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A CLASS OF INVERSE CURVATURE FLOWS IN $\mathbb{R}^{n+1}$, II

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Abstract. We consider closed, star-shaped, admissible hypersurfaces in $\mathbb{R}^{n+1}$ expanding along the flow $\dot{X} = |X|^{\alpha-1}F^{-\beta}$, $\alpha \leq 1$, $\beta > 0$, and prove that for the case $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$, this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a round sphere after rescaling. Besides, for the case $\alpha \leq 1$, $\alpha + \beta > 2$, if furthermore the initial closed hypersurface is strictly convex, then the strict convexity is preserved during the evolution process and the flow blows up at finite time.

1. Introduction

Recently, Chen, Mao, Tu and Wu [5] investigated the following inverse curvature flow (ICF for short)

$$\begin{cases} \dfrac{\partial}{\partial t}X = \dfrac{1}{|X|^{\alpha}H(X)}\nu, \\ X(\cdot,0) = M_0, \end{cases} \tag{1.1}$$

where $0 \leq \alpha$, M_0 is a closed, star-shaped and strictly mean convex $C^{2,\beta}$-hypersurface $(0 < \beta < 1)$ in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$, $X(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a one-parameter family of hypersurfaces immersed into $\mathbb{R}^{n+1}$ with $M_t = X(M_0,t)$, ν is the unit outward normal vector of M_t, and $|X|$ is the distance from the point $X(x,t)$ to the origin of $\mathbb{R}^{n+1}$. Clearly, (1.1) describes the deformation of M_0 along its unit outward normal vector with a speed $(|X|^{\alpha}H)^{-1}$ and generally it is an *expanding* flow. Besides, (1.1) is a non-scale-invariant flow except the case $\alpha = 0$, in which the ICF (1.1) degenerates into the classical inverse mean curvature flow (IMCF for short). For this non-scale-invariant flow (1.1), they have proven that the evolution exists for all the time, the evolving hypersurfaces remain star-shaped during the evolution, and converge smoothly to a round sphere after rescaling. This conclusion improves the long-time existence and the asymptotical behavior description of the IMCF firstly shown by Gerhardt [9] or Urbas [20] (IMCF is just a special case of the flow considered by them). In fact, Gerhardt [9] (or Urbas [20]) proved that if in

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the flow (1.1), $\alpha = 0$ and H was replaced by F, which is a positive, symmetric, monotone, homogeneous of degree one, concave function with respect to principal curvatures of the evolving hypersurfaces, that is, the evolution equation becomes

$$\frac{\partial}{\partial t}X = \frac{1}{F}\nu, \tag{1.2}$$

then in this case, similar conclusions can also be obtained. Because of the homogeneity of F, the flow (1.2) is scale-invariant. *What about the non-scale-invariant version of* (1.2)*? Can similar conclusions be obtained?* Gerhardt [11] (or Urbas [21]) has given a positive answer to these questions. In fact, if in (1.2) F was replaced by F^p with $p > 0$, then the new flow becomes non-scale-invariant, and the long-time existence, the asymptotical behavior ($0 < p \leq 1$) or the convergence ($p > 1$) of the new flow can be obtained (see, e.g., [11, Theorems 1.1, 1.2 and 4.1] for details).

The reason why geometers are interested in the study of the theory of ICFs is that it has important applications in Physics and Mathematics. For instance, by defining a notion of weak solutions to IMCF, Huisken and Ilmanen [12, 13] proved the Riemannian Penrose inequality by using the IMCF approach, which makes an important step to possibly and completely solve the famous Penrose Conjecture in the General Relativity. Also using the method of IMCF, Brendle, Hung and Wang [1] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurfaces in the n-dimensional ($n \geq 3$) anti-de Sitter-Schwarzschild manifold, which generalized the related conclusions in the Euclidean n-space. Besides, applying ICFs, Alexandrov-Fenchel type and other types inequalities in space forms and even in some warped products can be obtained - see, e.g., [7, 8, 15, 16, 18].

What happens if H was replaced by F^β, $\beta > 0$, in (1.1)*?* The purpose of this paper is to solve this problem.

Let $\Gamma \subset \mathbb{R}^n$ be an open, convex, symmetric cone with vertex at the origin, which contains the positive diagonal, i.e., all n-tuples of the form $(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$, $i = 1, 2, \ldots, n$. This is to say that Γ contains the positive cone Γ_+. Let F be a symmetric, positive function, homogeneous of degree one, defined on Γ, which also satisfies the following assumptions:

Regularity

$$F \in C^{m,\gamma}(\Gamma) \cap C^0(\overline{\Gamma}), \qquad \text{with } 4 \leq m \leq \infty \text{ and } 0 < \gamma < 1; \tag{1.3}$$

Monotonicity

$$\frac{\partial F}{\partial \lambda_i} > 0, \qquad i = 1, 2, \ldots, n, \text{ in } \Gamma; \tag{1.4}$$

Concavity

$$\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \leq 0; \tag{1.5}$$

Nonnegativity

$$F|_{\Gamma} > 0 \qquad \text{and} \qquad F|_{\partial\Gamma} = 0. \tag{1.6}$$

For convenience, we use the normalization convention

$$F(1, 1, \ldots, 1) = n \tag{1.7}$$

in the sequel. A hypersurface M_0 in $\mathbb{R}^{n+1}$ is said to be *admissible* if its principal curvatures lie in the interior of the cone Γ. That is, for any point $p \in M_0$, the principal curvatures κ_i, $i = 1, 2, \ldots, n$, of M_0 at the point p satisfies

$$(\kappa_1, \kappa_2, \ldots, \kappa_n) \in \operatorname{int}(\Gamma),$$

where $\operatorname{int}(\Gamma)$ represents the interior of Γ. In this paper, we consider the ICFs

$$\frac{\partial}{\partial t} X = \frac{1}{|X|^{1-\alpha} F^{\beta}} \nu, \qquad \alpha \leq 1,\ \beta > 0, \tag{1.8}$$

and can prove the followings:

Theorem 1.1. *Let $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$. Let M_0 be a closed, star-shaped and admissible $C^{m+2,\gamma}$-hypersurface in $\mathbb{R}^{n+1}$, and let F be a principal curvature function satisfying assumptions* (1.3)-(1.7). *Assume that*

$$M_0 = \operatorname{graph}_{\mathbb{S}^n} u_0$$

for a positive map $u_0 : \mathbb{S}^n \to \mathbb{R}$. Then

(i) *there exists a family of star-shaped and admissible hypersurfaces M_t given by the unique $C^{m+2+\gamma, \frac{m+2+\gamma}{2}}$-embedding*

$$X(\cdot, t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$$

for $t \geq 0$, satisfying the following system:

$$\begin{cases} \dfrac{\partial}{\partial t} X = \dfrac{1}{|X|^{1-\alpha} F^{\beta}} \nu & \text{on } \mathbb{S}^n \times (0, \infty), \\ X(\cdot, 0) = M_0 & \text{in } \mathbb{S}^n, \end{cases} \tag{1.9}$$

where ν is the unit outward normal vector of $M_t := X(\mathbb{S}^n, t)$, and $|X|$ is the distance from the point $X(x, t)$ to the origin.

(ii) *the leaves M_t are graphs over $\mathbb{S}^n$, i.e.,*

$$M_t = \operatorname{graph}_{\mathbb{S}^n} u(\cdot, t).$$

(iii) *Moreover, the evolving hypersurfaces converge smoothly, after rescaling, to a round sphere.*

Remark 1.1. (1) In order to avoid any potential confusion with the mean curvature H, we use $C^{m+2+\gamma, \frac{m+2+\gamma}{2}}$ not $H^{m+2+\gamma, \frac{m+2+\gamma}{2}}$ used in [11] to represent the parabolic Hölder norm.

(2) If $\alpha = \beta = 1$, then the flow (1.9) degenerates into the classical scale-invariant ICF considered in [9, 20]. If $\alpha = 1$, $0 < \beta \leq 1$, then (1.9) becomes

the non-scale-invariant ICF considered in [11] or [21], where, in this case, the one-parameter family $X(\cdot,t)$ satisfies

$$\frac{\partial}{\partial t}X = \frac{1}{F^{\beta}}\nu. \tag{1.10}$$

Hence, the long-time existence and the asymptotic behavior description of the flow (1.9) in Theorem 1.1 improve the corresponding conclusions shown in [9, 11, 20, 21].

