nonzero homogenous elements of R are not zero-divisors and R is called a *graded field* if nonzero homogenous elements of R are units. Let S be the set of nonzero homogenous elements of a graded domain R. Then the localization $K := R_S$ at S is called the graded quotient field of R, which is certainly a graded field. Notice that a graded field is not necessary an integral domain (refer to [21, Example 2]). In [20], let M be a graded R-module. $t^{\mathrm{gr}}(M) = \{x \in$

$M \mid Jx = 0$ for some $J \in \mathrm{GV}_{\mathrm{gr}}(R)\}$, where $\mathrm{GV}_{\mathrm{gr}}(R)$ is the set of graded GV-ideals. A graded R-module M is called a *graded GV-torsion-free module* if $t^{\mathrm{gr}}(M) = 0$. In [20], graded w-modules are introduced. The authors called a graded GV-torsion-free module a *graded w-module* if $\mathrm{EXT}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}_{\mathrm{gr}}(R)$, and in [20], some characterizations of graded w-modules are stated. For example, a GV-torsion-free module M is a graded w-module if and only if for a graded w-module N containing M, if $Jx \subseteq M$ where $J \in \mathrm{GV}_{\mathrm{gr}}(R)$ and $x \in N$, then $x \in M$. Following graded w-modules, graded w-envelopes are introduced in §2, the equivalent characterizations of grade w-envelopes are given. There are many papers discussing graded Noetherian rings. Correspondingly, we give the definition of graded w-Noetherian rings in §3. Naturally, for the requirements of characterizing graded w-Noetherian rings, we introduce the notions of graded w-exact sequences, graded finite type modules, and graded finitely presented modules. Besides, in §4, graded w-modules over strongly graded rings are studied, and so as the application of graded w-module theory, we prove graded principal ideal theorem over w-Noetherian domains.

Notations. Throughout this paper, G denotes a multiplicative Abelian group with identity element e and $R = \bigoplus_{\sigma \in G} R_\sigma$ is a commutative G-graded ring with identity 1. We also call R a graded ring for short. Let $M = \bigoplus_{\sigma \in G} M_\sigma$ be a graded module. For $x \in M$, we can write as $x = \sum_\sigma x_\sigma$, where $x_\sigma \in M_\sigma$ and only a finite number of $x_\sigma \neq 0$. If $x \in M_\sigma$ for some $\sigma \in G$, then x is said to be homogeneous. We denote by $h(M)$ the set of homogeneous elements of M. Clearly, $h(M) = \bigcup_{\sigma \in G} M_\sigma$. Sometimes we write $\deg(x) = \sigma$ when $x \in M_\sigma$ is a nonzero homogeneous element. Let M be a graded module and N be a graded submodule of M. If for every nonzero homogeneous element $m \in M$, we have $N \bigcap Rm \neq 0$, in other words, there is an $a \in h(R)$ such that $am \neq 0$, then N is graded essential. $E^g(M)$ is the graded injective envelop of a graded R-module M (refer to [14]).

Let $N = \bigoplus_{\sigma \in G} N_\sigma$ be another graded module and put

$$\mathrm{Hom}_{R\text{-gr}}(M, N) = \{f \in \mathrm{Hom}_R(M, N) \mid f(M_\sigma) \subseteq N_\sigma, \forall \sigma \in G\}.$$

And the elements of $\mathrm{Hom}_{R\text{-gr}}(M, N)$ are said to be graded homomorphisms. The category of graded R-modules is denoted by R-gr, in which the objects are graded R-modules and the morphisms are graded homomorphisms.

Fix $\sigma \in G$, write $N(\sigma)_\tau = N_{\tau\sigma}$ for any $\tau \in G$. Then $N(\sigma) := \bigoplus_{\tau \in G} N(\sigma)_\tau$ is a graded R-module, which is said to be *σ-suspended*. Let N' be a graded R-module and let $\varphi : N \to N'$ be a graded homomorphism. Let $\varphi^\sigma : N(\sigma) \to N'(\sigma)$ such that $\varphi^\sigma(x) = \varphi(x)$, $x \in N$. Clearly, if $0 \to A \to B \to C \to 0$ is a graded exact sequence, then the induced sequence $0 \to A(\sigma) \to B(\sigma) \to C(\sigma) \to 0$ is also a graded exact sequence.

Let $f : M \to N$ be an R-homomorphism and $\sigma \in G$. If $f(M_\tau) \subseteq N(\tau\sigma) = N(\sigma)_\tau$ for any $\tau \in G$, then f is called a homomorphism with degree σ. Clearly, we have $f \in \mathrm{Hom}_{R\text{-gr}}(M, N(\sigma))$.

Let $\mathrm{HOM}_R(M,N)_\sigma = \mathrm{Hom}_{R\text{-gr}}(M,N(\sigma))$. Then

$$\mathrm{HOM}_R(M,N) := \bigoplus_{\sigma\in G} \mathrm{HOM}_R(M,N)_\sigma = \bigoplus_{\sigma\in G} \mathrm{Hom}_{R\text{-gr}}(M,N(\sigma))$$

is a graded R-module.

The right derived functor of $\mathrm{Hom}_{R\text{-gr}}(-,-)$ is denoted by $\mathrm{Ext}^n_{R\text{-gr}}(-,-)$. Let $\mathrm{EXT}^n_R(M,N)_\sigma = \mathrm{Ext}^n_{R\text{-gr}}(M,N(\sigma))$ and let

$$\mathrm{EXT}^n_R(M,N) = \bigoplus_{\sigma\in G} \mathrm{EXT}^n_R(M,N)_\sigma = \bigoplus_{\sigma\in G} \mathrm{Ext}^n_{R\text{-gr}}(M,N(\sigma)).$$

Then $\mathrm{EXT}^n_R(M,N)$ is also a graded R-module.

Any undefined notions or notation are standard, we can refer to [13, 14, 19].

2. Graded w-envelopes and graded localization

In the ungraded cases, there is a w-envelope for any GV-torsion-free module. As it follows, we discuss the graded w-envelope over graded commutative rings.

Definition 2.1. Let M be a graded GV-torsion-free module. Set

$$M^g_w = \{x \in E^g(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}_{\mathrm{gr}}(R)\},$$

which is called the *graded w-envelope* of M.

Naturally, if M is a graded GV-torsion-free module, then $M \subseteq M^g_w$. Apparently, $M^g_w = 0$ if and only if $M = 0$.

