Bull. Korean Math. Soc. **57** (2020), No. 5, pp. 1299–1306
https://doi.org/10.4134/BKMS.b190981
pISSN: 1015-8634 / eISSN: 2234-3016

QUASI CONTACT METRIC MANIFOLDS WITH KILLING CHARACTERISTIC VECTOR FIELDS

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Abstract. An almost contact metric manifold is called a quasi contact metric manifold if the corresponding almost Hermitian cone is a quasi Kähler manifold, which was introduced by Y. Tashiro [9] as a contact O*-manifold. In this paper, we show that a quasi contact metric manifold with Killing characteristic vector field is a K-contact manifold. This provides an extension of the definition of K-contact manifold.

1. Introduction

A $(2n+1)$-dimensional smooth manifold is called an almost contact manifold if it admits a triple (ϕ, ξ, η) of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following condition [1]:

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \\ \phi\xi = 0, &\quad \eta \circ \phi = 0. \end{aligned} \tag{1}$$

From (1), we may deduce equality $\eta(\xi) = 1$. Further, an almost contact manifold $M = (M, \phi, \xi, \eta)$ equipped with a Riemannian metric g such that

$$\begin{aligned} g(\phi U, \phi V) &= g(U, V) - \eta(U)\eta(V), \\ \eta(U) &= g(\xi, U), \end{aligned} \tag{2}$$

for any smooth vector fields on U, V on M, is called an almost contact metric manifold. On the other hand, M admitting a 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M is called a contact manifold with the contact 1-form η. Now, an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the condition

$$d\eta(U, V) = g(U, \phi V), \tag{3}$$

for any smooth vector fields U, V on M, which is called a contact metric manifold with contact 1-form η. Then from (3), we can check that $\eta \wedge (d\eta)^n \neq 0$ on M, and hence, $M = (M, \eta)$ is a contact manifold with the contact 1-form η.

Received November 4, 2019; Accepted May 7, 2020.
2010 *Mathematics Subject Classification.* Primary 53B20, 53C25.
Key words and phrases. Quasi contact metric manifold, Killing vector field.
This paper was supported by Sungkyun Research Fund, Sungkyunkwan University, 2018.

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Now, let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $\tilde{M} = M \times \mathbb{R}$ be a product manifold of M and a real line $\mathbb{R}$ equipped with almost Hermitian structure $(\tilde{J}, \tilde{g})$ defined by

$$\begin{aligned} &\tilde{J}U = \phi U - \eta(U)\frac{\partial}{\partial t}, \quad \tilde{J}\frac{\partial}{\partial t} = \xi, \\ &\tilde{g}(U,V) = e^{-2t}g(U,V), \quad \tilde{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = e^{-2t}, \quad \tilde{g}(U, \frac{\partial}{\partial t}) = 0, \end{aligned} \tag{4}$$

for any $U, V \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M, and $t \in \mathbb{R}$. The almost Hermitian manifold $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is called an almost Hermitian cone corresponding to the almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ ([2], Remark 1.1). Gray and Hervella [4] classified 16 classes of almost Hermitian manifolds. Among their classes, the class of Kähler manifolds, almost Kähler manifolds and quasi Kähler manifolds (also known as O*-manifolds) have been examined extensively by many researchers. We remark that Kähler manifolds are included in almost Kähler manifolds and also almost Kähler manifolds are included in quasi Kähler manifolds [4].

Now, let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ be the corresponding almost Hermitian cone. Then, we may check that $M = (M, \phi, \xi, \eta, g)$ is a Sasakian manifold if and only if $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is a Kähler manifold, $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold if and only if $\tilde{M} = (\tilde{M}, \tilde{J}, \tilde{g})$ is an almost Kähler manifold. Thus, a contact metric manifold is necessary a quasi contact metric manifold.

Remark 1. Any 4-dimensional quasi Kähler manifold is necessarily an almost Kähler manifold.

Thus, taking account of [2] and Remark 1, the following question was raised by the third and fourth authors in the previous paper in [5]:

Question. Does there exist a $(2n+1)$ (≥ 5)-dimensional quasi contact metric manifold which is not a contact metric manifold?

Concerning the above question, several related results have been obtained [2, 5, 7].

Now, a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a K-contact manifold if the characteristic vector field ξ is a Killing vector field. The main purpose of the paper is to prove the following.