(3) It is interesting and important to see how the techniques used for the ICF (1.10) also apply for the anisotropic ICF (1.9), and what cannot be used.

Theorem 1.2. *Let $\alpha \leq 1$, $\alpha+\beta > 2$. Assume that the initial $C^{m+2,\gamma}$-hypersurface ($4 \leq m \leq \infty$, $0<\gamma<1$) is closed, strictly convex and $\Gamma = \Gamma_+$. Then the solution of the flow (1.8) exists on a maximal finite time interval $[0,T^*)$ and belongs to $C^{m+2+\gamma,\frac{m+2+\gamma}{2}}(\mathbb{S}^n \times [0,T^*))$. The leaves M_t are graphs over $\mathbb{S}^n$ and*

$$\lim_{t\to T^*}\inf_{\mathbb{S}^n} u(t,\cdot) = \infty.$$

Remark 1.2. Unlike what has shown in [11, Theorem 1.2], one cannot get the convergence for the rescaled flow of (1.8) in the case $\alpha < 1$, $\alpha+\beta > 2$ – for the reason, see Remark 4.1. This fact gives an example that some conclusion of the ICF (1.10) cannot be transferred to its anisotropic version (1.9).

The paper is organized as follows. In Section 2, we will firstly give some formulae for star-shaped hypersurfaces in $\mathbb{R}^{n+1}$, and then use these formulae to get the scalar version of the ICF (1.8), which leads to the short-time existence of the flow. In Sections 3 and 4, C^0-estimate, the gradient estimate, C^2-estimate will be given for the solution of the scalar flow equation. These estimates, together with the Krylov-Safonov estimate method for the second-order parabolic partial differential equations (PDEs for short), will give the long-time existence of the flow (1.8) if $\alpha \leq 1$, $\beta > 0$, $\alpha+\beta \leq 2$, or will show that the flow (1.8) blows up at the finite time $T^* < \infty$ if furthermore the initial hypersurface is strictly convex and $\alpha \leq 1$, $\alpha+\beta > 2$. In Section 5, the asymptotical behavior, after rescaling, of the ICF (1.8) will be revealed for the case $\alpha \leq 1$, $\beta > 0$, $\alpha+\beta \leq 2$.

2. The corresponding scalar equation

For a Riemannian manifold (M,g), the Riemann curvature (3,1)-tensor Rm is defined by

$$Rm(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

for $X,Y,Z \in \mathcal{X}(M)$, with $\mathcal{X}(M)$ the set of vector fields on M of class at least C^2. Pick a local coordinate chart $\{x_i\}_{i=1}^n$ of M, and then component of the (3,1)-tensor Rm is given by

$$Rm\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \doteq R^l_{ijk}\frac{\partial}{\partial x^l},$$

where $R_{ijkl} \doteq g_{lm} R^m_{ijk}$ and R_{ijkl} is the Riemannian curvature of M. It is well-known that we have the standard commutation formulas (i.e., Ricci identities)

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\alpha_{k_1 \cdots k_r} = \sum_{l=1}^{r} R^m_{ijk_l} \alpha_{k_1 \cdots k_{l-1} m k_{l+1} \cdots k_r}.$$

If furthermore (M, g) is an immersed hypersurface in $\mathbb{R}^{n+1}$. Let ν be a given unit outward normal and h_{ij} be the second fundamental form of the hypersurface M with respect to ν, that is,

$$h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle_{\mathbb{R}^{n+1}}.$$

Denote by $X_{ij} = \partial_i \partial_j X - \Gamma^k_{ij} X_k$, where Γ^k_{ij} is the Christoffel symbol of the metric on M. Recalling the following identities:

$$X_{ij} = -h_{ij}\nu, \qquad \text{Gauss formula;} \tag{2.1}$$

$$\nu_i = h_{ij} X^j, \qquad \text{Weingarten formula;} \tag{2.2}$$

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \qquad \text{Gauss equation;} \tag{2.3}$$

$$\nabla_k h_{ij} = \nabla_j h_{ik}, \qquad \text{Codazzi equation.} \tag{2.4}$$

Then, by the Codazzi equation we get

$$\nabla_i \nabla_j h_{kl} = \nabla_i (\nabla_j h_{lk}) = \nabla_i (\nabla_k h_{lj}) = \nabla_i \nabla_k h_{lj}.$$

By the Ricci identities, we have

$$\nabla_i \nabla_j h_{kl} = \nabla_k \nabla_i h_{lj} + R_{iklm} h^m_j + R_{ikjm} h^m_l.$$

Using the Codazzi equation again, it follows that

$$\begin{aligned} \nabla_i \nabla_j h_{kl} &= \nabla_k (\nabla_l h_{ji}) + R_{iklm} h^m_j + R_{ikjm} h^m_l \\ &= \nabla_k \nabla_l h_{ji} + R_{iklm} h^m_j + R_{ikjm} h^m_l. \end{aligned}$$

By the Gauss equation, we have

$$\nabla_i \nabla_j h_{kl} = \nabla_k \nabla_l h_{ij} + h^m_j (h_{il} h_{km} - h_{im} h_{kl}) + h^m_l (h_{ij} h_{km} - h_{im} h_{kj}). \tag{2.5}$$

Using coordinates on the unit sphere $\mathbb{S}^n$, we can equivalently formulate the problem by the corresponding scalar equation. Since the initial $C^{m+2,\gamma}$-hypersurface is star-sharped, there exists a scalar function $u_0 \in C^{m+2,\gamma}(\mathbb{S}^n)$ such that $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ has the form $x \mapsto (u_0(x), x)$. The hypersurface M_t given by the embedding

$$X(\cdot, t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$$

at time t may be represented as a graph over $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and then we can make ansatz

$$X(x, t) = (u(x, t), x)$$

for some function $u : \mathbb{S}^n \times [0, T) \to \mathbb{R}$.

Lemma 2.1. *Define $p := X(x,t)$ and assume that a point on $\mathbb{S}^n$ is described by local coordinates $\xi^1, \dots, \xi^n$, that is, $x = x(\xi^1, \dots, \xi^n)$. Let ∂_i be the corresponding coordinate fields on $\mathbb{S}^n$ and $\sigma_{ij} = g_{\mathbb{S}^n}(\partial_i, \partial_j)$ be the metric on $\mathbb{S}^n$. Let $u_i = D_i u$, $u_{ij} = D_j D_i u$, and $u_{ijk} = D_k D_j D_i u$ denote the covariant derivatives of u with respect to the round metric $g_{\mathbb{S}^n}$ and let ∇ be the Levi-Civita connection of M_t with respect to the metric g induced from the standard metric of $\mathbb{R}^{n+1}$. Then, the following formulas hold*:

(i) *The tangential vector on M_t is*

$$X_i = \partial_i + u_i \partial_r$$

and the corresponding outward unit normal vector is given by

$$\nu = \frac{1}{v}\left(\partial_r - \frac{1}{u^2} u^j \partial_j\right),$$

where $u^j = \sigma^{ij} u_i$, and $v := \sqrt{1 + u^{-2}|Du|^2}$ with the gradient Du of u.

(ii) *The induced metric g on M_t has the form*

$$g_{ij} = u^2 \sigma_{ij} + u_i u_j$$

and its inverse is given by

$$g^{ij} = \frac{1}{u^2}\left(\sigma^{ij} - \frac{u^i u^j}{u^2 v^2}\right).$$

(iii) *The second fundamental form of M_t is given by*

$$h_{ij} = \frac{1}{v}\left(-u_{ij} + u\sigma_{ij} + \frac{2}{u} u_i u_j\right)$$

and

$$h^i_j = g^{ik} h_{jk} = \frac{1}{uv}\delta^i_j - \frac{1}{uv}\widetilde{\sigma}^{ik}\varphi_{jk}, \quad \widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2},$$

where $\varphi = \log u$.