Theorem 2.2. *Let M be a graded R-module. If N is a graded w-submodule of $E^g(M)$ and $M \subseteq N$, then $M^g_w \subseteq N$. Therefore, M^g_w is the smallest graded w-submodule of $E^g(M)$ containing M.*

Proof. The proof is a similar manner as the ungraded cases (cf. [19, Theorem 6.2.2]). □

Proposition 2.3. *Let M be a graded GV-torsion-free module. Then M^g_w/M is a graded GV-torsion module.*

Proof. It's trivial. □

Theorem 2.4. *The following statements are equivalent for a graded* GV-*torsion-free module M:*

(1) *M is a graded w-module.*
(2) *Every graded homomorphism $f : A \to M$ can be extended to A^g_w for any graded* GV-*torsion-free module A.*
(3) $\mathrm{EXT}^1_R(A^g_w/A, M) = 0$ *for any graded* GV-*torsion-free module A.*
(4) $\mathrm{Ext}^1_{R\text{-gr}}(A^g_w/A, M) = 0$ *for any graded* GV-*torsion-free module A.*

Proof. (1)⇒(3) By [20, Theorem 3.8], the assertion follows.

(3)⇒(4)⇒(2) It is clear.

(2)⇒(1) For $J \in \mathrm{GV}_{\mathrm{gr}}(R)$, by Theorem 2.2, we have $J^g_w = R$. Then by [20, Theorem 3.8], we can finish the proof. □

Let M be a graded R-module. A proper graded submodule A of M is called a *graded prime submodule* of M if for $r \in h(R)$ and $x \in h(M)$, whenever $rx \in A$, then $x \in A$ or $rM \subseteq A$.

Theorem 2.5. *Let M be a graded module, M be a w-module, and N be a graded submodule of M. Then*:

(1) $N_w = N_w^g$.
(2) *If N is a maximal w-submodule of M, then N is a gr-maximal w-submodule of M. Especially, if A is a graded ideal of R and A is a maximal w-ideal of R, then A is a gr-maximal w-ideal of R.*

Proof. (1) By the definitions of w-envelopes as well as graded w-envelopes, we have $N_w^g \subseteq N_w$ since $E^g(N) \subseteq E(N)$. By [20, Theorem 3.17], we have N_w^g is a w-module, and thus $N_w \subseteq N_w^g$.

(2) Let N be a maximal w-submodule of M. According to [20, Theorem 3.17], N is a graded w-module of M. If A is a graded w-module containing N, then by [20, Theorem 3.17], A is also a w-submodule of M. So either $A = N$, or $A = M$. Therefore N is a gr-maximal w-submodule of M. $\square$

Let S be a multiplicative closed set of homogeneous elements of R, denote $S = \bigcup_{\sigma \in G} S_\sigma$, where $S_\sigma = R_\sigma \cap S$. Similarly to the ungraded case, let's construct a fractional ring R_S at S, where $R = \bigoplus_{\sigma \in G} R_\sigma$. By lucky coincidence, $R_S = \bigoplus_{\mu \in G} (R_S)_\mu$ is also a graded ring (cf. [13, §1.6]), where

$$(R_S)_\mu = \left\{ \frac{r}{s} \,|\, r \in R_\sigma,\, s \in S_\tau,\, \mu\tau = \sigma \right\}.$$

Moreover, we have M_S is a graded module over a graded ring R_S for any graded R-module M, and $M_S \cong R_S \otimes_R M$. Note that $S = h(R \setminus \mathfrak{p})$ is a multiplicative closed set of homogeneous elements of $R - \mathfrak{p}$, where $\mathfrak{p}$ is a graded prime ideal of R. Let's denote by $M_\mathfrak{p}^g$ the localization of graded R-module M at S. Also denote by gr-w-Max(R) the set of all gr-maximal w-ideals of R.

Theorem 2.6. *Let $\mathfrak{p}$ be a graded prime w-ideal and let M be a graded* GV-*torsion-free module. Then $M_\mathfrak{p}^g = (M_w^g)_\mathfrak{p}^g$.*

Proof. By Proposition 2.3, M_w^g/M is a graded GV-torsion module. According to [20, Theorem 3.15], we have $(M_w^g/M)_\mathfrak{p}^g = 0$, and therefore $(M_w^g)_\mathfrak{p}^g = M_\mathfrak{p}^g$. $\square$

If $\mathfrak{p}$ is a graded prime ideal of R, and $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$ is a chain of graded prime ideals of $\mathfrak{p}$, then we define the graded height of a graded prime ideal $\mathfrak{p}$, denoted by $\mathrm{ht}^g\, \mathfrak{p}$, which is the supremum of the lengths n of all strictly decreasing chains of graded prime ideals of $\mathfrak{p}$. Moreover, the graded Krull dimension of R is defined as

$$\dim^g(R) = \sup\{\mathrm{ht}^g\, \mathfrak{p} \,|\, \mathfrak{p} \text{ runs all graded prime ideals of } R\}.$$

Proposition 2.7. *Let I be a proper graded w-ideal of R and let $\mathfrak{p}$ be a minimal graded prime ideal of I. Then $\mathfrak{p}$ is a graded w-ideal.*

Proof. Assume $\mathfrak{p}$ is not a graded w-ideal. By Theorem 2.5 and [19, Theorem 6.2.9], we have $\mathfrak{p}_w^g = R$. Then there is $J \subseteq \mathfrak{p}$ for some $J \in \mathrm{GV}_{\mathrm{gr}}(R)$. Analogously to the ungraded case, $sJ^n \subseteq I$ for some positive integer n and some homogeneous element $s \in R \setminus \mathfrak{p}$. On the other hand, since $J^n \in \mathrm{GV}_{\mathrm{gr}}(R)$ and I is a graded w-ideal, we have $s \in I \subseteq \mathfrak{p}$, which is a contradiction. Thus $\mathfrak{p}$ is a graded w-ideal. $\square$

Theorem 2.8. *Let M be a graded w-module and let both A and B be graded submodules of M. Then $A_w^g = B_w^g$ if and only if $A_{\mathfrak{m}}^g = B_{\mathfrak{m}}^g$ for any $\mathfrak{m} \in$ gr-w-$\mathrm{Max}(R)$. Therefore, if both A and B are graded w-submodules of M, then $A = B$ if and only if $A_{\mathfrak{m}}^g = B_{\mathfrak{m}}^g$ for any $\mathfrak{m} \in$ gr-w-$\mathrm{Max}(R)$.*

Proof. Suppose $A_w^g = B_w^g$. By Theorem 2.6, $A_{\mathfrak{m}}^g = (A_w^g)_{\mathfrak{m}}^g = (B_w^g)_{\mathfrak{m}}^g = B_{\mathfrak{m}}^g$ for any gr-maximal w-ideal $\mathfrak{m}$ of R.