Theorem A. *A quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ with Killing characteristic vector field ξ is a K-contact manifold.*

We remark that the main theorem of the paper [6] follows immediately from Theorem A, taking account of the result by Tanno [8].

2. Preliminaries

In this section, we prepare some basic preliminaries and fundamental formulas on quasi contact metric manifolds. Unless otherwise stated, all manifolds are assumed be $(2n+1)$-dimensional smooth Riemannian manifolds. Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and h be the $(1,1)$-tensor field by

$$h = \frac{1}{2}\pounds_\xi \phi, \tag{5}$$

where $\pounds_\xi$ denotes the Lie derivative with respect to ξ. The tensor field h plays a vital role in the geometry of almost contact metric manifolds. From (5), we may easily check that the tensor field h satisfies the following properties:

$$h\xi = 0, \quad trh = 0. \tag{6}$$

Let ∇ be the Levi-Civita connection of g. We here recall the following results which characterize quasi contact metric manifold and contact metric manifold ([5] Theorems 3.2 and 4.2).

Theorem 2.1. *An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is a quasi contact metric manifold if and only if M satisfies the below equality*:

$$(\nabla_U \phi)V + (\nabla_{\phi U}\phi)\phi V = 2g(U,V)\xi - \eta(V)U - \eta(U)\eta(V)\xi - \eta(V)hU \tag{7}$$

for any $U, V \in \mathfrak{X}(M)$.

Further, we also have the following Theorem.

Theorem 2.2. *A quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold if h is symmetric with respect the metric g.*

In the remainder of this section, we suppose that $M = (M, \phi, \xi, \eta, g)$ is a quasi contact metric manifold. Then, the following equalities can be derived from the equality (7):

$$(\nabla_U \eta)(V) + (\nabla_{\phi U}\eta)(\phi V) + 2g(\phi U, V) = 0, \tag{8}$$

$$\nabla_\xi \phi = 0, \tag{9}$$

$$\nabla_\xi \xi = 0, \tag{10}$$

$$hU = \frac{1}{2}(-\nabla_{\phi U}\xi + \phi\nabla_U \xi), \tag{11}$$

for any $U, V \in \mathfrak{X}(M)$. From (11), taking account of (10), we obtain further the following equalities:

$$h\phi + \phi h = 0, \tag{12}$$

$$\nabla_U \xi = -\phi U - \phi h U, \tag{13}$$

for any $U \in \mathfrak{X}(M)$. From (12), we also get

$$\eta \circ h = 0. \tag{14}$$

The equalities (7) $\sim$ (13) play an central role in the proof of Theorem A.

3. Proof of Theorem A

Throughout this section, we suppose that $M = (M, \phi, \xi, \eta, g)$ is a $(2n+1)$-dimensional quasi contact metric manifold such that ξ is a Killing vector field. Here, thanks to account of Remark 1, for the prove Theorem A, it suffices to discuss in the case $n \geq 2$. Now, let $\nabla\xi$ be the $(1,1)$-tensor field on M defined by

$$(15) \qquad (\nabla\xi)U = \nabla_U \xi$$

for any $U \in \mathfrak{X}(M)$. Then, we can check that the $(1,1)$-tensor field $\nabla\xi$ is skew-symmetric with respect to g on M. So, from (13), it follows that ϕh is a skew-symmetric linear endomorphism with respect to g, and hence, taking account of (12), $h\phi$ is also skew-symmetric with respect to g:

$$(16) \qquad g(h(\phi U), V) + g(h(\phi V), U) = 0$$

for any $U, V \in \mathfrak{X}(M)$. Here, changing V by ϕV in (16), and taking account of (12), we get

$$(17) \qquad g(hU, V) + g(hV, U) = 0$$

for any $U, V \in \mathfrak{X}(M)$. Therefore, we obtain the following.

Lemma 3.1. *Under the hypothesis of Theorem A, the tensor field h is skew-symmetric with respect to the metric g.*