Proof. The formulas can be derived by direct calculation. The details can be found in [4]. □

Using techniques as in Ecker [6] (see also [9, 10]), the problem (1.9) can be reduced to solve the following scalar equation with the corresponding initial data:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{v}{u^{1-\alpha} F^\beta} & \text{in } \mathbb{S}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{S}^n. \end{cases} \tag{2.6}$$

Together with the homogeneous assumption on F, the first evolution equation in (2.6) can be rewritten as[1]

$$\frac{\partial}{\partial t}\varphi = e^{(\alpha+\beta-2)\varphi}(1+|D\varphi|^2)^{\frac{1}{2}}\frac{1}{F^{\beta}(\widetilde{h}^i_j)} := Q(\varphi, D\varphi, D^2\varphi), \tag{2.7}$$

where $\varphi(x,t) = \log u(x,t)$ and

$$\widetilde{h}^i_j = uh^i_j = \frac{1}{v}(\delta^i_j - \widetilde{\sigma}^{ik}\varphi_{jk}).$$

By using a similar method to [9, pp. 301–303], we know that the nonnegativity assumption (1.6) lets the flow equation in (1.9) or (2.6) makes sense at the beginning of the evolution process, and the monotonicity and concavity assumptions (1.4), (1.5) make sure that the flow equation is a nonlinear second-order parabolic PDE, which implies that the ICF (1.9) is reduced to the following scalar equation with the initial condition:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = Q(\varphi, D\varphi, D^2\varphi) & \text{in } \mathbb{S}^n \times (0,T), \\ \varphi(\cdot, 0) = \varphi_0 & \text{in } \mathbb{S}^n \end{cases} \tag{2.8}$$

for some $T > 0$. In fact, as in [9, 10], by the standard theory of second-order parabolic PDEs, we can get the following existence and uniqueness for the system (1.9).

Lemma 2.2. *Let $X_0(\mathbb{S}^n) = M_0$ be as in Theorem 1.1. Then there exist some $T > 0$, a unique solution $u \in C^{m+2+\gamma, \frac{m+2+\gamma}{2}}(\mathbb{S}^n \times [0,T])$, where $\varphi(x,t) = \log u(x,t)$, to the parabolic system* (2.8)*. Thus there exists a unique map $\psi : \mathbb{S}^n \times [0,T] \to \mathbb{S}^n$ such that the map $\widehat{X}$ defined by*

$$\widehat{X} : \mathbb{S}^n \times [0,T) \to \mathbb{R}^{n+1} : (x,t) \mapsto X(\psi(x,t), t)$$

has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.9)*.*

Let T^* be the maximal time such that there exists some

$$u \in C^{m+2+\gamma, \frac{m+2+\gamma}{2}}(\mathbb{S}^n \times [0,T^*))$$

which solves (2.8). In the sequel, we shall prove a priori estimates for those admissible solutions on $[0,T]$ where $T < T^*$.

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[1]Let $\mathcal{M}(\Gamma)$ be the set of all $n \times n$ matrices whose eigenvalues lie in the open cone $\Gamma \subset \mathbb{R}^n$. Then one can define a function $\mathcal{F}$ on $\mathcal{M}(\Gamma)$ such that $\mathcal{F}(a^{ij}) = F(\lambda_i)$, where (λ_i) are the eigenvalues of the matrix (a^{ij}). By the abuse of notations, we still use F to represent the function $\mathcal{F}$, which implies that, in (2.7), $F(\widetilde{h}^i_j)$ is essentially $\mathcal{F}(\widetilde{h}^i_j)$. As shown in [2], the monotonicity and concavity assumptions (1.4), (1.5) on F implies that $\left(\frac{\partial F}{\partial a^{ij}}\right)_{n\times n}$ is positive definite, and $\left(\frac{\partial^2 F}{\partial a^{ij}\partial a^{rs}}\right)_{n\times n}$ is negative semi-definite, which can be used to show that (2.7) is a second-order parabolic PDE.

3. C^0, $\dot{\varphi}$ and gradient estimates

3.1. C^0 estimates

We first show clearly the evolution of spheres under the flow (1.9). Fix a point $o \in \mathbb{R}^{n+1}$, consider the polar coordinates $\{r, \xi^i, \dots, \xi^{n-1}\}$ around o, which leads to the fact that the standard Euclidean metric of $\mathbb{R}^{n+1}$ can be expressed as

$$ds^2 = dr^2 + r^2 \sigma_{ij} d\xi^i d\xi^j,$$

where, as in Lemma 2.1, $\sigma_{ij} = g_{\mathbb{S}^n}(\partial_i, \partial_j)$ denotes the round metric on $\mathbb{S}^n$. Spheres with center p_0 and the radius r are umbilical, their second fundamental forms are given by

$$\bar{h}_{ij} = r^{-1} \bar{g}_{ij},$$

which implies, in this setting, the flow equation in (1.9) degenerates into

$$\frac{\partial r}{\partial t} = \frac{1}{r^{1-\alpha}(nr^{-1})^{\beta}} = r^{\alpha+\beta-1} n^{-\beta}. \tag{3.1}$$

If $\alpha + \beta \neq 2$, then solving the ODE (3.1) yields[2]

$$r(t) = \left(\frac{2-\alpha-\beta}{n^{\beta}} t + {r_0}^{2-\alpha-\beta} \right)^{\frac{1}{2-\alpha-\beta}},$$

where $r(0) = r_0$ is the radius of the initial sphere. Therefore, we have:

Lemma 3.1. *If the initial hypersurface is a sphere, the flow* (3.1) *exists for all the time if $\alpha + \beta \leq 2$, and converges to infinity, while in case $\alpha + \beta > 2$, the flow blows up at finite time*

$$T_s = \frac{n^{\beta}}{\alpha + \beta - 2} r_0^{2-\alpha-\beta}.$$

By using Lemma 3.1 and the maximum principle for second-order parabolic PDEs, we can get the following.

Corollary 3.2. *Assume that $\alpha \leq 1$ and $\beta > 0$. Let $M_0 = \mathrm{graph}_{\mathbb{S}^n} u_0$ be star-shaped, $u(\xi, t)$ be a solution of the flow* (2.6) *and r_1, r_2 be positive constants such that*

$$r_1 < u_0(\xi) < r_2, \quad \forall \xi \in \mathbb{S}^n.$$

Then $u(\xi, t)$ satisfies

$$\Theta(r_1, t) < u(\xi, t) < \Theta(r_2, t), \quad \forall\ 0 \leq t < \min\{T^*, T^*(r_1), T^*(r_2)\}, \tag{3.2}$$

[2]In fact, if $\alpha \leq 1$ and $\alpha + \beta = 2$, then we have $r(t) = r_0 e^{\frac{t}{n^{\beta}}}$ by solving the ODE (3.1) with the initial condition $r(0) = r_0$, which leads to the fact that the barrier function $\Theta(r, t)$ should be replaced by $\Theta(r, t) = r e^{\frac{t}{n^{\beta}}}$. Although $\Theta(r, t)$ has a different form, it is not difficult to check that Corollary 3.2 and all the estimates in the sequel for the case $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta < 2$ would be still true for the case $\alpha \leq 1$, $\alpha + \beta = 2$. Therefore, the ranges of α, β in Theorem 1.1 should be $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$.

where

$$\Theta(r,t) = \left(\frac{2-\alpha-\beta}{n^\beta}t + r^{2-\alpha-\beta}\right)^{\frac{1}{2-\alpha-\beta}}$$

and $T^*(r_i)$, $i = 1,2$, is the maximal time for which the spherical flow with initial sphere of radius r_i will exist.

Proof. On one hand, as shown at the beginning of this subsection, it is easy to know that spheres with radii $\Theta(r_i,t)$ are the spherical solutions of the flow (3.1) with the initial sphere of radius r_i.

On the other hand, as shown in (2.6), the ICF (1.8) can be reduced to the scalar parabolic equation

$$\frac{\partial u}{\partial t} = \frac{v}{u^{1-\alpha}F^\beta}, \tag{3.3}$$

which is obviously satisfied by $\Theta(r_i,t)$ and can be reduced to the ODE (3.1) provided the initial hypersurface M_0 is a sphere with radius r_i.

Therefore, applying the maximum principle for second-order parabolic PDEs to the difference $u(\xi,t) - \Theta(r_i,t)$, $i = 1,2$, and together with the linearization process, the conclusion of Corollary 3.2 follows. □

By applying Corollary 3.2 directly, we have the following.

Corollary 3.3. *Let $\alpha \le 1$, $\beta > 0$, $\alpha+\beta \le 2$, and $r_1 < r < r_2$. Then we have*

$$c_1 \le u(x,t)\Theta^{-1}(r,t) \le c_2, \quad \forall\, x \in \mathbb{S}^n,\ t \in [0,T^*),$$

for some positive constants c_1, c_2 depending only on r_1, r_2, α and β. The flow is compactly contained in $\mathbb{R}^{n+1}$ for finite $t > 0$.

