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ON THE C-PROJECTIVE VECTOR FIELDS ON RANDERS SPACES

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Abstract. A characterization of the C-projective vector fields on a Randers space is presented in terms of $\boldsymbol{\Xi}$-curvature. It is proved that the $\boldsymbol{\Xi}$-curvature is invariant for C-projective vector fields. The dimension of the algebra of the C-projective vector fields on an n-dimensional Randers space is at most $n(n+2)$. The generalized Funk metrics on the n-dimensional Euclidean unit ball $\mathbb{B}^n(1)$ are shown to be explicit examples of the Randers metrics with a C-projective algebra of maximum dimension $n(n+2)$. Then, it is also proved that an n-dimensional Randers space has a C-projective algebra of maximum dimension $n(n+2)$ if and only if it is locally Minkowskian or (up to re-scaling) locally isometric to the generalized Funk metric. A new projective invariant is also introduced.

1. Introduction

The projective and conformal structures on a manifolds determine the metric structure. The projective properties of a generic Finsler space may include sever inconveniencies in comparison to the Riemannian spaces. The Beltrami's theorem in Riemannian geometry is no longer valid in Finsler geometry (cf. [2]) and this turns the projective Finsler geometry different. In a Finsler space (M, F), the usual Ricci tensor $K_{jl} = K^i{}_{jil}$ (with respect to the Berwald connection) is not generally symmetric with respect to the indices j and l. Therefore, an infinitesimal projective transformation may convert or even preserves the anti-symmetric part of the Ricci tensor

$$\mathcal{R}_{jl} := \frac{1}{2}\big(K_{jl} - K_{lj}\big)$$

which is in turn a non-Riemannian quantity. Every projective vector field on a Riemannian background space preserves $\mathcal{R}$ trivially and it can be realized as the so called *closeness property*. However, in general a projective vector field on a generic Finsler space may refuse the have the closeness property. A projective vector field is called *C-projective* if it preserves $\mathcal{R}$ (or in other words it has the closeness property). In the recent work [4] the Lie algebra of the

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C-projective vector fields has been studied and a C-projective invariant has been also introduced. In [4], it is proved that the non-Riemannian curvature $\mathbf{H}$-curvature is an invariant quantity for the algebras of C-projective vector fields.

Let (M, F) be an n-dimensional Finsler manifold. Using the $\mathbf{S}$-curvature, one can define the non-Riemannian curvatures $\mathbf{\Xi} = \Xi_i dx^i$, $\mathbf{E} = E_{ij} dx^i \otimes dx^j$, $\mathbf{H} = H_{ij} dx^i \otimes dx^j$ and $\mathbf{\Omega} = \Omega_{ij} dx^i \otimes dx^j$ on the pullback tangent bundle $\pi^* TM$ as follows

$$\Xi_i := \mathbf{S}_{.i|m} y^m - \mathbf{S}_{|i}, \tag{1}$$

$$E_{ij} := \frac{1}{2} \mathbf{S}_{.i.j}, \tag{2}$$

$$H_{ij} := \frac{1}{2} \mathbf{S}_{.i.j|m} y^m, \tag{3}$$

$$\Omega_{ij} := \frac{1}{n+1} \Big(\mathbf{S}_{.i|j} - \mathbf{S}_{.j|i} \Big), \tag{4}$$

where $\mathbf{S}$ denotes the $\mathbf{S}$-curvature and "." and "|" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Berwald connection [6]. The quantity $\mathbf{\Xi}$ has been introduced by Shen in [13] and studied later in [15] and [21]. In fact, the above quantities do not depend to the choice of connection for performing horizontal derivatives and can be derived for the Finsler metric itself. Notice that the following implications for Randers metrics are useful (cf. [13]):

$$\mathbf{\Omega} = 0 \quad \Leftrightarrow \quad \mathbf{\Xi} = 0 \quad \Leftrightarrow \quad \mathbf{S} = (n+1)cF, \ (c \in \mathbb{R}).$$

For other non-Riemannian curvatures, see [5], [16], [18] and [19]. Here, we prove the following result:

Theorem 1.1. *Let $(M, F = \alpha + \beta)$ be a Randers space and V be a projective vector field on M. Then the following statements are equivalent*:
(1) *V is C-projective,*
(2) $\mathcal{L}_{\hat{V}} \mathbf{\Omega} = 0$,
(3) $\mathcal{L}_{\hat{V}} \mathbf{\Xi} = 0$.

Theorem 1.1 ensures that the $\mathbf{\Xi}$-curvature and $\mathbf{\Omega}$-curvature are C-projectively invariant quantities.

On an n-dimensional Randers space $(M, F = \alpha + \beta)$, denote the algebra of projective and C-projective vector fields by $p(M, F)$ and $cp(M, F)$, respectively. Then, $\dim \big(cp(M, F) \big) \leq n(n+2)$ as well as the projective algebra. However, the case of maximum dimension is also interesting: "The locally Minkowski spaces are trivial examples possessing the C-projective algebra of maximum dimension." The following Randers metric on the Euclidean unit ball $\mathbb{B}^n(1)$ is called the generalized Funk metric:

$$F(x, y) = \frac{\sqrt{|y|^2 - (|x|^2 |y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2} \pm \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \tag{5}$$

where, $\langle,\rangle$ and $|\cdot|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $a \in \mathbb{R}^n$, $|a| < 1$. The C-projective algebra takes its maximum dimension $n(n+2)$ for the generalized Funk metrics.