Since the characteristic vector field ξ is a Killing vector field, the $(1,1)$-tensor field $\nabla\xi$ is a skew-symmetric linear endomorphism on the surface $\{\xi_p\}^\perp$ in the tangent space T_pM at any point $p \in M$. Thus, we may choose a local orthonormal frame field $\{e_1, e_1^*, e_2, e_2^*, \dots, e_n, e_n^*\}$ in a neighborhood of the point p such that $e_a, e_a^* \perp \xi$ for any a $(1 \leq a \leq n)$, and further satisfies the condition:

$$(18) \qquad \nabla\xi(e_a) = -\lambda_a e_a^*, \qquad \nabla\xi(e_a^*) = \lambda_a e_a,$$

for some smooth functions λ_a defined on a neighborhood of the point p, for any a $(1 \leq a \leq n)$ [3]. Thus, from (15) and (18), taking account of (13), we have

$$(19) \qquad \begin{aligned} \phi(I+h)e_a &= \lambda_a e_a^*, \\ \phi(I+h)e_a^* &= -\lambda_a e_a, \end{aligned}$$

for $a = 1, 2, \dots, n$. Thus, from (19), taking account of (12), we get respectively

$$(20) \qquad \begin{aligned} \phi h e_a &= \lambda_a e_a^* - \phi e_a, \\ \phi h e_a^* &= -\lambda_a e_a - \phi e_a^*, \end{aligned}$$

for any a $(1 \leq a \leq n)$. Thus, from (20), taking account of (1) and (14), we have respectively

$$he_a = -e_a - \lambda_a \phi e_a^*, \quad he_a^* = -e_a^* + \lambda_a \phi e_a, \tag{21}$$

for any a $(1 \leq a \leq n)$. From the equalities of (21), taking account of (1) and (14), we have

$$g(he_a, e_a^*) = 0, \quad g(he_a^*, e_a) = 0. \tag{22}$$

Therefore, from (21), taking account of (20) together with (12) and (21), we have

$$\begin{aligned} h^2 e_a &= -he_a + \lambda_a \phi he_a^* \\ &= e_a + \lambda_a \phi e_a^* + \lambda_a(-\lambda_a e_a - \phi e_a^*) \\ &= (1 - \lambda_a^2) e_a, \\ h^2 e_a^* &= -he_a^* - \lambda_a \phi he_a \\ &= e_a^* - \lambda_a \phi e_a - \lambda_a(\lambda_a e_a^* - \phi e_a) \\ &= (1 - \lambda_a^2) e_a^*, \end{aligned} \tag{23}$$

for any a $(1 \leq a \leq n)$. Thus, from (23) taking account of (12), we see that the linear endomorphism h^2 is symmetric with respect to g, ϕ-invariant, and can be block-diagonalized as follows:

(24)

$$\begin{aligned} & h^2(e_1, e_1^*, e_2, e_2^*, \ldots, e_n, e_n^*) \\ &= (h^2 e_1, h^2 e_1^*, h^2 e_2, h^2 e_2^*, \ldots, h^2 e_n, h^2 e_n^*) \\ &= (e_1, e_1^*, \ldots, e_n, e_n^*) \begin{pmatrix} \begin{pmatrix} 1-\lambda_1^2 & 0 \\ 0 & 1-\lambda_1^2 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 1-\lambda_2^2 & 0 \\ 0 & 1-\lambda_2^2 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 1-\lambda_n^2 & 0 \\ 0 & 1-\lambda_n^2 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Here, from (23), taking account of Lemma 3.1, we have

$$\begin{aligned} 0 \leq g(he_a, he_a) &= -g(e_a, h^2 e_a) = \lambda_a^2 - 1, \\ 0 \leq g(he_a^*, he_a^*) &= -g(e_a^*, h^2 e_a^*) = \lambda_a^2 - 1, \end{aligned} \tag{25}$$

and hence, $\lambda_a^2 \geq 1$ for any a $(1 \leq a \leq n)$. Thus, from (24) and (25), taking account of (20)$\sim$(23), we may check that eigenvalues of the skew-symmetric linear endomorphism h are $i\sqrt{\lambda_a^2 - 1}$ or $-i\sqrt{\lambda_a^2 - 1}$ (possibly zero) for any a $(1 \leq a \leq n)$. Now, we here arrange the notational convention as follows:

$$\varepsilon_1 = e_1, \quad \varepsilon_2 = e_1^*, \quad \varepsilon_3 = e_2, \quad \varepsilon_4 = e_2^*, \quad \ldots, \quad \varepsilon_{2n-1} = e_n, \quad \varepsilon_{2n} = e_n^*. \tag{26}$$

Taking account of the arrangement of notational convention (26), we may identify $(\{\xi_p\}^{\perp}, g_p)$ with a $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with orthonormal basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}\}$ in the natural way. We also denote by $H = (H_{jk})$ $(j, k = 1, 2, \ldots, 2n)$ the $2n \times 2n$ real matrix defined by

$$H_{jk} = g(h\varepsilon_j, \varepsilon_k) \quad (j, k = 1, 2, \ldots, 2n). \tag{27}$$

Then, from (26) and (27), taking account of Lemma 3.1, the matrix $H = (H_{jk})$ is a $2n \times 2n$ skew-symmetric matrix. Thus, from the above discussion and the result in the paper [10] (Proposition 2.1, and Theorem 2.5), we have the following.