Conversely, we have the following:

Lemma 3.4. *If $\alpha \le 1$ and $\alpha+\beta > 2$, then the flow* (1.8) (*resp.,* (3.3)) *only exists in a finite time interval $[0,T^*)$, and*

$$\limsup_{t\to T^*} \max_{\mathbb{S}^n} u(\cdot,t) = \infty \tag{3.4}$$

holds.

Proof. For the situation $\alpha \le 1$ and $\alpha+\beta > 2$, the initial hypersurface is assumed to be strictly convex, which leads to the fact that, under the ICF (1.8) (resp., (3.3)), the evolving hypersurfaces are also convex. By the estimate (3.2) in Corollary 3.2, the maximal time T^* has to be finite for the case $\alpha \le 1$ and $\alpha+\beta > 2$. Also we know that the flow (1.8) (resp., (3.3)) will remain smooth with uniform estimates as long as it stays in a compact domain, which implies that (3.4) must be valid. □

Let $r_0 > 0$ be the radius such that for the function $\Theta(r_0,t)$, where $\alpha \le 1$ and $\alpha+\beta > 2$, the singularity is

$$T_s(r_0) = T^*. \tag{3.5}$$

We can prove the following:

Lemma 3.5. *Let u be the solution of the scalar flow equation* (3.3) *and assume $\alpha \leq 1$, $\alpha + \beta > 2$ and that* (3.4) *is valid. Then there exists a positive constant c_3 such that*

$$u(\xi, t) - c_3 \leq \Theta(r_0, t) \leq u(\xi, t) + c_3, \quad \forall\, \xi \in \mathbb{S}^n, \tag{3.6}$$

and therefore

$$\lim_{t \to T^*} u(\xi, t)\Theta^{-1}(r_0, t) = 1, \quad \forall\, \xi \in \mathbb{S}^n \tag{3.7}$$

holds.

Proof. Without loss of generality, assume that the origin is inside the convex body enclosed by M_0, since when $\alpha \leq 1$ and $\alpha + \beta > 2$, M_0 is assumed to be strictly convex. Inspired by the works of Gerhardt [11] and Urbas [21], we know that the support function

$$\bar{u} = \langle X, \nu \rangle \tag{3.8}$$

of the flow hypersurfaces M_t can be looked at as being on the Gauss image of M_t and $\bar{u}$ satisfies the parabolic equation

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} = (\sqrt{\bar{u}^2 + |D\bar{u}|^2})^{\alpha-1} \widetilde{F}^{\beta}(D^2\bar{u} + \bar{u}I), & \text{on } \mathbb{S}^n \times [0, T^*), \\ \bar{u}(\cdot, 0) = \langle X_0, \nu_0 \rangle, & \text{in } \mathbb{S}^n, \end{cases} \tag{3.9}$$

where $\widetilde{F}$ is defined as follows

$$\widetilde{F}(a_{ij}) = \frac{1}{F(\lambda_1^{-1}, \ldots, \lambda_n^{-1})}, \tag{3.10}$$

with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $[a_{ij}]$ and $(\lambda_1, \ldots, \lambda_n) \in \Gamma_+$.

First we need to show that there exists a constant $c_4 > 0$, depending only on $\bar{u}_0 := \bar{u}(\cdot, 0)$, such that

$$\operatorname{osc} \bar{u} \leq c_4. \tag{3.11}$$

For this purpose, we apply the Aleksandrov's reflection principle (see, e.g., [3]). Fixing a direction $a \in \mathbb{S}^n$ and $\lambda > 0$, we consider the reflection of x with respect to the hyperplane $\{z \in \mathbb{R}^{n+1} : z \cdot a = 0\} \subset \mathbb{R}^{n+1}$

$$x^* = x - 2\langle x, a \rangle a$$

and define a new function

$$u^*(x, t) = \bar{u}(x^*, t).$$

Given $\lambda > 0$, we define

$$u_\lambda(x, t) = u^*(x, t) + \lambda \langle x, a \rangle. \tag{3.12}$$

Let Ω_t, Ω_t^* and Ω_t^λ be the convex bodies whose support functions are respectively $\bar{u}(\cdot,t)$, $u^*(\cdot,t)$ and $u_\lambda(\cdot,t)$. Clearly, Ω_t^λ is a translation and a reflection of Ω_t and Ω_t^λ and Ω_t are symmetric with respect to the hyperplane $\Pi_\lambda = \{z \in \mathbb{R}^{n+1} : z \cdot a = \frac{\lambda}{2}\}$. Set

$$\Pi_\lambda^+ = \left\{ z \in \mathbb{R}^{n+1} : z \cdot a \geq \frac{\lambda}{2} \right\} \quad \text{and} \quad \Pi_\lambda^- = \left\{ z \in \mathbb{R}^{n+1} : z \cdot a \leq \frac{\lambda}{2} \right\}.$$

Since the initial data Ω_0 is compact, there exists $\lambda = \lambda(u_0) > 0$ which depends only on $\bar{u}_0$ (is independent of a) such that

$$\Omega_0 \in \operatorname{int}\left(\Pi_\lambda^-\right) \quad \text{and} \quad \Omega_0^\lambda \in \operatorname{int}\left(\Pi_\lambda^+\right). \tag{3.13}$$

Then, for any $x \in \mathbb{S}_+^n := \{y \in \mathbb{S}^n : y \cdot a \geq 0\}$, we have

$$u_\lambda(x,0) \geq \bar{u}(x,0)$$

and the equality holds only on $\partial\mathbb{S}_+^n$. We claim that for any $(x,t) \in \mathbb{S}_+^n \times [0,T)$,

$$u_\lambda(x,t) \geq \bar{u}(x,t). \tag{3.14}$$

In order to prove the claim (3.14), let $\hat{o} = \lambda a$ and $u_{\lambda,\,\hat{o}}(\cdot,t)$ be the support function of Ω_t^λ with respect to the center $\hat{o}$. Then

$$u_{\lambda,\,\hat{o}}(\cdot,t) = u^*(\cdot,t).$$

Thus, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} u_\lambda(x,t) &= \frac{\partial}{\partial t} u^*(x,t) = \left(\sqrt{(u^*)^2 + |Du^*|^2}\right)^{\alpha-1} \widetilde{F}^\beta(D^2u^* + u^*I) \\ &= \left(\sqrt{(u_{\lambda,\,\hat{o}})^2 + |Du_{\lambda,\,\hat{o}}|^2}\right)^{\alpha-1} \widetilde{F}^\beta(D^2u_{\lambda,\,\hat{o}} + u_{\lambda,\,\hat{o}}I) \end{aligned} \tag{3.15}$$

on $\mathbb{S}^n \times [0,T)$.

For Ω_t and Ω_t^λ, denote by $\nu_{\Omega_t}^{-1}$ and $\nu_{\Omega_t^\lambda}^{-1}$ the corresponding inverse Gauss map. Let $\mathcal{M}_t^+ = \nu_{\Omega_t}^{-1}(\mathbb{S}_+^n)$ and $\mathcal{M}_t^{\lambda,+} = \nu_{\Omega_t^\lambda}^{-1}(\mathbb{S}_+^n)$. For $x_0 \in \operatorname{int}\left(\mathbb{S}_+^n\right)$, let $t_0 \in [0,T)$ be the time such that

- $\nu_{\Omega_{t_0}}^{-1}(x_0) = \nu_{\Omega_{t_0}^\lambda}^{-1}(x_0) := z_0$;
- $\nu_{\Omega_t}^{-1}(x_0) \neq \nu_{\Omega_t^\lambda}^{-1}(x_0)$ for all $t \in [0,t_0)$ and $u_\lambda(\cdot,t_0) \geq u(\cdot,t_0)$ near x_0.

