J. Korean Math. Soc. **57** (2020), No. 4, pp. 973–986
https://doi.org/10.4134/JKMS.j190487
pISSN: 0304-9914 / eISSN: 2234-3008

# COMPUTATION OF HANKEL MATRICES IN TERMS OF CLASSICAL KERNEL FUNCTIONS IN POTENTIAL THEORY

Young-Bok Chung

Abstract. In this paper, we compute the Hankel matrix representation of the Hankel operator on the Hardy space of a general bounded domain with respect to special orthonormal bases for the Hardy space and its orthogonal complement. Moreover we obtain the compact form of the Hankel matrix for the unit disc case with respect to these bases. One can see that the Hankel matrix generated by this computation turns out to be a generalization of the case of the unit disc from the single simply connected domain to multiply connected domains with much diversities of bases.

## 1. Introduction

Suppose that $\Omega$ is a bounded domain in the complex plane with $C^\infty$ smooth boundary. For a function $\varphi \in L^\infty(b\Omega)$, the Hankel operator with the symbol $\varphi$ on the Hardy space $H^2(b\Omega)$ is the bounded operator $H_\varphi : H^2(b\Omega) \to H^2(b\Omega)^\perp$ defined by

$$H_\varphi(f) = P^\perp(\varphi f),$$

where $H^2(b\Omega)^\perp$ is the orthogonal complement of $H^2(b\Omega)$ in $L^2(b\Omega)$ and $P^\perp$ is the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)^\perp$.

The Hankel operators on the Hardy spaces belong to one of the most important classes of operators in function theory and in particular they are very important in systems theory and control theory. (See more in [7]).

The matrix representation $[H_\varphi]$ of the Hankel operator $H_\varphi$ with respect to given orthonormal bases for $H^2(b\Omega)$ and $H^2(b\Omega)^\perp$ is called the Hankel matrix associated to the Hankel operator $H_\varphi$ under the corresponding bases. In general, Hankel matrices are of the form

$$(\alpha_{l+m})$$

whose entries depend only on the sum of the coordinates when the two indices are nonnegative. Until now, the theory of Hankel operators on the Hardy spaces has been developed mainly for the case of the unit disc (and more extendibly the

Received July 11, 2019; Accepted December 12, 2019.
2010 *Mathematics Subject Classification.* Primary 47B35, 30C40.
*Key words and phrases.* Hankel operator, Hankel matrix, Hardy space.

unit ball or polydiscs in $\mathbb{C}^n$) which is the base domain. In fact, when $\Omega$ is the unit disc, it turns out that the Hankel matrix under the orthonormal bases of monomials $\frac{1}{\sqrt{2\pi}}z^p,\ p = 0, 1, 2, \ldots$ and their reciprocals $\frac{1}{\sqrt{2\pi}}z^p,\ p = -1, -2, \ldots,$ is determined explicitly in terms of the Fourier coefficients of the symbol $\varphi$. (See [4, 7]).

On the other hand, general theory of Hankel operators on arbitrary bounded domains is much more complicated than one for the unit disc case and even Hankel matrices represented by Hankel operators associated to general domains have been never computed before. So in this paper, I, as a starting point of work for general domains, would like to compute the Hankel matrix of the Hankel operator $H_\varphi : H^2(b\Omega) \to H^2(b\Omega)^\perp$ with respect to special orthonormal bases for $H^2(b\Omega)$ and for $H^2(b\Omega)^\perp$ which were constructed by the author in [5]. One can see that the Hankel matrix generated by this computation turns out to be a generalization of the case of the unit disc from the single simply connected domain to multiply connected domains with much diversities of bases.

The paper is outlined as follows. In §2 we introduce notations and notions used in the paper and list known results. In particular, we survey on important properties of the classical kernel functions and orthogonal projections to be used often in the paper. In §3 we, as main results of the paper, would like to compute the Hankel matrix of the Hankel operator on the Hardy space of a general bounded domain with special orthonormal bases for the Hardy space and its orthogonal complement. And finally we work on the unit disc and obtain the compact form of the Hankel matrices.

## 2. Preliminaries and notes

Throughout the paper, we assume that $n \in \mathbb{N}$ is fixed and $\Omega$ is a bounded $n$-connected region with $C^\infty$ smooth boundary unless otherwise specified. The Cauchy integral formula says that for any homomorphic function $f$ in a neighborhood of $\overline{\Omega}$ and for any point $w$ in $\Omega$, the value of $f$ at $w$ is represented by the boundary values of $f$ via

$$f(w) = \frac{1}{2\pi i}\int_{b\Omega}\frac{f(z)}{z-w}dz. \tag{1}$$

If we introduce the classical $L^2$ inner product $\langle\ ,\ \rangle$ defined by

$$\langle u, v\rangle = \int_{b\Omega} u\,\overline{v}\ ds,$$

where $ds$ is the differential element of arc length on the boundary $b\Omega$, the integral formula (1) is equivalent to the identity