Conversely, suppose $x \in A_w^g$. Denote $I = (B_w^g : x) = \{r \in R \,|\, rx \in B_w^g\}$. Then I is a graded w-ideal of R. Notice that $(B_w^g)_{\mathfrak{m}}^g = B_{\mathfrak{m}}^g = A_{\mathfrak{m}}^g = (A_w^g)_{\mathfrak{m}}^g$ for any gr-maximal w-ideal $\mathfrak{m}$. Therefore

$$I_{\mathfrak{m}} = \{a \in R_{\mathfrak{m}}^g \,|\, ax \in (B_w^g)_{\mathfrak{m}}^g\} = \{a \in R_{\mathfrak{m}}^g \,|\, ax \in (A_w^g)_{\mathfrak{m}}^g\} = R_{\mathfrak{m}}^g.$$

Hence $I \not\subseteq \mathfrak{m}$ for any gr-maximal w-ideal $\mathfrak{m}$ of R. Since I is a graded w-ideal, we have $I = R$. Then we get $x \in B_w^g$, i.e., $A_w^g \subseteq B_w^g$. In the same way, we have $B_w^g \subseteq A_w^g$. Therefore $A_w^g = B_w^g$. $\square$

3. Graded w-Noetherian rings

We begin this section with the definitions of graded w-morphisms.

Definition 3.1. Let both M and N be graded R-modules and let $f : M \to N$ be a graded homomorphism. Then f is called a *graded w-monomorphism* (resp., a *graded w-epimorphism*, a *graded w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a graded monomorphism (resp., a graded epimorphism, a graded isomorphism) for any gr-maximal w-ideal $\mathfrak{m}$.

Definition 3.2. A sequence of graded modules with graded homomorphisms $A \to B \to C$ is called a *graded w-exact sequence* if $A_{\mathfrak{m}}^g \to B_{\mathfrak{m}}^g \to C_{\mathfrak{m}}^g$ is a graded exact sequence for any gr-maximal w-ideal $\mathfrak{m}$ of R.

Definition 3.3. Let M be an R-module.

(1) M is said to be a *graded finite type* module if there exists a graded w-exact sequence $F \to M \to 0$, where F is a finitely generated gr-free module.
(2) M is said to be a *graded finitely presented type* module if there exists a graded w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_0 and F_1 are finitely generated gr-free modules.

It is natural that every finitely generated graded (resp., graded finitely presented) module is of graded finite type (resp., graded finitely presented type). By definition, it is clear that if M is of graded finite type (resp., graded finitely presented type), then $M^g_{\mathfrak{m}}$ is a finitely generated graded (resp., graded finitely presented type) $R_{\mathfrak{m}}$-module for any gr-maximal w-ideal $\mathfrak{m}$ of R. Clearly, every graded GV-torsion module must be a graded finitely presented type module.

Recall that a graded R-module M is said to be *graded Noetherian* if every graded submodule of M is finitely generated graded. A graded ring R is said to be *graded Noetherian* if every graded ideal of R is finitely generated graded. For the general theory of graded Noetherian modules and graded Noetherian rings, the reader may consult [8–11, 15, 17, 18].

Definition 3.4. Let R be a graded ring and let M be a graded R-module. Then M is called a *graded w-Noetherian module* if every graded submodule of M is of graded finite type. And a graded ring R is called a *graded w-Noetherian ring* if R as a graded R-module is a graded w-Noetherian module.

Since every finitely generated graded module must be of graded finite type, every graded Noetherian module must be a graded w-Noetherian module and every graded Noetherian ring must be a graded w-Noetherian ring. Especially, every graded GV-torsion module must be a graded w-Noetherian module.

Proposition 3.5. (1) *Let M be a graded w-Noetherian module. Then $M^g_{\mathfrak{m}}$ is a graded Noetherian module for any gr-maximal w-ideal $\mathfrak{m}$ of R.*

(2) *Let $M_1, \dots, M_k$ be graded w-Noetherian modules. Then $\bigoplus_{i=1}^k M_i$ is a graded w-Noetherian module.*

(3) *Let R be a graded w-Noetherian ring. Then $R^g_{\mathfrak{m}}$ is a graded Noetherian ring for any gr-maximal w-ideal $\mathfrak{m}$ of R.*

Proof. (1) Let A be a graded submodule of $M^g_{\mathfrak{m}}$ and $B = \{b \in R \mid \frac{b}{1} \in A\}$. Then it's easy to prove that B is a graded submodule of M and $A = B^g_{\mathfrak{m}}$. From assumption, B is of graded finite type, i.e., there is a graded exact sequence $F \to B \to 0$, where F is a finitely generated gr-free module. According to the ungraded case, $F^g_{\mathfrak{m}}$ is a finitely generated gr-free $R^g_{\mathfrak{m}}$-module and the sequence $F^g_{\mathfrak{m}} \to B^g_{\mathfrak{m}} \to 0$ is exact. Therefore A is a finitely generated graded $R^g_{\mathfrak{m}}$-module. Thus $M^g_{\mathfrak{m}}$ is a graded Noetherian module.

(2) This follows analogously to the ungraded case.

(3) This follows by (1). □

Let R be a graded domain. Denote by $K = R_S$ the graded quotient field of R, where S is a multiplicative closet of nonzero homogeneous elements of R. For a nonzero graded submodule A of K, we define $A_g^{-1} = \{x \in K \mid xA \subseteq R\}$. And we call A a *graded v-ideal* if A is a graded ideal of R and $A^g_v := (A_g^{-1})_g^{-1} = A$.

Let R be a graded domain and let M be a graded R-module. Recall that M is called a *graded torsion-free module* if $sx = 0$ implies $x = 0$, where $x \in h(M)$ and $0 \neq s \in h(R)$. And M is said to be *graded divisible* if for any nonzero homogeneous element $a \in R$ and $x \in M$, there is $y \in M$, such that $sy = x$. Note that both x and y, above mentioned, can be assumed as homogeneous elements. Obviously, K is a graded torsion-free module as well as a graded divisible module.

Lemma 3.6. *Let R be a graded domain, K be the graded quotient field of R, A be a graded submodule of K, and J be a nonzero graded ideal of R.*

1. $J_g^{-1} \cong \mathrm{HOM}_R(J,R)$.
2. *Let J be a finitely generated graded ideal of R. Then $J \in \mathrm{GV}_{\mathrm{gr}}(R)$ if and only if $J_g^{-1} = R$, equivalently, $J_v^g = R$.*

Proof. (1) Let $J_g^{-1} = \bigoplus_{\sigma\in G}(J_g^{-1})_\sigma$. Consider a natural graded homomorphism $\psi : J_g^{-1} \to \mathrm{HOM}_R(J,R)$, $\psi(x)(a) = ax$, $a \in J$, $x \in J_g^{-1}$. Let $\psi = \bigoplus \psi_\sigma$, where $\psi_\sigma : (J_g^{-1})_\sigma \to \mathrm{Hom}_{R\text{-gr}}(J, R(\sigma))$. In the following, we will show that ψ_σ is a bijective mapping, i.e., ψ is an isomorphism.