If for any $x_0 \in \operatorname{int}\left(\mathbb{S}_+^n\right)$, no such t_0 exists, then one infers by (3.13) that

$$\operatorname{int}\left(\mathcal{M}_t^+\right) \cap \operatorname{int}\left(\mathcal{M}_t^{\lambda,+}\right) = \emptyset$$

remains for all $t \in [0,T^*)$. Therefore, (3.14) follows immediately. Suppose t_0 exists. Then,

$$D^2u_\lambda(x_0,t_0) \geq D^2\bar{u}(x_0,t_0). \tag{3.16}$$

By the symmetry, it is easy to see that $z_0 \in P_\lambda^+$. Hence,

$$|z_0 - o| \geq |z_0 - \hat{o}|, \tag{3.17}$$

where o is the origin and $\hat{o} = \lambda a$ given as above. Since $\alpha \leq 1$ and $\beta > 0$, we have by using (3.15), (3.16) and (3.17)

$$\begin{aligned}\frac{\partial}{\partial t}u_\lambda(x_0,t_0) &= |z_0-\hat{o}|^{\alpha-1}\widetilde{F}^\beta(D^2u_\lambda + u_\lambda I)(x_0,t_0)\\ &\geq |z_0-o|^{\alpha-1}\widetilde{F}^\beta(D^2\bar{u} + \bar{u}I)(x_0,t_0)\\ &= \frac{\partial}{\partial t}u(x_0,t_0).\end{aligned}$$

This implies (3.14).

Then, given any two points $x_1, x_2 \in \mathbb{S}^n$ with $x_1 \neq x_2$, let

$$a = \frac{x_2 - x_1}{|x_2 - x_1|}.$$

Then, $x_2^* = x_2 - 2\langle x_2, a\rangle a = x_1$. Thus, we have in view of (3.14),

$$u_\lambda(x_2,t) = u(x_1,t) + \lambda\langle x_2, a\rangle \geq u(x_2,t),$$

which, by noticing $\langle x_2, a\rangle = \frac{|x_2-x_1|}{2}$, implies

$$\frac{u(x_2,t)-u(x_1,t)}{|x_2-x_1|} \leq \frac{\lambda}{2}.$$

Then (3.11) follows. Since $u(t,\xi) = \bar{u}(t,\xi)$ when ξ is an extremal point, we have

$$\mathrm{osc}\bar{u} \leq c_4.$$

Using a similar argument to that of [19, Lemma 5.1], it follows that for any $t \in [0, T^*)$, there exists ξ_t such that

$$u(\xi_t, t) = \Theta(r_0, t).$$

Conclusions (3.6) and (3.7) follow directly by combining the facts $\mathrm{osc}u \leq c_4$ and $u(\xi_t,t) = \Theta(r_0,t)$. This completes the proof. □

Proof of Theorem 1.2. Clearly, Theorem 1.2 is a consequence of Lemmas 2.2 and 3.5. □

3.2. $\dot{\varphi}$ estimate

We shall show that $\dot{\varphi}(x,t)\Theta(t)^{2-\alpha-\beta}$ keeps bounded during the flow evolution.

Lemma 3.6. *Assume that $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$, and let φ be a solution of* (2.8). *Then*

$$\begin{aligned}\min\left\{\inf_{\mathbb{S}^n}\dot{\varphi}(\cdot,0)\cdot\Theta(0)^{2-\alpha-\beta}, \frac{1}{n^\beta}\right\} &\leq \dot{\varphi}(x,t)\Theta(t)^{2-\alpha-\beta}\\ &\leq \max\left\{\sup_{\mathbb{S}^n}\dot{\varphi}(\cdot,0)\cdot\Theta(0)^{2-\alpha-\beta}, \frac{1}{n^\beta}\right\}.\end{aligned}$$

Proof. Set

$$M(x,t) = \dot{\varphi}(x,t)\Theta(t)^{2-\alpha-\beta}.$$

Differentiating both sides of the first evolution equation of (2.8), it is easy to get that $\dot{\varphi}$ satisfies

$$\begin{cases} \dfrac{\partial M}{\partial t} = Q^{ij}D_{ij}M + Q^k D_k M + (2-\alpha-\beta)\Theta^{\alpha+\beta-2}\left(\dfrac{1}{n^\beta} - M\right)M \\ \qquad\qquad \text{in } \mathbb{S}^n \times (0,T), \\ M(\cdot,0) = \dot{\varphi}_0 \cdot \Theta(0)^{2-\alpha-\beta} \quad \text{on } \mathbb{S}^n, \end{cases} \tag{3.18}$$

where $Q^{ij} = \frac{\partial Q}{\partial \varphi_{ij}}$ and $Q^k = \frac{\partial Q}{\partial \varphi_k}$. Then, we have

$$\frac{\partial M}{\partial t} = Q^{ij}D_{ij}M + Q^k D_k M + (2-\alpha-\beta)\Theta^{\alpha+\beta-2}\left(\frac{1}{n^\beta} - M\right)M.$$

For the lower bound, on the domain $\{(x,t) \in \mathbb{S}^n \times (0,T) \,|\, M(x,t) < \frac{1}{n^\beta}\}$, we have

$$(2-\alpha-\beta)\Theta^{\alpha+\beta-2}\left(\frac{1}{n^\beta} - M(x,t)\right) \geq 0,$$

which, by applying the maximum principle, implies

$$M(x,t) \geq \inf_{\mathbb{S}^n} \dot{\varphi}(\cdot,0) \cdot \Theta(0)^{2-\alpha-\beta}$$

for any $(x,t) \in \{(x,t) \in \mathbb{S}^n \times (0,T) \,|\, M(x,t) < \frac{1}{n^\beta}\}$. So

$$M(x,t) \geq \min\left\{\inf_{\mathbb{S}^n} \dot{\varphi}(\cdot,0) \cdot \Theta(0)^{2-\alpha-\beta}, \frac{1}{n^\beta}\right\}.$$

Similarly, we have

$$M(x,t) \leq \max\left\{\sup_{\mathbb{S}^n} \dot{\varphi}(\cdot,0) \cdot \Theta(0)^{2-\alpha-\beta}, \frac{1}{n^\beta}\right\}.$$

Therefore, we complete our proof. □

3.3. The gradient estimate

Lemma 3.7. *Let $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$, and φ be a solution of* (2.8)*. Then we have*

$$|D\varphi| \leq \left(\frac{c_3'}{\frac{2-\alpha-\beta}{n^\beta}t + c_3'}\right)^{n^\beta c_4'} \sup_{\mathbb{S}^n} |D\varphi(\cdot,0)|, \quad \forall\ x \in \mathbb{S}^n,\ t \in [0,T], \tag{3.19}$$

where c_3' and c_4' are positive constants.

Proof. Set $\psi = \frac{|D\varphi|^2}{2}$. By differentiating the function ψ, we have

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} D_m\varphi D^m\varphi = D_m\dot{\varphi}D^m\varphi = D_m Q D^m\varphi.$$

Then

$$\frac{\partial \psi}{\partial t} = Q^{ij} D_{ijm}\varphi D^m\varphi + Q^k D_{km}\varphi D^m\varphi + (\alpha+\beta-2)Q|D\varphi|^2.$$

Interchanging the covariant derivatives, we have

$$\begin{aligned} D_{ij}\psi &= D_j(D_{mi}\varphi D^m\varphi) \\ &= D_{mij}\varphi D^m\varphi + D_{mi}\varphi D_j^m\varphi \\ &= (D_{ijm}\varphi + R^l_{jmi}D_l\varphi)D^m\varphi + D_{mi}D_j^m\varphi. \end{aligned}$$

Therefore, we can express $D_{ijm}\varphi D^m\varphi$ as

$$D_{ijm}\varphi D^m\varphi = D_{ij}\psi - R^l_{jmi}D_l\varphi D^m\varphi - D_{mi}\varphi D_j^m\varphi.$$

Then, in view of the fact $R_{jmil} = \sigma_{ji}\sigma_{ml} - \sigma_{lj}\sigma_{im}$ on $\mathbb{S}^n$, we have

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= Q^{ij}D_{ij}\psi + Q^k D_k\psi - Q^{ij}(\sigma_{ij}|D\varphi|^2 - D_i\varphi D_j\varphi) \\ &\quad - Q^{ij}D_{mi}\varphi D_j^m\varphi + (\alpha+\beta-2)Q|D\varphi|^2. \end{aligned} \tag{3.20}$$

Since the matrix Q^{ij} is positive definite, the third and the fourth terms in the RHS of (3.20) are non-positive. The last term in the RHS of (3.20) can be estimated if $\alpha+\beta \le 2$ by using Lemma 3.6, i.e.,

$$\begin{aligned} (2-\alpha-\beta)Q|D\varphi|^2 &= 2(2-\alpha-\beta)\psi\Theta^{\alpha+\beta-2}Q\Theta^{2-\alpha-\beta} \\ &\ge 2(2-\alpha-\beta)\frac{c_4}{\frac{2-\alpha-\beta}{n^\beta}t + c_3}\psi. \end{aligned}$$