Theorem 1.2. *Let $(M, F = \alpha+\beta)$ be a Randers space of dimension $n \geq 3$. The C-projective algebra $cp(M, F)$ takes the maximum (local) dimension $n(n+2)$, if and only if F is a locally Minkowski metric or it is (up to re-scaling) locally isometric to the generalized Funk metric on the Euclidean unit ball $\mathbb{B}^n(1)$.*

The horizontal derivatives with respect to the Berwald connection D is denoted by "$|$". The subscripts $_{;i}$ and $_{.i}$ stand for the partial derivations $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ are respectively. The complete lift of any vector field V on M to TM is denoted by $\hat{V}$ and $\mathcal{L}_{\hat{V}}$ denotes the Lie derivative operator with respect to $\hat{V}$. Moreover, we deal with pure Randers metrics, i.e., $\beta \neq 0$ and the underlying manifolds are supposed to be connected.

2. Preliminaries

Let M be an n-dimensional C^∞ connected manifold. The tangent space of M at $x \in M$ is denoted by T_xM and the tangent manifold of M is the disjoint union of tangent spaces $TM := \cup_{x\in M} T_xM$. Every element of TM is a pair (x, y) where $x \in M$ and $y \in T_xM$. Denote the slit tangent manifold by $TM_0 = TM \setminus \{\mathbf{o}\}$, where $\mathbf{o}$ denotes the zero section of the tangent bundle. The natural projection $\pi : TM \to M$ given by $\pi(x, y) := x$ makes TM a vector bundle of rank n over M and TM_0 a fiber bundle over M with fiber type $\mathbb{R}^n \setminus \{\mathbf{o}\}$. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ satisfying following conditions: (i) F is C^∞ on TM_0, (ii) $F(x, y)$ is positively 1-homogeneous y and (iii) the Hessian matrix of F^2 with entries

$$g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^iy^j}$$

is positively defined on TM_0. Given any Finsler metric F on M, the pair (M, F) is called a *Finsler space*. Traditionally, we denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$.

The geodesic spray G is naturally induced by F on TM_0 given in any standard coordinate (x^i, y^i) for TM_0 by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 given by

$$G^i = \frac{1}{4}g^{ih}\Big\{y^kF^2_{x^ky^h} - F^2_{x^h}\Big\}.$$

Assume the following conventions:

$$G^i{}_j := \frac{\partial G^i}{\partial y^i}, \quad G^i{}_{jk} := \frac{\partial G^i{}_j}{\partial y^k}$$

the local functions ${G^i}_j$ are coefficients of a connection in the pullback bundle $\pi^*TM \longrightarrow M$ which is called the *Berwald connection* denoted by D. Recall that for instance, the derivatives of a vector field V and a 2-covariant tensor $T = T_{ij}dx^i \otimes dx^j$ is given by:

$$
\begin{aligned}
(6) \qquad & V^i_{\ |k} = \frac{\delta V^i}{\delta x^k} + V^r G^i_{rk}, \quad V_{i|k} = \frac{\delta V_i}{\delta x^k} - V_r G^r_{ki}, \\
& T_{ij|k} = \frac{\delta T_{ij}}{\delta x^k} - T_{rj} G^r_{ik} - T_{ir} G^r_{kj},
\end{aligned}
$$

where

$$
\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - {G^i}_k \frac{\partial}{\partial y^i}.
$$

The *Busemann-Hausdorff volume form* $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ on any Finsler space (M, F) is defined by

$$
\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}}.
$$

The **S**-curvature with respect to the Busemann-Hausdorff volume form is denoted by **S** and is defined by

$$
\mathbf{S} = G^m_{\ m} - y^m \frac{\partial}{\partial x^m} \ln \sigma_F. \tag{7}
$$

Given any Randers space $(M, F = \alpha + \beta)$, the **S**-curvature takes the following form:

$$
\mathbf{S} = (n+1)\Big\{\frac{e_{00}}{2F} - s_0 - \rho_0\Big\}, \tag{8}
$$

where $\rho = \ln(\sqrt{1 - \|\beta_x\|^2_\alpha})$ and $\rho_0 = y^i c_{x^i}$. We may consider several quantities using the **S**-curvature. The $\mathbf{\Xi}$, $\mathbf{\Omega}$ and **H**-curvatures are denoted by $\mathbf{\Xi}$ and $\mathbf{\Omega}$ and **H** respectively, and are defined at every point $x \in M$ by

$$
\begin{aligned}
\mathbf{\Xi}_y &= \Xi_i(y)dx^i, \\
\mathbf{E}_y &= E_{ij}dx^i \otimes dx^j, \\
\mathbf{H}_y &= H_{ij}(y)dx^i \otimes dx^j, \\
\mathbf{\Omega}_y &= \Omega_{ij}(y)dx^i \otimes dx^j,
\end{aligned}
$$

where $y \in T_xM - \{0\}$ and

$$
\begin{aligned}
\Xi_i &:= y^m \mathbf{S}_{.i|m} - \mathbf{S}_{|i}, \\
E_{ij} &:= \frac{1}{2}\mathbf{S}_{.i.j}, \\
H_{ij} &:= \frac{1}{2} y^m \mathbf{S}_{.i.j|m}, \\
\Omega_{ij} &:= \frac{1}{n+1}\Big\{\mathbf{S}_{.i|j} - \mathbf{S}_{.j|i}\Big\}.
\end{aligned}
$$