If $x \in (J_g^{-1})_\sigma$ with $\psi(x) = 0$, then $\psi(x)(a) = ax = 0$ for any nonzero homogeneous element $a \in J$. Since K is a graded field, we have $x = 0$. Thus ψ_σ is a monomorphism.

Let $f \in \mathrm{Hom}_{R\text{-gr}}(J, R(\sigma))$ and let $a, b \in J$ be nonzero homogeneous elements. Since $f(ab) = af(b) = bf(a)$, we have $\frac{f(a)}{a} = \frac{f(b)}{b}$, which shows that $x_f := \frac{f(a)}{a}$ is independent of the choice of a nonzero homogeneous element a. If the degree of a is τ, then $f(a) \in R(\sigma)_\tau = R_{\tau\sigma}$, and thus $x_f \in K_\sigma$. So we have $x_f a = f(a) \in R_{\tau\sigma} \subseteq R$ for any nonzero τ-homogeneous element $a \in J$. Hence $x_f \in J_g^{-1}$. Then clearly $x_f \in (J_g^{-1})_\sigma$. Since $\psi_\sigma(x_f)(a) = x_f a = f(a)$ for any nonzero homogeneous element $a \in J$, we get $\psi_\sigma(x_f) = f$. That is, ψ_σ is an epimorphism, too.

(2) Clearly $R_g^{-1} = R$, and so $R \subseteq J_g^{-1}$. Assume $J \in \mathrm{GV}_{\mathrm{gr}}(R)$ and let $x \in J_g^{-1}$. Then $Jx \subseteq R$. Since R is a graded w-module, $x \in R$. Therefore $J_g^{-1} = R$.

For the converse, assume that $J_g^{-1} = R$. Consider the following commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & J_g^{-1} \arrow[d] \\ R \arrow[r, "\phi"] & \mathrm{HOM}_R(J,R) \end{tikzcd}$$

By (3), the vertical arrow on the right is an isomorphism, and hence ϕ is an isomorphism. Therefore $J \in \mathrm{GV}_{\mathrm{gr}}(R)$. □

Comparing with [2,3,16], it's worth noting that G here can be any Abelian group and graded domains need not be integral domains, and therefore it has more general applicability.

Lemma 3.7. *Let R be a graded w-Noetherian domain.*

1. *Let $\mathfrak{m}$ be a gr-maximal w-ideal of R. Then $\mathfrak{m}$ is a graded v-ideal.*
2. *Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending chain on graded v-ideals of R with $\bigcap_{n=1}^{\infty} A_n \neq 0$. Then there exists a positive integer m such that $A_n = A_m$ for all $n \geq m$.*

Proof. (1) Suppose $\mathfrak{m} = (x_1, \dots, x_n)^g_w$ and denote $J = (x_1, \dots, x_n)$. If $\mathfrak{m}_g^{-1} = R$, then by Lemma 3.6, we have $J_g^{-1} = R$. Hence $J \in \mathrm{GV}_{\mathrm{gr}}(R)$ and $J \subseteq \mathfrak{m}$, which contradicts to the fact that $\mathfrak{m}$ is a gr-maximal w-ideal. So $\mathfrak{m}_g^{-1} \neq R$, and then $\mathfrak{m}^g_v \neq R$. Since $\mathfrak{m} \subseteq \mathfrak{m}^g_v$, we have $\mathfrak{m}^g_v = \mathfrak{m}$.

(2) Consider a nonzero homogeneous element $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x(A_n)_g^{-1}$ is a graded w-ideal of R. As a consequence,

$$x(A_1)_g^{-1} \subseteq x(A_2)_g^{-1} \subseteq \cdots \subseteq x(A_n)_g^{-1} \subseteq \cdots$$

is an ascending chain of graded w-ideals of R. Therefore there exists a positive integer m such that $x(A_n)_g^{-1} = x(A_m)_g^{-1}$ for all $n \geq m$. In this case, we have $A_n = A_m$ for all $n \geq m$. □

Theorem 3.8. *A graded domain R is a graded w-Noetherian domain if and only if $R^g_{\mathfrak{m}}$ is a graded Noetherian ring for any gr-maximal w-ideal $\mathfrak{m}$ of R and there are only finite numbers of gr-maximal w-ideals containing a for any nonzero homogeneous element a.*

Proof. Suppose R is a graded w-Noetherian domain and $\mathfrak{m}$ is a gr-maximal w-ideal of R. Then by Proposition 3.5, $R^g_{\mathfrak{m}}$ is a graded Noetherian ring.

Assume on the contrary that there are infinite numbers of gr-maximal w-ideals containing a. Then we may pick $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n, \dots$ as distinct gr-maximal w-ideals including a. Thus $a \in \bigcap_{n=1}^{\infty} \mathfrak{m}_n$. Since $\mathfrak{m}_1 \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \supseteq \cdots \supseteq \bigcap_{i=1}^{n} \mathfrak{m}_i \supseteq \cdots$ is a descending chain on graded v-ideals of R, by Lemma 3.7 there exists some n such that $\bigcap_{i=1}^{n} \mathfrak{m}_i = \bigcap_{i=1}^{n+1} \mathfrak{m}_i$. Thus we have $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{m}_{n+1}$. Hence $\mathfrak{m}_i \subseteq \mathfrak{m}_{n+1}$ for some i, where $1 \leq i \leq n$. So $\mathfrak{m}_i = \mathfrak{m}_{n+1}$, which is a contradiction. That is, there are only finite numbers of gr-maximal w-ideals including a.