$$f(w) = \langle f, C_w\rangle,$$

where $C_w(z) = \overline{\frac{1}{2\pi i}\frac{T(z)}{z-w}}$ is the Cauchy kernel and $T$ is the unit tangent vector function on $b\Omega$ pointing in the direction of the standard orientation of $b\Omega$. The function $T$ relates to the arc length via the identity

$$ds = \overline{T(z)}dz \quad \text{on b}\Omega. \tag{2}$$

Now let $L^2(b\Omega)$ be the Hilbert space completion of $C^\infty(b\Omega)$ with respect to the inner product $\langle\ ,\ \rangle$ and let $H^2(b\Omega)$ denote the classical Hardy space associated to $\Omega$ which is the space of holomorphic functions on $\Omega$ with $L^2$-boundary values in $b\Omega$. Since $H^2(b\Omega)$ can be regarded as the completion of the restrictions of holomorphic functions in $C^\infty(\overline{\Omega})$ to $b\Omega$ in $L^2(b\Omega)$, it follows from the inequality $|f(w)| \leq \|f\|_{L^2(b\Omega)}\|C_w\|_{L^2(b\Omega)}$ that the evaluation function at $a \in \Omega$ is a continuous linear functional on $H^2(b\Omega)$. Thus, given $w \in \Omega$, we can apply the Riesz Representation Theorem to the linear functional on $H^2(b\Omega)$ to get a unique function $S_w \in H^2(b\Omega)$ such that for all $f \in H^2(b\Omega)$,

$$f(w) = \langle f, S_w\rangle = \int_{b\Omega} f\,\overline{S_w}\ ds.$$

On the other hand, since $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$, there exists the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ called the Szegő projection which is denoted by

$$P : L^2(b\Omega) \to H^2(b\Omega).$$

Since for all $f \in H^2(b\Omega)$,

$$\langle f, S_w\rangle = f(w) = \langle f, C_w\rangle = \langle f, P(C_w)\rangle$$

and $P(C_w) \in H^2(b\Omega)$, the uniqueness property for the function $S_w$ implies that

$$P(C_w) = S_w$$

and we call $S_w$ the Szegő kernel for the Szegő projection $P$ and $S_w$ is denoted by $S_w(z) = S(z,w)$ when it is considered as a function of two variables $z$ and $w$.

It is well known (see [1, 2]) that any $u \in L^2(b\Omega)$ has an orthogonal decomposition as a direct sum of the Hardy space $H^2(b\Omega)$ and the orthogonal complement $H^2(b\Omega)^\perp$ of the Hardy space via

$$u = P_\Omega(u) + \overline{T}\ \overline{P(\,\overline{uT}\,)}. \tag{3}$$

There is also a special kernel function which is the kernel for the orthogonal projection $P^\perp$ of the Szegő projection $P$ in some sense. The Garabedian kernel function $G(z,w)$ is defined by

$$G(z,w) = \frac{1}{2\pi(z-w)} + P\left(\overline{iC_wT}\right)(z) = \frac{1}{2\pi(z-w)} + \langle \overline{iC_wT}, S_z\rangle. \tag{4}$$

It is easy to see from (4) that for fixed $w = a \in \Omega$, $G(z,a)$ is a meromorphic function on $\Omega$ with a single simple pole at $z = a$ having residue $\frac{1}{2\pi}$ which extends $C^\infty$ smoothly up to the boundary of $\Omega$. It is also known (see [1]) that $L(z,a)$ never vanishes for all $(z,a) \in \overline{\Omega} \times \Omega$ with $z \neq a$. An important property about the Szegő kernel and the Garabedian kernel to which we often refer in this paper is

$$G(z,w) = i\,\overline{S(z,w)}\,\overline{T(z)}, \quad (z,w) \in b\Omega \times \Omega. \tag{5}$$

It is very interesting to see that given $a \in \Omega$, the quotient map

$$f_a(z) = \frac{S(z,a)}{G(z,a)} \tag{6}$$

is the Ahlfors map (which is a generalization of the Riemann mapping function in the case of simply connected regions) associated to the pair $(\Omega, a)$ which is a proper holomorphic mapping of $\Omega$ onto the unit disc with $f_a(a) = 0$ and $f_a'(a) > 0$, having the extremal property as follows: the function $f_a$ maximizes $h'(a)$ among all holomorphic functions $h$ mapping $\Omega$ into the unit disc making $h'(a)$ real valued (see [6]).

## 3. Hankel matrices

In this main section, we would like to compute the Hankel matrix of the Hankel operator on the Hardy space of a general bounded domain with special orthonormal bases for the domain and the codomain spaces.

