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CORRIGENDUM TO "THE IDEAL OF WEAKLY p-NUCLEAR OPERATORS AND ITS INJECTIVE AND SURJECTIVE HULLS" [J. KOREAN MATH. SOC. 56 (2019), NO. 1, PP. 225–237]

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Abstract. We indicate that some results in [2] are wrong, and obtain some new results on them.

1. Weakly 1-nuclear operators

We use all notations, terminologies and definitions in [2]. Let us recall the concept of a *weakly* 1-*nuclear operator* from a Banach space X to a Banach space Y as any operator which can be represented as

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \in \mathcal{N}_{w1}(X, Y),$$

where $(x_n^*)_n \in \ell_1^w(X^*)$ and $(y_n)_n \in c_0^w(Y)$. Every weakly 1-nuclear operator $T : X \to Y$ is weakly compact because $T(B_X)$ is contained in the convex hull of a weakly null sequence in Y.

Proposition 1.1 ([2, Proposition 2.2]). *Let $1 \leq p \leq \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{N}_{wp}(X, Y)$ if and only if there exist $R \in \mathcal{L}(X, \ell_p)$ and $S \in \mathcal{L}(\ell_p, Y)$ (ℓ_p is replaced by c_0 if $p = \infty$) such that $T = SR$. In this case, $\|T\|_{\mathcal{N}_{wp}} = \inf \|S\|\|R\|$, where the infimum is taken over all such factorizations.*

The case $p = 1$ in Proposition 1.1 is wrong. Indeed, if that statement would be true, then the identity map $id_{\ell_1} : \ell_1 \to \ell_1$ should be a weakly compact operator. This is a contradiction because ℓ_1 has the Schur property.

The following lemma is well known but we provide a proof for the sake of completeness of our presentation.

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Lemma 1.2. *Let X and Y be Banach spaces. An operator $T : X^* \to Y$ is weak* to weak continuous if and only if $T^*(Y^*) \subset i_X(X)$, where $i_X : X \to X^{**}$ is the canonical isometry.*

Proof. Assume that T is weak* to weak continuous and let $y^* \in Y^*$. To show that T^*y^* is a weak* continuous functional, let $(x^*_\alpha)_\alpha$ be a net in X^* and let $x^* \in X^*$ be such that $\lim_\alpha x^*_\alpha = x^*$ in the weak* topology on X^*. Since T is weak* to weak continuous,

$$\lim_\alpha T^*y^*(x^*_\alpha) = \lim_\alpha y^*(Tx^*_\alpha) = y^*(Tx^*) = T^*y^*(x^*).$$

To show the converse, let $(x^*_\alpha)_\alpha$ be a net in X^* and let $x^* \in X^*$ be such that $\lim_\alpha x^*_\alpha = x^*$ in the weak* topology on X^*. By assumption, for every $y^* \in Y^*$,

$$\lim_\alpha y^*(Tx^*_\alpha) = \lim_\alpha T^*y^*(x^*_\alpha) = T^*y^*(x^*) = y^*(Tx^*).$$

Hence T is weak* to weak continuous. □

We now obtain some factorizations of weakly 1-nuclear operators.

Theorem 1.3. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear map. Then the following statements are equivalent.*

(a) $T \in \mathcal{N}_{w1}(X, Y)$.
(b) *There exist an operator $R : X \to \ell_1$ and a weak* to weak continuous operator $S : \ell_1 \to Y$ such that $T = SR$.*
(c) *There exist operators $R : X \to \ell_1$ and $S \in \mathcal{N}_{w1}(\ell_1, Y)$ such that $T = SR$.*

In this case, $\|T\|_{\mathcal{N}_{w1}} = \inf \|S\|\|R\| = \inf \|S\|_{\mathcal{N}_{w1}}\|R\|$, where the infimums are taken over all such factorizations.

Proof. (c)⇒(a) is clear and $\|T\|_{\mathcal{N}_{w1}} \le \inf \|\cdot\|_{\mathcal{N}_{w1}}\|\cdot\|$.
(a)⇒(b): Let $T \in \mathcal{N}_{w1}(X, Y)$ and let

$$T = \sum_{n=1}^{\infty} x^*_n \underline{\otimes} y_n$$

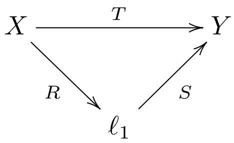
be an arbitrary weakly 1-nuclear representation. Consider the maps

$$R : X \to \ell_1, x \mapsto (x^*_n(x))_n \quad \text{and} \quad S : \ell_1 \to Y, (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n y_n.$$

Then we see that $\|R\| = \|(x^*_n)_n\|^w_1$ and $\|S\| = \|(y_n)_n\|_\infty$.
Also, for every $y^* \in Y^*$ and $(\alpha_n)_n \in \ell_1$,

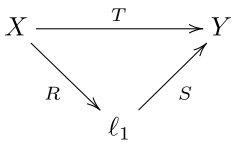
$$(S^*y^*)((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n y^*(y_n) = \langle (\alpha_n)_n, (y^*(y_n))_n \rangle.$$

Since $(y_n)_n \in c_0^w(Y)$, $S^*(y^*) \in i_{c_0}(c_0)$. Thus by Lemma 1.2, S is weak* to weak continuous and the following diagram is commutative.



Since the weakly 1-nuclear representation of T was arbitrary, $\inf \|\cdot\|\|\cdot\| \leq \|T\|_{\mathcal{N}_{w1}}$.

(b)⇒(c): Let T have the following factorization in (b).