Conversely, let I be any nonzero graded w-ideal of R. Select a nonzero homogeneous element $a \in I$. Then there are only finite numbers of gr-maximal w-ideals containing a, denoted by $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Since $R^g_{\mathfrak{m}_i}$ is a graded Noetherian ring, we have $I^g_{\mathfrak{m}_i} = (x_{i1}, \dots, x_{it})^g_{\mathfrak{m}_i}$ for homogeneous elements $x_{i1}, \dots, x_{it} \in I$, where $i = 1, \dots, n$. Denote by B the ideal generated by a and those x_{ij}, where $i = 1, \dots, n$, $j = 1, \dots, t$. Then $B \subseteq I$. If there is some i $(1 \leqslant i \leqslant n)$ such that $\mathfrak{m} = \mathfrak{m}_i$ for any gr-maximal w-ideal $\mathfrak{m}$ of R, then $I^g_{\mathfrak{m}} = (x_{i1}, \dots, x_{it})^g_{\mathfrak{m}} \subseteq B^g_{\mathfrak{m}}$, and so $I^g_{\mathfrak{m}} = B^g_{\mathfrak{m}}$. And if $\mathfrak{m} \neq \mathfrak{m}_i$, $i = 1, \dots, n$, then $a \notin \mathfrak{m}$, and hence $B^g_{\mathfrak{m}} = R^g_{\mathfrak{m}} = I^g_{\mathfrak{m}}$. That is, by Theorem 2.8, $B_w = I_w = I$ for any $\mathfrak{m} \in \text{gr-}w\text{-Max}(R)$ and $B^g_{\mathfrak{m}} = I^g_{\mathfrak{m}}$, which shows that I is of graded finite type. Therefore R is a graded w-Noetherian domain. □

4. Graded w-modules over strongly graded rings

Let R be a graded ring. We define $\mathfrak{F}(A) = R \otimes_{R_e} A$ for any R_e-module A, and define $\mathfrak{F}(f) = \mathbf{1} \otimes f$ for $f \in \mathrm{Hom}_{R_e}(A, B)$. Then $\mathfrak{F} : {}_{R_e}\mathfrak{M} \to R\text{-gr}$ is an additive functor. On the other hand, we define $\mathfrak{G}(M) = M_\sigma$ for a fixed $\sigma \in G$ and any $M \in R\text{-gr}$. Let both M and N be graded modules and let $g : N \to M$

be a graded homomorphism. Define $\mathfrak{G}(g) = g_\sigma := g|_{N_\sigma}$. Then $\mathfrak{G} : R\text{-gr} \to {}_{R_e}\mathfrak{M}$ is also an additive functor. When $\sigma = e$, we denote $\mathfrak{G}(g) = \rho$, i.e.,

$$\rho : \mathrm{Hom}_{R\text{-gr}}(N, M) \to \mathrm{Hom}_{R_e}(N_e, M_e), \quad \rho(g) = g|_{N_e}, \quad g \in \mathrm{Hom}_{R\text{-gr}}(N, M).$$

Clearly ρ is an R_e-homomorphism.

Recall that a graded ring R is said to be a *strongly graded ring* if $R_\sigma R_\tau = R_{\sigma\tau}$ for any $\sigma, \tau \in G$. Then $R_\sigma R_{\sigma^{-1}} = R_e$. Therefore, if R is a strongly graded ring, then there is an inverse element in each R_σ. Let D be a commutative ring and x be an indeterminate. Then $D[x, x^{-1}]$ is an example of a strongly ($\mathbb{Z}$-)graded ring. By Dade's theorem (cf. [14, Theorem 3.1.1]), if R is a strongly graded ring, then the same $\mathfrak{F}$ as $\mathfrak{G}$ is an equivalent functor. Thus we have:

Lemma 4.1. *Let R be a strongly graded ring and let M, N be graded R-modules. Then*:

1. *ρ is a graded isomorphism.*
2. *If $n \geqslant 1$, then there exists a natural graded isomorphism*

$$\mathrm{Ext}^n_{R\text{-gr}}(N, M) \cong \mathrm{Ext}^n_{R_e}(N_e, M_e).$$

Proposition 4.2. *Let M be a graded R-module. Then M is a graded GV-torsion-free module if and only if the natural graded homomorphism $\phi : M \to \mathrm{HOM}_R(J, M)$ is a monomorphism for any $J \in \mathrm{GV}_{\mathrm{gr}}(R)$, that is,*

$$\mathrm{HOM}_R(R/J, M) = 0.$$

Proof. Let $J \in \mathrm{GV}_{\mathrm{gr}}(R)$. Notice that ϕ is a graded monomorphism if and only if $\varphi : M \to \mathrm{Hom}_R(J, M)$ is a monomorphism. So the rest of the proof is good enough for φ.

Let M be a graded GV-torsion-free module. Suppose $x \in M$ and $\varphi(x) = 0$. Then $\varphi(x)(a) = ax = 0$ for any $a \in J$, and so $Jx = 0$. Since M is a graded GV-torsion-free module, we have $x = 0$, that is, φ is a monomorphism.

Conversely, let $J \in \mathrm{GV}_{\mathrm{gr}}(R)$, $x \in M$ and $Jx = 0$. Define $f(\bar{r}) = rx$ for $r \in R$. Then f is well-defined, and therefore $f \in \mathrm{Hom}_R(R/J, M)$. By hypothesis, φ is a monomorphism. Hence we get $\mathrm{Hom}_R(R/J, M) = 0$, and thus $f = 0$. So $x = f(\bar{1}) = 0$, and then M is a graded GV-torsion-free module. □

We use the notation w_e (instead of w) whenever we treat w-modules as well as w-envelopes over the ring R_e.

Theorem 4.3. *Let R be a strongly graded ring, let J be a finitely generated graded ideal, and let M be a graded R-module. Then*:

1. *$J \in \mathrm{GV}_{\mathrm{gr}}(R)$ if and only if $J_e \in \mathrm{GV}(R_e)$.*
2. *Let $I \in \mathrm{GV}(R_e)$. Then $J := IR \in \mathrm{GV}_{\mathrm{gr}}(R)$, and thus R is a w-linked extension of R_e, i.e., R as an R_e-module is a w_e-module.*
3. *M is a graded GV-torsion-free module if and only if M_e is a GV-torsion-free R_e-module.*
4. *M is a graded w-module if and only if M_e is a w_e-module.*

Proof. (1) Because J is a finitely generated graded ideal, it is clear that J_e is a finitely generated ideal of R_e. If $J \in \mathrm{GV}_{\mathrm{gr}}(R)$, then $\mathrm{HOM}_R(R/J,R) = 0$ and $\mathrm{EXT}^1_R(R/J,R) = 0$. By Lemma 4.1, we have $\mathrm{Hom}_{R_e}(R_e/J_e,R_e) = 0$ and $\mathrm{Ext}^1_{R_e}(R_e/J_e,R_e) = 0$. Therefore, $J_e \in \mathrm{GV}(R_e)$.