See [13, 17, 20]. Although the **S**-curvature depends to the chosen volume form, it not hard to show that $\mathbf{\Xi}$, $\mathbf{E}$, $\mathbf{\Omega}$ and $\mathbf{H}$ are independent from choosing any volume form for the **S**-curvature, for example, we may see it for $\mathbf{\Omega}$ below: By the definition of the **S**-curvature $\mathbf{S} = \Pi - y^m \frac{\partial}{\partial x^m} \ln \sigma_F$, we have:

$$
\begin{aligned}
(n+1)\Omega_{ij} &= \mathbf{S}_{.i|j} - \mathbf{S}_{.j|i} = \frac{\delta}{\delta x^j}\mathbf{S}_{.i} - \mathbf{S}_{.k}G^k{}_{ji} - \frac{\delta}{\delta x^i}\mathbf{S}_{.j} + \mathbf{S}_{.k}G^k{}_{ij} \\
&= \frac{\delta}{\delta x^j}\mathbf{S}_{.i} - \frac{\delta}{\delta x^i}\mathbf{S}_{.j} \\
&= \frac{\delta}{\delta x^j}\big(\Pi_{y^i} - \frac{\partial}{\partial x^i}\ln \sigma_F\big) - \frac{\delta}{\delta x^i}\big(\Pi_{y^j} - \frac{\partial}{\partial x^j}\ln \sigma_F\big) \\
&= \big(\frac{\delta}{\delta x^j}\Pi_{y^i} - \frac{\delta}{\delta x^i}\Pi_{y^j}\big) + \big(\frac{\partial^2}{\partial x^i x^j}\ln \sigma_F - \frac{\partial^2}{\partial x^j x^i}\ln \sigma_F\big) \\
&= \Pi_{y^i x^j} - G^r{}_j \Pi_{y^r y^i} - \Pi_{y^j x^i} - G^r{}_i \Pi_{y^r y^j},
\end{aligned}
$$

which follows

$$
\Omega_{ij} = \frac{1}{n+1}\Big\{\Pi_{y^i x^j} - \Pi_{y^r y^j}G^r{}_i - \Pi_{y^j x^i} + \Pi_{y^r y^i}G^r{}_j\Big\}
$$

and $\Pi := G^m_{\ m}$.

It is not hard to show that there are fine relations between the above four quantities given below:

$$\Omega_{ij} = -\Omega_{ji}, \tag{9}$$

$$y^i\Omega_{ij} = -\frac{1}{n+1}\Xi_j, \tag{10}$$

$$y^j\Xi_{j.k} = -\Xi_k, \tag{11}$$

$$\Xi_{i.j} + \Xi_{j.i} = 4H_{ij}, \tag{12}$$

$$\Xi_{i.j} - \Xi_{j.i} = 2(n+1)\Omega_{ij}, \tag{13}$$

$$y^i\Omega_{ij.k} = -\frac{2}{n+1}H_{jk}, \tag{14}$$

$$\Xi_{j.k} = 2H_{jk} + (n+1)\Omega_{jk}. \tag{15}$$

And from (11) and (15) the following equations result:

$$-\Xi_k = y^j\Xi_{j.k} = (n+1)y^j\Omega_{jk}. \tag{16}$$

A Finsler space is said to be *of isotropic* **S**-*curvature* if there is a function $c = c(x)$ defined on M such that $\mathbf{S} = (n+1)c(x)F$. It is called a Finsler space *of constant* **S**-*curvature* once c is a constant. Every Berwald space is of vanishing **S**-curvature [12]. The following result proved by Z. Shen shows that constancy of **S**-curvature and vanishing of $\mathbf{\Xi}$ are the equivalent for Randers metrics:

Theorem 2.1 (Z. Shen, [13])**.** *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Then $\mathbf{S} = (n+1)cF$ for some constant c if and only if $\mathbf{\Xi} = 0$.*

Let (M, α) be a Riemannian space and $\beta = b_i(x)y^i$ be a 1-form defined on M such that $\|\beta_x\|_\alpha := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric on a manifold M. Denote the geodesic spray coefficients of α and F by the notations G^i_α and G^i, respectively and the Levi-Civita connection of α by $\widetilde{\nabla}$. Define $\widetilde{\nabla}_j b_i$ by $(\widetilde{\nabla}_j b_i)\theta^j := db_i - b_j \theta_i^{\ j}$, where $\theta^i := dx^i$ and $\theta_i^{\ j} := \tilde{\Gamma}^j_{ik} dx^k$ denote the Levi-Civita connection forms and $\widetilde{\nabla}$ denotes its associated covariant derivation of α. Recall the conventional standard notations for Randers metrics given by

$$r_{ij} := \frac{1}{2}(\widetilde{\nabla}_j b_i + \widetilde{\nabla}_i b_j), \quad s_{ij} := \frac{1}{2}(\widetilde{\nabla}_j b_i - \widetilde{\nabla}_i b_j),$$
$$s^i_{\ j} := a^{ih} s_{hj}, \quad s_j := b_i s^i_{\ j}, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then, we get