It follows that

$$S = \sum_{n=1}^{\infty} e_n^* \underline{\otimes} S e_n$$

and $\|(e_n^*)_n\|_1^w = 1$, where e_n and e_n^* are the standard unit vectors in ℓ_1 and c_0, respectively. Since S is weak* to weak continuous and $\lim_{n\to\infty} e_n = 0$ in the weak* topology on ℓ_1, $(Se_n)_n \in c_0^w(Y)$ and $\|(Se_n)_n\|_\infty \leq \|S\|$.

Consequently, $S \in \mathcal{N}_{w1}(\ell_1, Y)$ and

$$\inf \|\cdot\|_{\mathcal{N}_{w1}} \|\cdot\| \leq \|S\|\|R\|. \qquad \square$$

It was shown in [2, Lemma 2.3] that if $1 < p \leq \infty$, then for every Banach space X, $\mathcal{N}_{wp}(X, \ell_p)$ (respectively, $\mathcal{N}_{wp}(\ell_p, X)$) is isometrically equal to $\mathcal{L}(X, \ell_p)$ (respectively, $\mathcal{L}(\ell_p, X)$) ($\ell_p = c_0$ when $p = \infty$). For the case $p = 1$, we have:

Proposition 1.4. *For every Banach space X,*

$$\mathcal{N}_{w1}(X, \ell_1) = \mathcal{K}(X, \ell_1)$$

holds isometrically.

Proof. Note that

$$\mathcal{N}_{w1}(X, \ell_1) \subset \mathcal{W}(X, \ell_1) = \mathcal{K}(X, \ell_1).$$

To show the reverse inclusion, let $T = \sum_{n=1}^{\infty} e_n^* T \underline{\otimes} e_n \in \mathcal{K}(X, \ell_1)$ and let $\varepsilon > 0$. Since $T(B_X)$ is a relatively compact subset of ℓ_1,

$$\lim_{l\to\infty} \sup_{x\in B_X} \sum_{n\geq l} |e_n^* T x| = 0.$$

Then there exists a sequence $(\beta_n)_n$ with $\beta_n > 1$ and $\lim_{n\to\infty} \beta_n = \infty$ such that

$$\lim_{l\to\infty} \sup_{x\in B_X} \sum_{n\geq l} |\beta_n e_n^* Tx| = 0 \quad \text{and} \quad \sup_{x\in B_X} \sum_{n=1}^{\infty} |\beta_n e_n^* Tx| \leq (1+\varepsilon) \sup_{x\in B_X} \sum_{n=1}^{\infty} |e_n^* Tx|$$

(cf. [3, Lemma 3.1]). Now, we see that

$$T = \sum_{n=1}^{\infty} \beta_n e_n^* T \underline{\otimes} (e_n/\beta_n) \in \mathcal{N}_{w1}(X, \ell_1)$$

and

$$\|T\|_{\mathcal{N}_{w1}} \leq (1+\varepsilon) \sup_{x\in B_X} \sum_{n=1}^{\infty} |e_n^* Tx| = (1+\varepsilon)\|T\|. \qquad \square$$

2. Weakly 1-compact sets

A subset K of a Banach space X is called *weakly* 1*-compact* if there exists $(x_n)_n \in \ell_1^w(X)$ such that

$$K \subset 1\text{-}co(x_n)_n := \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{c_0} \Big\}.$$

Proposition 2.1 ([2, Lemma 3.5(a)])**.** *Let X be a Banach space. For $1 \leq p < \infty$, if $(x_n)_n \in \ell_p^w(X)$, then the set p-$co(x_n)_n$ is balanced, convex and weakly compact.*

The case $p = 1$ in Proposition 2.1 is wrong. Indeed, let $(e_n)_n$ be the sequence of standard unit vectors in c_0. Then we see that $(e_n)_n \in \ell_1^w(c_0)$ and 1-$co(e_n)_n = B_{c_0}$. Consequently, B_{c_0} is a weakly 1-compact subset of c_0. But it is not weakly compact. Generally, we have:

Proposition 2.2. *The following statements are equivalent for a Banach space X.*

(a) *X does not have an isomorphic copy of c_0.*
(b) *Every weakly* 1*-compact set in X is relatively compact.*
(c) *Every weakly* 1*-compact set in X is relatively weakly compact.*
(d) *For every $(x_n)_n \in \ell_1^w(X)$, the set* 1*-$co(x_n)_n$ is relatively weakly compact.*

Proof. (b)⇒(c) and (c)⇒(d) are trivial.

It is well known that a Banach space X does not have an isomorphic copy of c_0 if and only if every weakly 1-summable sequence in X is unconditionally summable (cf. [4, Theorem 4.3.12]). Also a sequence $(x_n)_n$ in X is unconditionally summable if and only if

$$\lim_{l\to\infty} \sup_{x^*\in B_{X^*}} \sum_{n\geq l} |x^*(x_n)| = 0$$

(cf. [1, Theorem 1.9]).

(a)⇒(b): Let $(x_n)_n \in \ell_1^w(X)$. By (a), $(x_n)_n$ is unconditionally summable. Hence by [1, Theorem 1.9], $1\text{-}co(x_n)_n$ is relatively compact.

(d)⇒(a): Let $(x_n)_n \in \ell_1^w(X)$. Define the map

$$S : c_0 \to X \quad \text{by} \quad S(\alpha_n)_n = \sum_{n=1}^{\infty} \alpha_n x_n.$$

By (d), S is a weakly compact operator. We see that the adjoint operator $S^* : X^* \to \ell_1$ is defined by

$$S^* x^* = (x^*(x_n))_n.$$

Since S^* is weakly compact, by the Schur property S^* is compact. Consequently, $(x_n)_n$ is unconditionally summable. □

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