So we get the equation about ψ as follows:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} \le Q^{ij}D_{ij}\psi + Q^k D_k\psi - 2(2-\alpha-\beta)\dfrac{c_4'}{\frac{2-\alpha-\beta}{n^\beta}t + c_3'}\psi \quad \text{in } \mathbb{S}^n \times (0,T], \\ \psi(\cdot,0) = \dfrac{|D\varphi(\cdot,0)|^2}{2} \quad \text{in } \mathbb{S}^n. \end{cases}$$

Using the maximum principle, we get the gradient estimate of φ in Lemma 3.7. □

Corollary 3.8. *Under the assumptions of Theorem 1.1, the evolving hypersurface M_t is always star-shaped.*

Proof. We just need to show

$$\left\langle \frac{X}{|X|}, \nu \right\rangle = \frac{1}{v}$$

is bounded from below by some positive constant, which is clearly implied by the estimate (3.19) in Lemma 3.7. □

Combining the gradient estimate with $\dot{\varphi}$ estimate, we can obtain:

Corollary 3.9. *Under the assumptions of Theorem 1.1, if φ satisfies* (2.8), *then we have*

$$0 < c_5 \leq F(\widetilde{h}^i_j) \leq c_6 < +\infty, \tag{3.21}$$

where c_5 and c_6 are positive constants independent of φ.

For the case $\alpha \leq 1$, $\alpha+\beta > 2$, one can also get a gradient estimate as follows.

Lemma 3.10. *Let $\alpha \leq 1$, $\alpha+\beta > 2$ and assume* (3.4) *to be satisfied. Then*

$$v - 1 \leq c_7 \Theta^{-1},$$

i.e.,

$$\lim_{t \to T^*} |Du| = 0.$$

Proof. Let $\bar{u}$ be the support function defined as (3.8) and let $\widetilde{u} = u\Theta^{-1}$, $\widetilde{\bar{u}} = \bar{u}\Theta^{-1}$. Then by Lemma 3.5, we have

$$v - 1 = (u - \bar{u})\bar{u}^{-1} = (\widetilde{u} - 1)\widetilde{\bar{u}}^{-1} + (1 - \widetilde{\bar{u}})\widetilde{\bar{u}}^{-1} \leq c\Theta^{-1},$$

which implies the conclusion of Lemma 3.10. □

4. C^2 estimates

Set $\Psi = \frac{1}{|X|^{1-\alpha}F^\beta}$, $\chi = \langle X, \nu\rangle^{-1}$, $F^{ij} = \frac{\partial F}{\partial h_{ij}}$ and $F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}$.

Lemma 4.1. *Under the flow*

$$\frac{\partial}{\partial t}X = \frac{1}{|X|^{1-\alpha}F^\beta}\nu,$$

we have the following evolution equations:

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= 2\Psi h_{ij},\\
\frac{\partial}{\partial t}g^{ij} &= -2\Psi h^{ij},\\
\frac{\partial}{\partial t}\nu &= -\nabla\Psi,
\end{aligned}$$

$$\begin{aligned}
\partial_t h^j_i - \beta\Psi F^{-1}F^{kl}h^j_{i,kl} = {} & \beta\Psi F^{-1}h^j_i F^{kl}h^m_l h_{km} - \Psi\frac{\beta(\beta+1)F_iF^j}{F^2}\\
& - (1+\beta)\Psi h_{ik}h^{kj}\\
& - (1-\alpha)\beta\Psi(\nabla_i \log u \nabla^j \log F + \nabla^j \log u \nabla_i \log F)\\
& - (\alpha-1)\Psi u^{-1}u^j_i - (\alpha-1)(\alpha-2)\Psi\nabla_i \log u \nabla^j \log u\\
& + \beta\Psi F^{-1}F^{kl,rs}h_{kl,}{}^j h_{rs,i} + \beta\Psi F^{-1}F^{kl}h^m_k h_{im}h^j_l\\
& - \beta\Psi F^{-1}F^{kl}h^{mj}h_{ik}h_{lm},
\end{aligned}$$

and

$$\partial_t\chi - \beta\Psi F^{-1}F^{kl}\chi_{kl} = (\beta-1)\chi^2\Psi - 2\beta\Psi F^{-1}\chi^{-1}F^{kl}\chi_k\chi_l$$

$$- \beta\Psi F^{-1}F^{kl}h_l^m h_{km}\chi + (\alpha-1)\Psi\chi^2\nabla^l \log u\langle X, X_l\rangle.$$

Proof. The first three evolution equations are easy to get, and we omit the derivation process here. Now, we show how to get the last two equations.

On one hand, using the Gauss formula (2.1), we have

$$\begin{aligned}
\partial_t h_{ij} &= \partial_t\langle \partial_i\partial_j X, -\nu\rangle \\
&= \langle \partial_i\partial_j(\Psi\nu), -\nu\rangle - \langle \Gamma_{ij}^k \partial_k X - h_{ij}\nu, \partial_t\nu\rangle \\
&= -\partial_i\partial_j\Psi - \Psi\langle \partial_i\partial_j\nu, \nu\rangle + \Gamma_{ij}^k\Psi_k \\
&= -\nabla_{ij}^2\Psi - \Psi\langle \partial_i(h_j^k\partial_k X), \nu\rangle \\
&= -\nabla_{ij}^2\Psi + \Psi h_{ik}h_j^k. \qquad (4.1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla_{ij}^2\Psi = \Psi &\left(-\frac{\beta}{F}F_{ij} + \frac{\beta(\beta+1)F_iF_j}{F^2}\right) \\
&+ (1-\alpha)\beta\Psi(\nabla_i \log u\nabla_j \log F + \nabla_j \log u\nabla_i \log F) \\
&+ (\alpha-1)\Psi u^{-1}u_{ij} + (\alpha-1)(\alpha-2)\Psi\nabla_i \log u\nabla_j \log u,
\end{aligned}$$

where $F_{ij} = F^{kl}h_{kl,ij} + F^{kl,rs}h_{kl,i}h_{rs,j}$.

Since

$$\begin{aligned}
F^{kl}h_{ij,kl} = F_{ij} &+ Fh_{im}h_j^m - h_{ij}F^{kl}h_l^m h_{km} - F^{kl,rs}h_{kl,i}h_{rs,j} \\
&+ F^{kl}h_k^m h_{im}h_{lj} - F^{kl}h_j^m h_{ik}h_{lm},
\end{aligned}$$

we have

$$\begin{aligned}
\nabla_{ij}^2\Psi = &(1-\alpha)\beta\Psi(\nabla_i \log u\nabla_j \log F + \nabla_j \log u\nabla_i \log F) \\
&+ (\alpha-1)\Psi u^{-1}u_{ij} + \Psi\frac{\beta(\beta+1)F_iF_j}{F^2} \\
&+ (\alpha-1)(\alpha-2)\Psi\nabla_i \log u\nabla_j \log u \\
&- \beta\Psi F^{-1}(-Fh_{ik}h_j^k + h_{ij}F^{kl}h_l^m h_{km} + F^{kl}h_{ij,kl} + F^{kl,rs}h_{kl,i}h_{rs,j} \\
&- F^{kl}h_k^m h_{im}h_{lj} + F^{kl}h_j^m h_{ik}h_{lm}). \qquad (4.2)
\end{aligned}$$