$$G^i = G^i_\alpha + \Big(\frac{e_{00}}{2F} - s_0\Big) y^i + \alpha s^i_{\ 0}, \tag{17}$$

where $e_{00} := e_{ij}y^iy^j$, $s_0 := s_i y^i$, $s^i_{\ 0} := s^i_{\ j} y^j$ and G^i_α denote the geodesic coefficients of α, see [12]. It is well-known that a Randers metric F is of isotropic $\mathbf{S}$-curvature $\mathbf{S} = (n+1)c(x)F$ if and only if $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ (see [14]). Therefore, for a Randers metric of isotropic $\mathbf{S}$-curvature the spray coefficients G^i are of the form

$$G^i = G^i_\alpha + \Big(c(\alpha - \beta) - s_0\Big) y^i + \alpha s^i_{\ 0}. \tag{18}$$

Notice that, due to (8) the coefficients G^i can be written in terms the $\mathbf{S}$-curvature as follows:

$$G^i = G^i_\alpha + \Big(\frac{\mathbf{S}}{n+1} + \rho_0\Big) y^i + \alpha s^i_{\ 0}. \tag{19}$$

The Riemann curvature tensor is defined by $\mathbf{R}_y = R^i_{\ k}(x, y) dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \longrightarrow T_xM$ by

$$R^i_{\ k} := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \Big\{\mathbf{R}_{\pi z}\Big\}_{z \in TM_0}$ is called the Riemann curvature [12]. The Berwald-Riemann curvature tensor $K^i_{\ jkl}$ is defined by

$$K^i_{\ jkl} = \frac{1}{3}\Big\{R^i_{\ k.l.j} - R^i_{\ l.k.j}\Big\}.$$

The *Ricci scalar* is denoted by $\mathbf{Ric}$ it is defined by $\mathbf{Ric} := K^k_{\ k}$. A Finsler space (M, F) is called an *Einstein space* if there is a function λ defined on M such that $\mathbf{Ric} = \lambda(x)F^2$. Notice that, the usual Ricci tensor $K_{jl} dx^j \otimes dx^l$ has an antisymmetric part denoted by $\mathcal{R}$ and defined by $\mathcal{R}_{jl} = \big(K_{jl} - K_{lj}\big)/2$.

The locally projectively flat Einstein Randers metric are locally characterized in the following theorem, see [3].

Theorem 2.2 ([3]). *Let $F=\alpha+\beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that F has constant Ricci curvature* $\mathbf{Ric}=(n-1)\lambda F^2$. *Then $\lambda \leq 0$. Further, if $\lambda=0$, F is locally Minkowskian. If $\lambda=-1/4$, F can be expressed in the following form*:

$$F(x,y)=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2}\pm\frac{\langle x,y\rangle}{1-|x|^2}\pm\frac{\langle a,y\rangle}{1+\langle a,x\rangle},$$

where $y\in T_x\mathbb{R}^n$, $a\in\mathbb{R}^n$ and $|a|<1$.

3. C-projective vector fields on Finsler spaces

Every vector field V on a manifold M induces naturally an infinitesimal coordinate transformations on TM given by $(x^i,y^i)\longrightarrow(\bar{x}^i,\bar{y}^i)$ is given locally by

$$\bar{x}^i=x^i+V^i dt,\qquad \bar{y}^i=y^i+y^k\frac{\partial V^i}{\partial x^k}dt. \tag{20}$$

This is in fact *the complete lift* of V (see for example [22]) to a vector field on TM_0 denoted by $\hat{V}$ and given by

$$\hat{V}=V^i\frac{\partial}{\partial x^i}+y^k\frac{\partial V^i}{\partial x^k}\frac{\partial}{\partial y^i}. \tag{21}$$

Notice that, $\mathcal{L}_{\hat{V}}y^i=0$, $\mathcal{L}_{\hat{V}}dx^i=0$ and the differential operators $\mathcal{L}_{\hat{V}}$, $\frac{\partial}{\partial x^i}$, exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute.

Now, let us suppose that (M,F) be a Finsler space. A vector field V on M is said to be *projective*, if there is a function P (called the *projective factor*) on TM_0 such that

$$\mathcal{L}_{\hat{V}}G^i=Py^i.$$

See [1]. It is known that, given any projective vector field V, its local flow $\{\phi_t\}$ associated to V is a projective transformation, denoted by ϕ_t sends forward geodesics to forward geodesics and vice-versa: It may be regarded as sort of a geometric meaning. The collection of the projective vector fields on a Finsler space (M,F) is denoted by $proj(M,F)$ and is classically known to be a finite dimensional Lie algebra with respect to the usual Lie bracket $[,]$. If V is a projective vector field, then the following identities are known by [1,4]:

$$\mathcal{L}_{\hat{V}}G^i{}_k = P\delta^i{}_k+P_ky^i, \tag{22}$$

$$\begin{aligned}\mathcal{L}_{\hat{V}}G^i{}_{jk} &= \delta^i{}_jP_k+\delta^i{}_kP_j+y^iP_{kj}, \qquad (23)\\ 2\mathcal{L}_{\hat{V}}\mathbf{E}_{jl} &= (n+1)P_{jl},\\ 2\mathcal{L}_{\hat{V}}\mathbf{H}_{jl} &= (n+1)P_{jl|m}y^m,\\ \mathcal{L}_{\hat{V}}K^i{}_{jkl} &= \delta^i{}_j(P_{l|k}-P_{k|l})+\delta^i{}_lP_{j|k}-\delta^i{}_kP_{j|l}+y^i(P_{l|k}-P_{k|l})_{.j},\end{aligned}$$

$$\mathcal{L}_{\hat{V}}K\ {}_{jl} = P_{l|j}-nP_{j|l}+P_{lj|0}, \tag{24}$$

where $P_i=P_{.i}$ and $P_{ij}=P_{i.j}$.

A projective vector field is said to be *affine* if $P = 0$. The collections of the affine and Killing vector fields on a Finsler space (M, F) are denoted by $aff(M, F)$ and $k(M, F)$, respectively. It is well-know that every Killing vector field is affine and every affine vector field is projective. Thus, it is clear that $k(M, F) \subseteq aff(M, F) \subseteq proj(M, F)$. Recall that, given any projective vector field V on a Riemannian spaces, the projective factor $P = P(x, y)$ is linear with respect to y and thus, it is the natural lift of a 1-form on M to a function on TM_0: Notice that, by (24), the projective factor is actually a closed 1-form for any Riemannian space (that is to say that $P_{i|j} = P_{j|i}$), while these issue is a non-Riemannian feature in a Finslerian media. Consider the following conventional definitions of a projective vector field V; V is said to be (cf. [7,8])

(i) *special* if $\mathcal{L}_{\hat{V}}\mathbf{E} = 0$, or equivalently, $P(x, y) = P_i(x)y^i$.
(ii) *C-projective* if $P_{i|j} = P_{j|i}$.
(iii) $\mathbf{H}$-*invariant* if $\mathcal{L}_{\hat{V}}\mathbf{H} = 0$, equivalently, $P_{jk|l} = P_{jl|k}$.

A projective vector field on a Riemannian manifold is simultaneously special and C-projective. Every projective vector field on a weakly Berwald space (i.e., $\mathbf{E} = 0$) is special and every special projective vector field on a Randers space of constant non-zero $\mathbf{S}$-curvature is C-projective (cf. [7, 8]). Now let us suppose that $\mathcal{R}_{lj} := \big(K_{lj} - K_{jl}\big)/2$ denotes the anti-symmetric part of the usual Ricci curvature $K_{jl} = K^i_{\ jil}$. Then by (24), the following equations result immediately:

$$\begin{aligned} \mathcal{L}_{\hat{V}}\mathcal{R}_{jl} &= \frac{1}{2}\Big(P_{l|j} - nP_{j|l} + P_{lj|0} - P_{j|l} + nP_{l|j} - P_{jl|0}\Big) \\ &= \frac{(n+1)}{2}\big(P_{l|j} - P_{j|l}\big). \end{aligned} \tag{25}$$

Using (25), we obtain the following characterization of the C-projective vector fields:

Corollary 3.1. *Let us suppose that $(M, F = \alpha + \beta)$ be a Randers space. Then a projective vector field V is C-projective if and only if $\mathcal{L}_{\hat{V}}\mathcal{R}_{jl} = 0$.*

The collection of the C-projective vector fields on a Finsler space (M, F) is denoted by $cproj(M, F)$. It is clear that $cproj(M, F) \subseteq proj(M, F)$ and thus

$$\dim\Big(cp(M, F)\Big) \leq \dim\Big(p(M, F)\Big) \leq n(n + 2).$$

The case of maximum dimension $n(n + 2)$ is known on Randers paces by the following result:

Theorem 3.2 (Rafie-Rad and Rezaei [10])**.** *A Randers metric $F = \alpha + \beta$ on a manifold M of dimension n, $(n \geq 3)$ is projective if and only if $proj(M, F)$ has (locally) dimension $n(n + 2)$.*

The quotient $Q(M, F) := proj(M, F)/cproj(M, F)$ demonstrates a non-Riemannian feature. The following result shows that the generalized Funk metrics on the Euclidean unit ball $\mathbb{B}^n(1)$ are examples whose projective algebra consists only of the C-projective vector fields, namely

$$Q(M,F) := \frac{proj(M,F)}{cproj(M,F)} = \{0\}.$$

More precisely, we have the following.

Theorem 3.3 (Rafie-Rad [7]). *Let $F=\alpha+\beta$ be an n-dimensional Randers space of nonzero constant* **S***-curvature. If F is projective (i.e., locally projectively flat), then every projective vector field is C-projective.*

Remark 3.4. The generalized Funk metric is of nonzero constant **S**-curvature $\mathbf{S}=\pm(n+1)/2F$ and Theorem 3.3 ensures that they are explicit examples for which, very projective vector field is C-projective. This fact also may complete the half part of the proof of Theorem 1.2.