Conversely, suppose $J_e \in \mathrm{GV}(R_e)$. By Lemma 4.1, we have

$$\mathrm{Hom}_{R\text{-gr}}(R/J,R) = \mathrm{Hom}_{R_e}(R_e/J_e,R_e) = 0.$$

By Dade's theorem, we get

$$\mathrm{HOM}_R(R/J,R) = \bigoplus_{\sigma\in G} \mathrm{Hom}_{R\text{-gr}}(R/J,R) = 0.$$

In a similar way, $\mathrm{EXT}^1_R(R/J,R) = 0$. Therefore, $J \in \mathrm{GV}_{\mathrm{gr}}(R)$.

(2) Applying (1), clearly we have $J_e = I$.

(3) According to Lemma 4.1, M is a graded GV-torsion-free module if and only if $\mathrm{HOM}_R(R/J,M) = 0$ for any $J \in \mathrm{GV}_{\mathrm{gr}}(R)$; according to Dade's theorem, if and only if $\mathrm{Hom}_{R\text{-gr}}(R/J,M) = 0$; by Lemma 4.1(1), if and only if $\mathrm{Hom}_{R_e}(R_e/I,M_e) = 0$ for any $I \in \mathrm{GV}(R_e)$; and if and only if M_e is a GV-torsion-free R_e-module.

(4) Similarly to (3), we have $\mathrm{EXT}^1_R(R/J,M) = 0$ for any $J \in \mathrm{GV}_{\mathrm{gr}}(R)$ if and only if $\mathrm{Ext}^1_{R_e}(R_e/I,M_e) = 0$ for any $I \in \mathrm{GV}(R_e)$. Then M is a graded w-module if and only if M_e is a w_e-module. □

Lemma 4.4. *Let R be a strongly graded ring and let N be a graded submodule of graded R-module M. Then $N = M$ if and only if $N_e = M_e$.*

Proof. Consider $M/N = \bigoplus_{\sigma\in G}(M_\sigma/N_\sigma)$. Then the assertion follows by [14, Theorem 3.1.1(5)]. □

In [8], Goto and Yamagishi proved that let H be a finitely generated abelian group and $A = \bigoplus_{h\in H} A_h$ an H-graded ring. Then A is a Noetherian ring if and only if the ring A_0 is Noetherian and A is a finitely generated A_0-algebra, if and only if every homogeneous ideal of A is finitely generated. In the following, we will give the relationship between R and R_e over a graded w-Noetherian ring R.

Theorem 4.5. *Let R be a strongly graded ring. Then R is a graded w-Noetherian ring if and only if R_e is a w-Noetherian ring.*

Proof. Assume that R is a graded w-Noetherian ring and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an ascending chain of w_e-ideals of R_e. Then RI_n is a graded ideal of R and $(RI_n)_e = I_n$ for each n. By Theorem 4.3, we get $RI_1 \subseteq RI_2 \subseteq \cdots \subseteq RI_n \subseteq \cdots$ is an ascending chain on w-ideals of R. Thus there exists a positive integer m such that if $n \geq m$, then $RI_n = RI_m$. Then, applying Lemma 4.4, we have $I_n = I_m$ for all $n \geq m$. Therefore R_e is a w-Noetherian ring.

Conversely, suppose R_e is a w-Noetherian ring and let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ be an ascending chain on graded w-ideals of R. Set $I_n = (A_n)_e$, $n \in \mathbb{N}$. By Theorem 4.3, we have $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an ascending chain of w_e-ideals of R_e. Thus there exists a positive integer m such that if $n \geq m$, then $RI_n = RI_m$. Applying Lemma 4.4, when $n \geq m$, we get $A_n = A_m$. Hence R is a graded w-Noetherian ring. $\square$

There are some classic results of Krull's principal ideal theorem. In [5], the authors proved the principal ideal theorem over SM domains. Let R be an integral domain and let G be a torsion-free monoid. Then the principal ideal theorem over graded SM domains was verified in [17]. And in [22], they showed the principal ideal theorem over w-Noetherian rings. In 2011, C. H. Park and M. H. Park provided the graded version of principal ideal theorem over graded Noetherian rings as follows: Let R be a graded Noetherian ring (under convention G be a torsion-free monoid). Then the hight of the minimal graded prime ideal of a nonunit homogeneous element is at most 1 (cf. [17, Theorem 3.5]). In the following, we will prove the principal ideal theorem over graded Noetherian rings (without the assumption that G is a torsion-free group), where the method derives from [12] and [17], and we only make somewhat adjustment if necessary. Based on the principal ideal theorem over graded Noetherian rings, we will show the principal ideal theorem over graded w-Noetherian domain, without the assumption that R is an integral domain and G is a torsion-free group.

Lemma 4.6. *Let R be a graded integral domain and let a, b be nonzero homogeneous elements of R with* $\deg(a) = \alpha$, $\deg(b) = \beta$. *Then*:

(1) *There exists a graded isomorphism $(a, b)/(a) \cong ((a^2, ab)/(a^2))(\alpha)$.*
(2) *If $ra^2 \in (b)$ implies $ra \in (b)$, then there exists a graded isomorphism $(a)/(a^2) \cong ((a^2, b)/(a^2, ab))(\sigma)$, where $\sigma = \alpha^{-1}\beta$.*
(3) *If $a = sa^2 + tb$, where $s, t \in R$, then there exists homogeneous element $u, v \in R$ such that $a = ua^2 + vb$.*

Proof. (1) Define $f(rb+(a)) = rab+(a^2)$, $r \in R$. Then f is an isomorphism with degree α. Therefore $f : (a, b)/(a) \to ((a^2, ab)/(a^2))(\alpha)$ is a graded isomorphism.

(2) Define $g : (a) \to (a^2, b)/(a^2, ab)$, $g(ra) = rb + (a^2, ab)$, $r \in R$. Then g is an isomorphism with degree σ. Thus $g : (a)/(a^2) \to ((a^2, b)/(a^2, ab))(\sigma)$ is a graded isomorphism.

(3) Let $s = \sum_{\sigma\in G} s_\sigma$, $t = \sum_{\mu\in G} t_\mu$. Then $a = \sum_{\sigma\in G} a^2 + \sum_{\mu\in G} t_\mu b$. Let $u = s_{\alpha^{-1}}$, $v = t_{\alpha\beta^{-1}}$. Hence u, v are homogeneous elements. It is clear that $a = ua^2 + vb$ by comparing the degree. $\square$

Lemma 4.7. *Let I be an ideal of a graded ring R and let S be a multiplicative set of homogeneous elements of R. Suppose that $I \cap S = \varnothing$. Then there exists a graded prime ideal $\mathfrak{p}$ such that $I \subseteq \mathfrak{p}$ as well as $\mathfrak{p} \cap S = \varnothing$.*

Proof. Let $\Gamma = \{J \mid J$ is a graded ideal of R, $I \subseteq J$, and $J \cap S = \varnothing\}$. Clearly, $I \in \Gamma$, and hence Γ is not empty. By Zorn's lemma, there exists a maximal element in Γ, which is denoted by $\mathfrak{p}$. Then it is trivial to conclude that $\mathfrak{p}$ is a graded prime ideal of R. $\square$

Let M be a graded R-module. Recall that M is said to *have a graded composition series* if there exists an ascending chain of graded submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M,$$

in which M_i/M_{i-1} is a graded simple module for $i = 1, 2, \ldots, n$. When M has a graded composition series, the index n is fixed, which called *the graded length* of M, denoted by $l^g(M)$. Apparently, if M has a graded composition series, then $M(\sigma)$ also has a graded composition series for any $\sigma \in G$ and $l^g(M(\sigma)) = l^g(M)$.