Lemma 3.2. *There exists a $2n \times 2n$ orthogonal matrix $A \in O(2n)$ satisfying the equality*:

(28)

$$AHA^{-1} = \begin{pmatrix} \begin{pmatrix} 0 & \sqrt{\lambda_1^2 - 1} \\ -\sqrt{\lambda_1^2 - 1} & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & \sqrt{\lambda_2^2 - 1} \\ -\sqrt{\lambda_2^2 - 1} & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & \sqrt{\lambda_n^2 - 1} \\ -\sqrt{\lambda_n^2 - 1} & 0 \end{pmatrix} \end{pmatrix}.$$

We here denote by $\bar{H} = (\bar{H}_{jk})$ $(j, k = 1, 2, \ldots, 2n)$ the $2n \times 2n$ real matrix of the right-hand side of the equality (28), which is called the normal real form for the skew-symmetric $H = (H_{jk})$. Then, we can check that $\bar{H} = (\bar{H}_{jk})$ is also a $2n \times 2n$ skew-symmetric matrix. Now, we denote by α the orthogonal transformation on the $2n$-dimensional Euclidean space $\mathbb{R}^{2n} = (\{\xi_p\}^{\perp}, g_p)$ corresponding to the matrix A, we now set

$$\alpha e_a = \bar{e_a}, \quad \alpha e_a^* = \bar{e_a^*} \quad (a = 1, 2, \ldots, n), \tag{29}$$

and also

$$\alpha \varepsilon_l = \bar{\varepsilon}_l \quad (l = 1, 2, \ldots, 2n). \tag{30}$$

Further, we denote by $\bar{h}$ the linear endomorphism on $\mathbb{R}^{2n} = (\{\xi_p\}^{\perp}, g_p)$ corresponding to the $2n \times 2n$ skew-symmetric matrix $\bar{H}$ defined by

$$\bar{H}_{jk} = g(\bar{h}\bar{\varepsilon}_j, \bar{\varepsilon}_k) \quad (j, k = 1, 2, \ldots, 2n). \tag{31}$$

Then, from (29) $\sim$ (31), we may easily check that

$$\alpha h \alpha^{-1} = \bar{h}, \tag{32}$$

and hence $\alpha h = \bar{h}\alpha$ holds. Thus, from (28) $\sim$ (32), taking account of (27), we have

$$\alpha h e_a = \bar{h}\alpha e_a = \bar{h}\bar{e_a},$$

and hence

$$\begin{aligned} he_a &= \alpha^{-1}(\bar{h}\bar{e_a}) \\ &= \alpha^{-1}(-\sqrt{\lambda_a^2 - 1}\bar{e}_a^*) \\ &= -\sqrt{\lambda_a^2 - 1}e_a^* \end{aligned} \tag{33}$$

for any a $(1 \leq a \leq n)$. Similarly we have also

$$he_a^* = \sqrt{\lambda_a^2 - 1}e_a \tag{34}$$

for any a $(1 \leq a \leq n)$. Thus, from (34) and (22), we get

$$\sqrt{\lambda_a^2 - 1} = 0 \tag{35}$$

for any a $(1 \leq a \leq n)$, and similarly from (33) and (22), we get also

$$\sqrt{\lambda_a^2 - 1} = 0 \tag{36}$$

for any a $(1 \leq a \leq n)$. Therefore, from (23), taking account of (35) and (36), we see that $h^2 e_a = 0$, and also $h^2 e_a^* = 0$ for any a $(1 \leq a \leq n)$, and hence $h^2 = 0$ everywhere on M since $h^2\xi = 0$. We here recall that h is skew-symmetric with respect to the metric g by Lemma 3.1, and hence, it follows that $\|h\|^2 = -trh^2 = 0$. Thus, we see that h vanishes identically on M. Especially, h is symmetric with respect to the metric g. Therefore, $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold by virtue of Theorem 2.2, and hence, a K-contact manifold. This completes the proof of Theorem A.

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