The projective vector fields have been characterized in several contexts such as [1] in general and [11] for compact case. Some characterization of projective vector fields in a Randers space $(M,F=\alpha+\beta)$ are give by in terms of α and β as follows:

Theorem 3.5 (Rafie-Rad and Rezaei [9]). *A vector field V is projective on a Randers space $(M,F=\alpha+\beta)$ if and only if it is projective in (M,α) and $\mathcal{L}_{\hat{V}}(\alpha {s^i}_j)=0$.*

Remark 3.6. The quantity $\alpha {s^i}_j \frac{\partial}{\partial x^i}\otimes dx^j$ is a projective invariant tensor.

Let us suppose that V is a projective vector field on a Randers space $(M,F=\alpha+\beta$. Therefore,

$$\mathcal{L}_{\hat{V}}G^i = Py^i, \qquad \mathcal{L}_{\hat{V}}G^i_\alpha = \eta y^i,$$

where P and η denote the projective factors for F and α, respectively. Now, by (19) and Remark 3.6, we have

$$\begin{aligned}\mathcal{L}_{\hat{V}}G^i = Py^i &= \mathcal{L}_{\hat{V}}\Big\{G^i_\alpha + (\frac{\mathbf{S}}{n+1}+\rho_0)y^i + \alpha {s^i}_0\Big\}\\ &= \Big\{\eta + \mathcal{L}_{\hat{V}}(\frac{\mathbf{S}}{n+1}+\rho_0)\Big\}y^i = Py^i.\end{aligned}$$

Remark 3.7. The projective factor for every projective vector field V on a Randers space $(M,F=\alpha+\beta)$ is given by

$$P = \eta + \mathcal{L}_{\hat{V}}\Big(\frac{\mathbf{S}}{n+1}+\rho_0\Big),$$

where P and η denote the projective factors for F and α, respectively.

Remark 3.8. Notice that, given any projective (i.e., locally projectively flat) Finsler metric on an open subset $U\subseteq\mathbb{R}^n$, the geodesic spray coefficients G^i are given by $G^i=\Psi y^i$, where $P\in C^\infty(TU)$. Therefore, the projective factor P for a projective vector field V is given by $P=\mathcal{L}_{\hat{V}}\Psi$, since $\mathcal{L}_{\hat{V}}y^i=0$.

4. Proof of main Theorems

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) Let us suppose that V is a C-projective vector field, i.e., $\mathcal{L}_{\hat{V}}G^i = Py^i$ and $P_{i|j} = P_{j|i}$. By Remark 3.7, the projective factor P is given by $P = \eta + \mathcal{L}_{\hat{V}}\Big(\frac{\mathbf{S}}{n+1} + \rho_0\Big)$, where, η is the projective factor for α. Let us recall that η is a closed form. Hence, it results that

$$P_{i|j} = \eta_{i|j} + \Big\{\mathcal{L}_{\hat{V}}\Big(\frac{\mathbf{S}_{.i}}{n+1} + \rho_i\Big)\Big\}_{|j} = \eta_{i|j} + \frac{1}{n+1}\Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j} + \Big(V.\rho\Big)_{;i|j}. \tag{26}$$

On the other hand, following (6), (22) and (23), the term $\mathcal{L}_{\hat{V}}\mathbf{S}_{.i|j}$ can be obtained as follows:

$$\begin{aligned}\mathcal{L}_{\hat{V}}\mathbf{S}_{.i|j} &= \mathcal{L}_{\hat{V}}\Big\{\frac{\partial}{\partial x^j}\mathbf{S}_{.i} - \mathbf{S}_{.i.r}G^r{}_j - \mathbf{S}_{.r}G^r_{ij}\Big\}\\ &= \frac{\partial}{\partial x^j}\mathcal{L}_{\hat{V}}\mathbf{S}_{.i} - \big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i.r}\big)G^r{}_j - \mathbf{S}_{.i.r}\mathcal{L}_{\hat{V}}G^r{}_j - \big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.r}\big)G^r_{ij} - \mathbf{S}_{.r}\mathcal{L}_{\hat{V}}G^r_{ij}\\ &= \Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j} - \mathbf{S}_{.i.r}\big(P\delta^r_j + P_jy^r\big) - \mathbf{S}_{.r}\big(P_{ij}y^r + P_i\delta^r{}_j + P_j\delta^r{}_i\big)\\ &= \Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j} - P\mathbf{S}_{.i.j} - \mathbf{S}P_{ij} - \mathbf{S}_{.i}P_j - \mathbf{S}_{.j}P_i\end{aligned}$$

and finally we obtain $\Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j}$ as follows:

$$\Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j} = \mathcal{L}_{\hat{V}}\mathbf{S}_{.i|j} + P\mathbf{S}_{.i.j} + \mathbf{S}P_{ij} + \mathbf{S}_{.i}P_j + \mathbf{S}_{.j}P_i. \tag{27}$$

By closeness of η (i.e., $\eta_{i|j} = \eta_{j|i}$) and $\Big(V.\rho\Big)_{;i|j} = \Big(V.\rho\Big)_{;j|i}$ and also plugging $\Big(\mathcal{L}_{\hat{V}}\mathbf{S}_{.i}\Big)_{|j}$ from (27) in (26), it results:

$$P_{i|j} - P_{j|i} = \mathcal{L}_{\hat{V}}\Big\{\frac{1}{n+1}\big(\mathbf{S}_{.i|j} - \mathbf{S}_{.j|i}\big)\Big\} = \mathcal{L}_{\hat{V}}\Omega_{ij}. \tag{28}$$