Let R be a strongly graded ring. By Lemma 4.1, we have that a graded R-module M is a graded projective module if and only if M_e is a graded projective R_e-module. Hence R is a graded semisimple ring if and only if R_e is a semisimple ring. Let K be a graded field, then K is a graded semisimple ring by [13, Theorem I.5.8]. Thus every finitely generated K-module has a graded composition series.

Lemma 4.8 (cf. [17, Lemma 3.1])**.** *Let* $(R, \mathfrak{p})$ *be a graded Noetherian ring with the unique graded prime ideal* $\mathfrak{p}$*. Then*:

(1) $\mathfrak{p}$ *is a nilpotent ideal.*

(2) *Every finitely generated graded module has a graded composition series.*

Proof. (1) Since $\mathfrak{p}$ is generated by homogeneous elements, it is good enough to prove that the homogeneous elements of $\mathfrak{p}$ are nilpotent.

Suppose on the contrary that $x \in \mathfrak{p}$ is a homogeneous element, but not a nilpotent. Consider a multiplicative set S of the homogeneous elements of R: $S = \{x^n \mid n \in \mathbb{N}\}$. By Lemma 4.7, there exists a graded prime ideal $\mathfrak{q}$ of R such that $x \notin \mathfrak{q}$, which contradicts to the fact that $\mathfrak{p}$ is the unique graded prime ideal of R.

(2) By (1), we can let $\mathfrak{p}^n = 0$ for some positive integer n. Let M be a finitely generated graded R-module. Then M has an ascending chain of graded submodules

$$0 = \mathfrak{p}^n M \subseteq \mathfrak{p}^{n-1} M \subseteq \mathfrak{p}^{n-2} M \subseteq \cdots \subseteq \mathfrak{p} M \subseteq M.$$

What we want is that every $\mathfrak{p}^k M$ has a graded composition series for $k = 0, 1, \ldots, n-1$. For this purpose, it is good enough to prove that every $M_k := \mathfrak{p}^{k-1} M/\mathfrak{p}^k M$ has a graded composition series. Since M_k is a finitely generated graded $R/\mathfrak{p}$-module and $R/\mathfrak{p}$ is a graded field, we have M_k is a graded semisimple ring. Therefore M_k has a graded composition series. □

Proposition 4.9 (The principal ideal theorem over a graded Noetherian ring)**.** *Let R be a graded Noetherian ring and $a \in h(R)$. Assume* $\mathfrak{p}$ *is a graded minimal prime ideal of* (a)*. Then* $\operatorname{ht}^g \mathfrak{p} \leq 1$*.*

Proof. With the localization method, we may assume that $(R, \mathfrak{p})$ is a graded local ring and $\mathfrak{p}_0$ is a graded minimal prime ideal of R which is contained in $\mathfrak{p}$. Then in the graded integral domain $\overline{R} = R/\mathfrak{p}_0$, $\mathfrak{p}/\mathfrak{p}_0$ is a graded minimal prime ideal of $\overline{a}$. Replacing $\overline{R}$ with R and $\overline{a}$ with a respectively, we may assume that R is an integral domain. If $\mathfrak{p} = \mathfrak{p}_0$, then $\operatorname{ht}\mathfrak{p} = 0$. Now, suppose $\mathfrak{p} \neq \mathfrak{p}_0$. Then $a \neq 0$. Let $\mathfrak{q}$ be a graded prime ideal of R with $\mathfrak{q} \subset \mathfrak{p}$. Assume on the contrary that $\mathfrak{q} \neq 0$. Consider a homogeneous element $b \in \mathfrak{q}$, $b \neq 0$. Then $I_k = ((b) : a^k)$ is a graded ideal of R and $I_k \subseteq I_{k+1}$. Since R is a graded Noetherian domain, there exists a positive n such that $I_k = I_n$ for all $k \geq n$. Therefore, $ra^{2n} \in (b)$ can imply that $ra^n \in (b)$. Since $\mathfrak{p}$ is a minimal prime ideal of a^n, for the sake of convenience in writing, we replace a^n with a. Then $ra^2 \in (b)$ can imply $ra \in (b)$. Let $\overline{R} = R/(a^2)$. Then $\overline{R}$ is a graded Noetherian ring. Obviously, $\mathfrak{p}/(a^2)$ is the unique graded prime ideal of $\overline{R}$. Now let $A = (a, b)/(a^2)$ and $B = (a^2, b)/(a^2)$. Then both A and B are finitely generated graded $\overline{R}$-modules. By Lemma 4.8, both A and B have graded composition series. Consider the following graded exact sequences

$$0 \to (a)/(a^2) \to A \to (a, b)/(a) \to 0$$

and

$$0 \to (a^2, ab)/(a^2) \to B \to (a^2, b)/(a^2, ab) \to 0.$$

After counting the graded lengths of graded composition series, we have $l(A) = l(B)$ by Lemma 4.6(1) and (2). Notice that $B \subseteq A$, and so $A = B$. Thus $(a, b) = (a^2, b)$, and so there exist $s, t \in R$ such that $a = sa^2 + tb$. Applying Lemma 4.6, we may choose homogeneous elements s, t. Hence $(tb : a)$ is a graded ideal. Since $(1 - sa)a = tb$, we have $1 - sa \in (tb : a)$. By the fact that $1 - sa \notin \mathfrak{p}$ and $\mathfrak{p}$ is the unique graded maximal ideal of R, $1 - sa$ is a unit. Thus $a \in (b) \subseteq \mathfrak{q}$, which contradicts to the fact that $\mathfrak{p}$ is a minimal prime ideal of a. Therefore $\mathfrak{q} = 0$, and hence $\operatorname{ht}\mathfrak{p} = 1$. □

In 2011, C. H. Park and M. H. Park provided the graded version of principal ideal theorem over graded Noetherian rings as follows:

Let x be a nonunit homogeneous element in a graded Noetherian ring R and let $\mathfrak{p}$ be a prime ideal minimal over (x). Then $\operatorname{ht}^g \mathfrak{p} \leq 1$.