Combining (4.1) and (4.2) yields

$$\begin{aligned}
\partial_t h_{ij} = &(\alpha-1)\beta\Psi(\nabla_i \log u\nabla_j \log F + \nabla_j \log u\nabla_i \log F) \\
&- (\alpha-1)\Psi u^{-1}u_{ij} - \Psi\frac{\beta(\beta+1)F_iF_j}{F^2} \\
&- (\alpha-1)(\alpha-2)\Psi\nabla_i \log u\nabla_j \log u \\
&- \beta\Psi h_{ik}h_j^k + \beta\Psi F^{-1}h_{ij}F^{kl}h_l^m h_{km} + \beta\Psi F^{-1}F^{kl}h_{ij,kl} \\
&+ \Psi h_{ik}h_j^k + F^{kl,rs}h_{kl,i}h_{rs,j}\beta\Psi F^{-1} \\
&+ \beta\Psi F^{-1}F^{kl}h_k^m h_{im}h_{lj} - \beta\Psi F^{-1}F^{kl}h_j^m h_{ik}h_{lm},
\end{aligned}$$

which implies

$$\begin{aligned}
&\partial_t h_{ij} - \beta\Psi F^{-1}F^{kl}h_{ij,kl}\\
&= \beta\Psi F^{-1}h_{ij}F^{kl}h_l^m h_{km}\\
&\quad + (1-\beta)\Psi h_{ik}h_j^k - \Psi\frac{\beta(\beta+1)F_iF_j}{F^2}\\
&\quad - (1-\alpha)\beta\Psi(\nabla_i \log u\nabla_j \log F + \nabla_j \log u\nabla_i \log F)\\
&\quad - (\alpha-1)\Psi u^{-1}u_{ij} - (\alpha-1)(\alpha-2)\Psi\nabla_i \log u\nabla_j \log u\\
&\quad + \beta\Psi F^{-1}F^{kl,rs}h_{kl,i}h_{rs,j} + \beta\Psi F^{-1}F^{kl}h_k^m h_{im}h_{lj} - \beta\Psi F^{-1}F^{kl}h_j^m h_{ik}h_{lm}.
\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}
&\partial_t h_i^j - \beta\Psi F^{-1}F^{kl}h_{i,kl}^j\\
&= \beta\Psi F^{-1}h_i^j F^{kl}h_l^m h_{km}\\
&\quad - \Psi\frac{\beta(\beta+1)F_iF^j}{F^2} - (1+\beta)\Psi h_{ik}h^{kj}\\
&\quad - (1-\alpha)\beta\Psi(\nabla_i \log u\nabla^j \log F + \nabla^j \log u\nabla_i \log F)\\
&\quad - (\alpha-1)\Psi u^{-1}u_i^j - (\alpha-1)(\alpha-2)\Psi\nabla_i \log u\nabla^j \log u\\
&\quad + \beta\Psi F^{-1}F^{kl,rs}h_{kl,}{}^j h_{rs,i} + \beta\Psi F^{-1}F^{kl}h_k^m h_{im}h_l^j - \beta\Psi F^{-1}F^{kl}h^{mj}h_{ik}h_{lm}.
\end{aligned}$$

By direct calculation, one has

$$\partial_t\chi = -\chi^2\Psi + \chi^2\Psi((\alpha-1)\nabla^i \log u - \beta\nabla^i \log F)\langle X, X_i\rangle, \tag{4.3}$$

$$\chi_{ij} = 2\chi^3\langle X,\nu\rangle_i\langle X,\nu\rangle_j - \chi^2\langle X,\nu\rangle_{ij}. \tag{4.4}$$

Using the Weingarten equation (2.2), we have

$$\langle X,\nu\rangle_i = h_i^k\langle X, X_k\rangle, \tag{4.5}$$

and

$$\begin{aligned}
\langle X,\nu\rangle_{ij} &= h_{i,\ j}^k\langle X, X_k\rangle + h_{ij} - h_i^k h_{kj}\langle X,\nu\rangle\\
&= h_{ij,k}\langle X, X^k\rangle + h_{ij} - h_i^k h_{kj}\langle X,\nu\rangle.
\end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) results in

$$\chi_{ij} = -\chi^2 h_{ij} + 2\chi^{-1}\chi_i\chi_j + \chi h_i^k h_{kj} - \chi^2 h_{ij,k}\langle X, X^k\rangle,$$

which, together with (4.3), implies

$$\begin{aligned}
\partial_t\chi - \beta\Psi F^{-1}F^{kl}\chi_{kl} &= (\beta-1)\chi^2\Psi - 2\beta F^{-1}\chi^{-1}\Psi F^{kl}\chi_k\chi_l\\
&\quad - \beta\Psi F^{-1}F^{kl}h_l^m h_{km}\chi + (\alpha-1)\Psi\chi^2\nabla^i \log u\langle X, X_i\rangle.
\end{aligned}$$

This completes the proof. □

We try to get a priori estimate for the second order derivatives of φ.

Theorem 4.2. *Let φ be a solution of the flow* (2.8) *and $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$. Then, there exists a constant $C := C(\alpha, \beta, n, M_0)$, depending only on α, β, n and M_0, such that*

$$|\Theta\kappa_i| \leq C(\alpha, \beta, n, M_0), \quad \forall (x,t) \in \mathbb{S}^n \times [0, T^*).$$

Proof. Define functions

$$\zeta = \sup\{h_{ij}\eta^i\eta^j \mid g_{ij}\eta^i\eta^j = 1\}$$

and

$$w = \log\zeta + \log\chi + 2\log\Theta.$$

We claim that w is bounded. Fix $0 < T < T^*$. Suppose that w attains a maximal value at (t_0, ξ_0), that is,

$$\sup_{\mathbb{S}^n \times [0,T]} w = w(t_0, \xi_0), \quad t_0 > 0.$$

Choose Riemannian normal coordinates at (t_0, ξ_0) such that at this point we have

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}, \quad \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n. \tag{4.7}$$

Since ζ is only continuous in general, we need to find a differential version instead. Set

$$\widetilde{\zeta} = \frac{h_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j},$$

where $\eta = (0, \ldots, 0, 1)$. There holds at (t_0, ξ_0),

$$h_{nn} = h_n^n = \kappa_n = \zeta = \widetilde{\zeta}.$$

By a simple calculation, we get

$$\frac{\partial}{\partial t}\widetilde{\zeta} = \frac{(\frac{\partial}{\partial t}h_{ij})\eta^i\eta^j}{g_{ij}\eta^i\eta^j} - \frac{h_{ij}\eta^i\eta^j}{(g_{ij}\eta^i\eta^j)^2}\left(\frac{\partial}{\partial t}g_{ij}\right)\eta^i\eta^j$$

and

$$\frac{\partial}{\partial t}h_n^n = \frac{\partial}{\partial t}(h_{nk}g^{kn}) = \left(\frac{\partial}{\partial t}h_{nk}\right)g^{kn} - g^{ki}\left(\frac{\partial}{\partial t}g_{ij}\right)g^{jn}h_{nk}.$$

Clearly, in a neighborhood of (t_0, ξ_0),

$$\widetilde{\zeta} \leq \zeta$$

holds, and we find at (t_0, ξ_0),

$$\frac{\partial}{\partial t}\widetilde{\zeta} = \frac{\partial}{\partial t}h_n^n$$

and the spatial derivatives also coincide. This implies that $\widetilde{\zeta}$ satisfies the same evolution as h_n^n. Without loss of generality, we treat h_n^n as a scalar and pretend that w is defined by

$$w = \log h_n^n + \log\chi + 2\log\Theta.$$

By Lemma 4.1, we have

$$\begin{aligned}\partial_t h_n^n - \beta\Psi F^{-1}F^{kl}h_{n,kl}^n = {} & \beta\Psi F^{-1}h_n^n F^{kl}h_l^m h_{km} \\ & - \Psi\frac{\beta(\beta+1)F_nF^n}{F^2} - (1+\beta)\Psi(h_{nn})^2 \\ & - 2(1-\alpha)\beta\Psi\nabla_n\varphi\nabla^n\log F \\ & - (\alpha-1)\Psi u^{-1}u_n^n - (\alpha-1)(\alpha-2)\Psi\nabla_n\varphi\nabla^n\varphi \\ & + \beta\Psi F^{-1}F^{rs,kl}h_{rs,}{}^n h_{kl,n} + \beta\Psi F^{-1}F^{kl}h_k^m h_{nm}h_l^n \\ & - \beta\Psi F^{-1}F^{kl}h^{mn}h_{nk}h_{lm}.\end{aligned}$$

Writing it in another form, and using the (4.7), we have

$$\begin{aligned}\partial_t\log h_n^n - \beta\Psi F^{-1}F^{kl}(\log h_n^n)_{kl} \le {} & \beta\Psi F^{-1}F^{kl}\nabla_k\log h_n^n\nabla_l\log h_n^n \\ & + \beta\Psi F^{-1}F^{kl}h_l^m h_{km} - (1+\beta)\Psi h_n^n \\ & - \beta(\beta+1)\Psi(h_n^n)^{-1}\nabla_n\log F\nabla^n\log F \\ & - 2(1-\alpha)\beta\Psi(h_n^n)^{-1}\nabla_n\varphi\nabla^n\log F \\ & - (\alpha-1)\Psi(h_n^n)^{-1}u^{-1}u_n^n \\ & - (h_n^n)^{-1}(\alpha-1)(\alpha-2)\Psi|\nabla_n\varphi|^2.\end{aligned}$$