(2) $\Rightarrow$ (3) Let us suppose that $\mathcal{L}_{\hat{V}}\Omega_{ij} = 0$. Derivation with respect to y^k commutes with the Lie derivative operator: This yields $\mathcal{L}_{\hat{V}}\Omega_{ij.k} = 0$. Due to (14), it follows

$$-\frac{2}{n+1}\mathcal{L}_{\hat{V}}H_{jk} = \mathcal{L}_{\hat{V}}\big(y^i\Omega_{ij.k}\big) = y^i\mathcal{L}_{\hat{V}}\Omega_{ij.k} = 0.$$

Taking into account $\mathcal{L}_{\hat{V}}H_{jk} = 0$ and using (15), we obtain $\mathcal{L}_{\hat{V}}\Xi_{j.k} = 0$. Finally, by (11) it results

$$-\frac{1}{n+1}\mathcal{L}_{\hat{V}}\Xi_k = \mathcal{L}_{\hat{V}}\big(y^j\Xi_{j.k}\big) = -y^j\mathcal{L}_{\hat{V}}\Xi_{j.k} = 0.$$

(3) $\Rightarrow$ (1) Let us suppose that $\mathcal{L}_{\hat{V}}G^i = Py^i$ and $\mathcal{L}_{\hat{V}}\Xi_i = 0$. By Remark 3.7, $P = \eta + \mathcal{L}_{\hat{V}}\Big(\frac{\mathbf{S}}{n+1} + \rho_0\Big)$, where η is the projective factor for α and is known to

be closed. Now by (11), (16), (26) and (27) it follows that

$$y^m P_{i|m} - P_{|i} = y^m \Big(P_{i|m} - P_{m|i} \Big) = -y^m \mathcal{L}_{\hat{V}} \Omega_{mi} = -\mathcal{L}_{\hat{V}} \big(y^m \Omega_{mi} \big)$$
$$= -\frac{1}{n+1} \mathcal{L}_{\hat{V}} \big(y^m \Xi_{mi} \big) = \frac{1}{n+1} \mathcal{L}_{\hat{V}} \Xi_i = 0.$$

Recall that, $y^m P_{i|m} - P_{|i} = 0$ and a derivation of the two sides with respect to y^j yields

$$P_{i|j} - P_{ij|m} y^m - P_{j|i} = 0.$$

It follows that $P_{i|j} - P_{j|i} = P_{ij|m} y^m$. The left hand is anti-symmetric while the right hand is symmetric with respect to indices i and j. Thus, $P_{i|j} - P_{j|i} = 0$ and V is a C-projective vector field. □

Remark 4.1. Suppose that V is an arbitrary projective vector fields on the Finsler space (M, F). Taking into account the equation (25) (as it appeared in the proof of the implication (1) ⇒ (2)) and using (28), we obtain

$$\mathcal{L}_{\hat{V}} \Big\{ \mathcal{R}_{jl} + \frac{(n+1)}{2} \Omega_{jl} \Big\} = 0.$$

Thus, the tensor $\mathbf{Z} = Z_{jl} dx^j \otimes dx^l$ given by $Z_{jl} := \mathcal{R}_{jl} + \frac{(n+1)}{2} \Omega_{jl}$ is projectively invariant. Notice that, Z is in fact a non-Riemannian quantity as it vanishes for the Riemannian metrics.

Proof of Theorem 1.2. Suppose that the C-projective algebra $cp(M, F)$ has maximum dimension $n(n+2)$. It follows immediately that the whole projective algebra $p(M, F)$ has dimension $n(n+2)$ and thus $p(M, F) = cp(M, F)$. By Theorem 3.2, it follows that F is a projective metric; Equivalently, α has constant sectional curvature and $s_{ij} = 0$. Notice that, in this case F and α are projectively equivalent. Now by Theorem 1.1, given any projective vector field V − which is now a C-projective vector field too − we have $\mathcal{L}_{\hat{V}} \Omega_{ij} = 0$. In particular, every Killing vector field for α is also projective for F (and also is C-projective). α has constant sectional curvature say k, hence, the algebra of Killing vectors of (M, α) has maximal dimension $n(n+1)/2$. It is also well-known that the Killing vector field V is locally of the form

$$V^i = Q^i{}_k x^k + C^i + k\langle x, C\rangle x^i, \tag{29}$$

where C is an arbitrary constant vector and $Q^i{}_k$ is an arbitrary constant skew-symmetry bilinear form. On the other hand, $\mathcal{L}_{\hat{V}} \Omega_{ij} = 0$ gives

$$\mathcal{L}_{\hat{V}} \Omega_{ij} = \frac{\partial V^k}{\partial x^j} \Omega_{ik} + \frac{\partial V^k}{\partial x^i} \Omega_{kj} + V^k \Omega_{ij,k} + y^m \frac{\partial V^k}{\partial x^m} \Omega_{ij.k} = 0, \tag{30}$$

where the subscript $_{,k}$ denotes the derivative with respect to $\frac{\delta}{\delta x^k}$. From (29), we obtain

$$\frac{\partial V^k}{\partial x^j} = Q^k{}_j + k\langle x, C\rangle \delta^k{}_j + kC^j x^k, \tag{31}$$

$$\frac{\partial V^k}{\partial x^i} = Q^k{}_i + k\langle x, C\rangle \delta^k{}_i + kC^i x^k. \tag{32}$$