The following is the w-analogue of the above result.

Theorem 4.10 (The principal ideal theorem over a graded w-Noetherian domain)**.** *Let R be a graded w-Noetherian domain and let $a \in R$ be a homogeneous element. Suppose $\mathfrak{p}$ is a minimal graded prime ideal of (a). Then $\operatorname{ht}^g \mathfrak{p} \leq 1$.*

Proof. Since R is a graded domain, (a) is a graded w-ideal. By Proposition 2.7, $\mathfrak{p}$ is a graded w-ideal. By Proposition 3.5, $R^g_{\mathfrak{p}}$ is a graded Noetherian ring. By Proposition 4.9, there is a one-to-one correspondence between the set of graded prime ideals of R which is contained $\mathfrak{p}$ and the set of graded prime ideals of $R^g_{\mathfrak{p}}$. Therefore $\operatorname{ht}^g \mathfrak{p} = \dim^g(R^g_{\mathfrak{p}}) \leq 1$. □

In [19], according to the definition of SM domains, a w-Noetherian domain must be an SM domain. In [16], a graded SM domain is an integral domain. Therefore let R be a graded ring and G be a multiplicative Abelian group. Then a graded w-Noetherian graded domain is not necessary an integral domain.

Hence a graded w-Noetherian graded domain is not necessary a graded SM domain. Now, we will show the example.

Example 4.11. Let $G = \{e, \sigma\}$ be a torsion group of order 2 and $R = \mathbb{Z}_2[X]/(X^2 - 1)$. Then $R = R_e \bigoplus R_\sigma$ is a G-graded ring, where $R_e = \mathbb{Z}_2$, $R_\sigma = \{0, x\}$, and x is the image of X in R. So the unit element of R is 1 and nonzero homogeneous elements of R are 1 and x. Since $x^2 = 1$, we have that R is a graded field. So R is a graded integral domain. Notice that $(1+x)^2 = 1 + 1 = 0$, and thus R is not an integral domain. Besides R is also a Noetherian ring. Clearly R is a w-Noetherian ring, and so R is a graded w-Noetherian graded domain. Since R is not an integral domain, we have that R is not a graded SM domain.

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References

[1] D. F. Anderson, *Graded Krull domains*, Comm. Algebra **7** (1979), no. 1, 79–106. https://doi.org/10.1080/00927877908822334

[2] D. D. Anderson and D. F. Anderson, *Divisibility properties of graded domains*, Canadian J. Math. **34** (1982), no. 1, 196–215. https://doi.org/10.4153/CJM-1982-013-3

[3] ———, *Divisorial ideals and invertible ideals in a graded integral domain*, J. Algebra **76** (1982), no. 2, 549–569. https://doi.org/10.1016/0021-8693(82)90232-0

[4] S. E. Atani and U. Tekir, *On the graded primary avoidance theorem*, Chiang Mai J. Sci. **34** (2007), no. 2, 161–164.

[5] W. Fanggui and R. L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165. https://doi.org/10.1016/S0022-4049(97)00150-3

[6] S. Goto and K. Watanabe, *On graded rings. I*, J. Math. Soc. Japan **30** (1978), no. 2, 179–213. https://doi.org/10.2969/jmsj/03020179

[7] ———, *On graded rings. II. ($\mathbf{Z}^n$-graded rings)*, Tokyo J. Math. **1** (1978), no. 2, 237–261. https://doi.org/10.3836/tjm/1270216496

[8] S. Goto and K. Yamagishi, *Finite generation of Noetherian graded rings*, Proc. Amer. Math. Soc. **89** (1983), no. 1, 41–44. https://doi.org/10.2307/2045060

[9] W. Heinzer, *On Krull overrings of a Noetherian domain*, Proc. Amer. Math. Soc. **22** (1969), 217–222. https://doi.org/10.2307/2036956

[10] Z. Huang, *On the grade of modules over Noetherian rings*, Comm. Algebra **36** (2008), no. 10, 3616–3631. https://doi.org/10.1080/00927870802157756

[11] Y. Kamoi, *Noetherian rings graded by an abelian group*, Tokyo J. Math. **18** (1995), no. 1, 31–48. https://doi.org/10.3836/tjm/1270043606

[12] I. Kaplansky, *Commutative Rings*, revised edition, The University of Chicago Press, Chicago, IL, 1974.

[13] C. Năstăsescu and F. van Oystaeyen, *Graded Ring Theory*, North-Holland Mathematical Library, 28, North-Holland Publishing Co., Amsterdam, 1982.

[14] ———, *Methods of graded rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004. https://doi.org/10.1007/b94904

[15] J. Nishimura, *Note on integral closures of a Noetherian integral domain*, J. Math. Kyoto Univ. **16** (1976), no. 1, 117–122. https://doi.org/10.1215/kjm/1250522963

[16] M. H. Park, *Integral closure of graded integral domains*, Comm. Algebra **35** (2007), no. 12, 3965–3978. https://doi.org/10.1080/00927870701509511

[17] C. H. Park and M. H. Park, *Integral closure of a graded Noetherian domain*, J. Korean Math. Soc. **48** (2011), no. 3, 449–464. https://doi.org/10.4134/JKMS.2011.48.3.449

[18] M. Refai, *Augmented Noetherian graded modules*, Turkish J. Math. **23** (1999), no. 3, 355–360.

[19] F. Wang and H. Kim, *Foundations of commutative rings and their modules*, Algebra and Applications, 22, Springer, Singapore, 2016. https://doi.org/10.1007/978-981-10-3337-7

[20] X. Y. Wu, F. G. Wang, and C. M. Lang, *Graded w-modules over graded rings*, J. Sichuan Normal Univ. **4** (2019), 450–459.

[21] X. Y. Wu, F. G. Wang, and Y. J. Xie, *Graded Matlis cotorsion modules and graded Matlis domain*, J. Heilongjiang Univ. **36** (2019), 253–261.

[22] H. Yin and F. Wang, *The principal ideal theorem for w-Noetherian rings*, Studia Sci. Math. Hungar. **53** (2016), no. 4, 525–531. https://doi.org/10.1556/012.2016.53.4.1349

[23] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222. https://doi.org/10.4134/JKMS.2011.48.1.207

[24] A. Yousefian Darani, *Graded primal submodules of graded modules*, J. Korean Math. Soc. **48** (2011), no. 5, 927–938. https://doi.org/10.4134/JKMS.2011.48.5.927

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