Clearly,

$$(1-\alpha)\nabla_n\varphi\nabla^n\log F \le \frac{\beta+1}{2}|\nabla_n\log F|^2 + \frac{(1-\alpha)^2}{2(\beta+1)}|\nabla_n\varphi|^2.$$

A direct computation gives

$$u_n = \frac{1}{u}\langle X_n, X\rangle$$

and

$$u_{nn} = \frac{1}{u}(-\chi h_{nn} + 1 - (u_{nn})^2) \le \frac{1}{u}.$$

Therefore, we have

$$\begin{aligned}\partial_t\log h_n^n - \beta\Psi F^{-1}F^{kl}(\log h_n^n)_{kl} \le {} & \beta\Psi F^{-1}F^{kl}\nabla_k\log h_n^n\nabla_l\log h_n^n \\ & + \beta\Psi F^{-1}F^{kl}h_l^m h_{km} - (1+\beta)\Psi h_n^n \\ & - (\alpha-1)\Psi(h_n^n)^{-1}u^{-2} \\ & + \frac{(\alpha-1)}{\beta+1}(2+\beta-\alpha)\Psi(h_n^n)^{-1}\nabla_n\varphi\nabla^n\varphi. \qquad (4.8)\end{aligned}$$

Writing the evolution equation of χ in Lemma 4.1 in another form

$$\begin{aligned}(4.9)\qquad & \partial_t\log\chi - \beta\Psi F^{-1}F^{kl}(\log\chi)_{kl} \\ = {} & (\beta-1)\chi\Psi - \beta\Psi F^{-1}F^{kl}\nabla_k\log\chi\nabla_l\log\chi \\ & - \beta\Psi F^{-1}F^{kl}h_l^m h_{mk} + (\alpha-1)\Psi\chi\nabla^l\log u\langle X, X_l\rangle.\end{aligned}$$

Note that, at (t_0, ξ_0),

$$\nabla \log h_n^n + \nabla \log \chi = 0. \tag{4.10}$$

Thus, at (t_0, ξ_0), combining (4.8), (4.9) and (4.10) yields

$$\begin{aligned} 0 &\le \partial_t w - \beta \Psi F^{-1} F^{kl} w_{kl} \\ &\le (\beta - 1)\chi\Psi - (1+\beta)\Psi h_n^n - (\alpha - 1)\Psi (h_n^n)^{-1} u^{-2} \\ &\quad + (\alpha - 1)\Psi\chi \nabla^k \log u \langle X, X_k \rangle + 2\frac{1}{n^\beta}\Theta^{\alpha+\beta-2}. \end{aligned} \tag{4.11}$$

By Lemma 3.7 and Corollary 3.9, we know that

$$|\nabla^k \log u \langle X, X_k \rangle| \le |\nabla u| \le C.$$

Substituting the above estimate into (4.11) results in

$$(1+\beta)\Psi h_n^n \le (\alpha - 1)\Psi (h_n^n)^{-1} u^{-2} + C\Psi\chi + \frac{2}{n^\beta}\Theta^{\alpha+\beta-2},$$

which implies

$$u h_{nn} \le C.$$

This completes the proof. □

Theorem 4.3. *Under the hypothesis of Theorem 1.1, we have*

$$T^* = +\infty.$$

Proof. From the first evolution equation in (2.8), we have

$$\frac{\partial \varphi}{\partial t} = Q(x, \varphi, D\varphi, D^2\varphi).$$

Set $\widetilde{F}_j^i = \frac{\partial F}{\partial \widetilde{h}_i^j}$. By a simple calculation, we get

$$\frac{\partial Q}{\partial \varphi_{ij}} = \beta e^{(\alpha+\beta-2)\varphi} F^{-(\beta+1)} \widetilde{F}_l^i \left(\sigma^{jl} - \frac{\varphi^j \varphi^l}{v^2} \right),$$

which is uniformly parabolic on finite intervals from C^0-estimate (3.2), C^1-estimate (3.19) and the estimate (3.21). Then by Krylov-Safonov estimate [14] (or the results in [17, Chapter 14]), we have

$$|\varphi|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(\mathbb{S}^n \times [0, T^*])} \le C(n, M_0, T^*),$$

which implies the maximal time interval is unbounded, i.e., $T^* = +\infty$. □

Remark 4.1. Considering the case $\alpha \le 1$, $\alpha + \beta > 2$, if furthermore $\alpha = 1$, then $\beta > 1$ and the anisotropic ICF (1.8) degenerates into the ICF (1.10). Gerhardt successfully applied the evolution equation of $\Psi = u^{\alpha-1} F^{-\beta} = F^{-\beta}$ (see [11, (3.64)]) to get lower bounds for F and the rescaled curvature function $F\Theta$ respectively, and then obtained upper bound estimates for principal curvatures and also rescaled ones, which leads to the convergence of the rescaled flow. However, this method is invalid for $\alpha < 1$, $\alpha + \beta > 2$. The reason is the

following: under the situation $\alpha < 1$, $\alpha+\beta > 2$, the evolution equation of Ψ has terms involving the second-order derivatives of u, which cannot be controlled. This leads to the fact that it is impossible to give lower bounds for F, $F\Theta$, and meanwhile principal curvatures' estimates cannot be obtained also.

5. Convergence of the rescaled flow for the case $\alpha \leq 1$, $\beta > 0$, $\alpha + \beta \leq 2$

Now, we define the rescaled flow by

$$\widetilde{X} = X\Theta^{-1}.$$

Thus

$$\widetilde{u} = u\Theta^{-1},$$

$$\widetilde{\varphi} = \varphi - \log\Theta,$$

and the rescaled curvature function is given by

$$\widetilde{F} = F\Theta.$$

Then by a direct computation, we have

$$\frac{\partial}{\partial t}\widetilde{u} = \frac{v}{\widetilde{u}^{1-\alpha}\widetilde{F}^{\beta}}\Theta^{\alpha+\beta-2} - \frac{\widetilde{u}}{n^{\beta}}\Theta^{\alpha+\beta-2}.$$

Defining $s = s(t)$ by the relation

$$\frac{ds}{dt} = \Theta^{\alpha+\beta-2}$$

such that $s(0) = 0$, we conclude that s ranges from 0 to $+\infty$ and $\widetilde{u}$ satisfies

$$\frac{\partial}{\partial s}\widetilde{u} = \frac{v}{\widetilde{u}^{1-\alpha}\widetilde{F}^{\beta}} - \frac{\widetilde{u}}{n^{\beta}},$$

or equivalently,

$$\frac{\partial}{\partial s}\widetilde{\varphi} = \frac{v}{\widetilde{u}^{2-\alpha}\widetilde{F}^{\beta}} - \frac{1}{n^{\beta}} = \widetilde{Q}(\widetilde{\varphi}, D\widetilde{\varphi}, D^2\widetilde{\varphi}), \tag{5.1}$$

with $\widetilde{\varphi} = \log\widetilde{u}$. Since the spatial derivatives of $\widetilde{\varphi}$ are identical to those of φ, (5.1) is a nonlinear second-order parabolic PDE with a uniformly parabolic and concave operator $\widetilde{F}$. Then, similar to what we have done in (3.19), we can deduce a decay estimate of $\widetilde{\varphi}(\cdot, s)$ as follows:

Lemma 5.1. *Let φ be a solution of* (2.8)*, and $\alpha \leq 1$, $\beta > 0$, $\alpha+\beta \leq 2$. Then we have*

$$|D\widetilde{\varphi}(x,s)| \leq e^{-\lambda(2-\alpha-\beta)s}\sup_{\mathbb{S}^n}|D\widetilde{\varphi}(\cdot,0)|, \tag{5.2}$$

where λ is a positive constant.

Thus, we can apply the Krylov-Safonov estimate [14] and thereafter the parabolic Schauder estimate to conclude:

Lemma 5.2. *Let φ be a solution of the flow* (2.8)*. Then*

$$\widetilde{\varphi}(\cdot, s)$$

converges to a real number for $s \to +\infty$.

So, we have:

Theorem 5.3. *The rescaled flow*

$$\frac{d\widetilde{X}}{ds} = \frac{1}{|\widetilde{X}|^{1-\alpha}\widetilde{F}^{\beta}}\nu - \frac{\widetilde{X}}{n^{\beta}}$$

exists for all time and the leaves converge in C^{∞} to a round sphere.

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