Let us put

$$T^l{}_{ijk} := x^l \Omega_{ij,k} + y^l \Omega_{ij.k}.$$

Plugging the terms $\partial V^k/\partial x^j$ and $\partial V^k/\partial x^i$ from (31) and (32) in (30) and taking into account $C = 0$, it results that

$$Q^k{}_j \Omega_{ik} + Q^k{}_i \Omega_{kj} + Q^k{}_l T^l{}_{ijk} = 0, \tag{33}$$

where $Q = \left(Q^k{}_j\right)$ is an arbitrary skew-symmetric matrix. Consider two fixed distinct indices l_0 and k_0 such that $Q^{k_0}_{\ l_0} = -Q^{l_0}_{\ k_0} = 1$ and $Q^k{}_l = 0$ if $k \neq k_0$ or $l \neq l_0$. Given any indices i and j such that $i, j \neq l_0$, we have

$$Q^k{}_j \Omega_{ik} = 0, \quad Q^k{}_i \Omega_{kj} = 0,$$

$$Q^k{}_l x^l = \begin{cases} x^{l_0}, & k = k_0, \\ -x^{k_0}, & k = l_0, \\ 0, & \text{otherwise}, \end{cases} \qquad Q^k{}_l y^l = \begin{cases} y^{l_0}, & k = k_0, \\ -y^{k_0}, & k = l_0, \\ 0, & \text{otherwise}. \end{cases}$$

The equation (33) becomes

$$T^{l_0}_{ijk_0} - T^{k_0}_{ijl_0} = 0. \tag{34}$$

It follows that, (34) holds if $i, j \neq k$. Now, fix two distinct indices i and j and consider the matrix Q given by $Q^i{}_j = -Q^j{}_i = 1$ and $Q^k{}_l = 0$ if $k \neq i$ or $l \neq l_0$. Observe that for the matrix Q we have

$$Q^k{}_j \Omega_{ik} = Q^i{}_j \Omega_{ii} = 0, \quad Q^k{}_i \Omega_{kj} = Q^j{}_i \Omega_{jj} = 0,$$

$$Q^k{}_l x^l = \begin{cases} x^j, & k = i, \\ -x^i, & k = j, \\ 0, & \text{otherwise}, \end{cases} \qquad Q^k{}_l y^l = \begin{cases} y^j, & k = i, \\ -y^i, & k = j, \\ 0, & \text{otherwise}, \end{cases}$$

and the equation (33) becomes

$$T^j{}_{iji} - T^i{}_{ijj} = 0. \tag{35}$$

Therefore, from (34) and (35), it follows that given any skew-symmetric matrix $Q = \left(Q^k{}_l\right)$, we have $T^l{}_{ijk} = T^k{}_{ijl}$ and thus,

$$Q^k{}_l T^l{}_{ijk} = 0. \tag{36}$$

Plugging (36) in (33), we obtain

$$Q^k{}_j \Omega_{ik} + Q^k{}_i \Omega_{kj} = 0, \tag{37}$$

Now, let $i \neq j$ and $k_0 \neq i, j$ and $Q^{k_0}_{\ i} = -Q^i{}_{k_0} = 1$ and $Q^k{}_l = 0$ if $k \neq k_0$ or $l \neq i$. Thus, the equation (37) can be written as follows: $Q^k{}_j \Omega_{ik} + Q^k{}_i \Omega_{kj} = \Omega_{k_0 j} = 0$. Since i and j are arbitrarily chosen, hence $\Omega_{ij} = 0$. Moreover, $\Xi_i = -y^j \Omega_{ij} = 0$. By (14), it results that $H_{jk} = -y^i \Omega_{ij.k} = 0$ and the **H**-curvature vanishes. We summarize the results as F is a projective Randers metric with constant flag curvature $\mathbf{K} = \lambda$ and subsequently, it is of constant Ricci curvature $\mathbf{Ric} = (n-1)\lambda F^2$. Now, by Theorem 2.2, F is locally Minkowskian or up to a re-scaling, F is locally isometric to the generalized Funk metric given in (5).

Conversely, the generalized Funk metrics F on the Euclidean unit ball $\mathbb{B}^n(1)$ have constant $\mathbf{S}$-curvature $\mathbf{S} = \pm\frac{(n+1)}{2}F$. This follows immediately that given any projective vector field V on $\mathbb{B}^n(1)$, we have

$$\Omega_{ij} = \frac{1}{n+1}\Big\{\mathbf{S}_{.i|j} - \mathbf{S}_{.j|i}\Big\} = \pm\frac{1}{2}\Big\{F_{.i|j} - F_{.j|i}\Big\} = 0.$$

Now by Remark 4.1 it follows that $P_{i|j} = P_{j|i}$ and V is a C-projective vector field; Thus, $proj(M,F) = cproj(M,F)$. The generalized Funk metrics are projective and it is known that $proj(M,F) = proj(M,\alpha)$. α is of constant sectional curvature and in this case, $proj(M,\alpha)$ has local dimension $n(n+2)$ and so does $cproj(M,F) = proj(M,F)$. $\square$

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