Now let $\Omega$ is $n$-connected and let $a \in \Omega$ be fixed. We may assume that the Ahlfors map $f_a$ has exactly $n$ distinct simple zeroes $a_0, a_1, \ldots, a_{n-1}$ in $\Omega$ with $a_0 = a$ (see [3] for this fact). Notice from (6) that $a_1, a_2, \ldots, a_{n-1}$ are all zeroes of $S_a$ in $\Omega$. For $k \geq 0$ and $j = 0, 1, \ldots, n-1$, we define

$$E_{kn} := c_{00} S(z,a) {f_a}^k, \ c_{00} = \frac{1}{\sqrt{S(a,a)}}, \tag{7}$$

$$E_{kn+j} := \sum_{i=1}^{j} c_{ij} S(z,a_i) {f_a}^k \text{ for } j \geq 1,$$

$$\begin{aligned} E_{-kn-j} := {} & (1-\delta_0^j) \sum_{i=0}^{j-1} \overline{c_{i,j-1}} G(z,a_i)/{f_a}^k \\ & + \delta_0^j \sum_{i=0}^{n-1} \overline{c_{i,n-1}} G(z,a_i)/{f_a}^{k-1} \text{ for } kn+j \geq 1, \end{aligned} \tag{8}$$

where the numbers $c_{ij}$ are the constants obtained in the process of normalizing the ordered basis

$$\{S(z,a){f_a}^k, S(z,a_1){f_a}^k, \ldots, S(z,a_{n-1}){f_a}^k\ ;$$
$$G(z,a)/{f_a}^k, G(z,a_1)/{f_a}^k, \ldots, G(z,a_{n-1})/{f_a}^k \mid k \geq 0, j = 0, 1, \ldots, n-1\}$$

for the space $L^2(b\Omega)$. See more details in [5]. Then we obtain the orthonormal bases

$$\mathcal{B}_+ := \{E_{kn+j} \mid k \geq 0,\ j = 0, 1, \ldots, n-1\},$$

$$\mathcal{B}_- := \{E_{-kn-j} \mid k \geq 0,\ j = 0, 1, \ldots, n-1;\ kn + j \geq 1\},$$
$$\mathcal{B} := \mathcal{B}_+ \cup \mathcal{B}_-$$

for $H^2(b\Omega), H^2(b\Omega)^{\perp}$, $L^2(b\Omega)$, respectively. We want to make sure that we are using indices according to the Euclidean division modulo $n$ with $k \geq 0$ first and then $j$ through $n-1$ for nonnegative integers. And for negative integers we use indices in the backward direction starting from $-1$ for convenience.

Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be the Fourier series representation associated to the base $\mathcal{B}$. We are then ready to compute the matrix representation of the Hankel operator $H_\varphi : H^2(b\Omega) \to H^2(b\Omega)^{\perp}$ with respect to the orthonormal bases $\mathcal{B}_+$ and $\mathcal{B}_-$.

Suppose that $m = k_m n + j_m,\ k_m \geq 0,\ j_m = 0, 1, \ldots, n-1$ is a nonnegative integer with the standard form of Euclidean division and for $l \leq -1$, let $l = -k_{-l} n - j_{-l},\ k_{-l} \geq 0,\ j_{-l} = 0, 1, \ldots, n-1$.

Then the $l$-th and $m$-th entry of the Hankel matrix $[H_\varphi]$ is obtained by

$$[H_\varphi]_{lm} = \langle H_\varphi(E_m), E_l \rangle = \sum_{p=-\infty}^{\infty} \alpha_p \langle E_p E_m, E_l \rangle. \tag{9}$$

Remember again that the row column runs in the negative direction starting from $-1$. Thus we need to compute the triple inner product of $E_p, E_m$ and $E_l$ of the form $\langle E_p E_m, E_l \rangle$ in order to get the entry of Hankel matrix. Let $p = k_p n + j_p$ be the standard form.

**Lemma 3.1.** *For $p \geq 0, m \geq 0$ and $l \leq -1$, $\langle E_p E_m, E_l \rangle = 0$.*

*Proof.* If $p = k_p n + j_p \geq 0,\ m = k_m n + j_m \geq 0$ and $l = -k_{-l} n \leq -1$, it follows from identities (5) and (2) that

$$\begin{aligned}
&\langle E_p E_m, E_l \rangle \\
&= \sum_{i=0}^{j_p} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} c_{i,j_p} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} S_{a_i} S_{a_\mu} {f_a}^{k_p + k_m + k_{-l} - 1} \overline{G_{a_\nu}}\, ds \\
&= \sqrt{-1} \sum_{i=0}^{j_p} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} c_{i,j_p} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} S_{a_i} S_{a_\mu} {f_a}^{k_p + k_m + k_{-l} - 1} S_{a_\nu}\, dz
\end{aligned}$$

which equals zero because the power $k_p + k_m + k_{-l} - 1 \geq 0$ and hence the integrand is holomorphic.

If $p = k_p n + j_p \geq 0,\ m = k_m n + j_m \geq 0$ and $l = -k_{-l} n - j_{-l} \leq -1$ with $j_{-l} \geq 1$, it follows from identities (5) and (2) that

$$\begin{aligned}
&\langle E_p E_m, E_l \rangle \\
&= \sum_{i=0}^{j_p} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} c_{i,j_p} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} S_{a_i} S_{a_\mu} {f_a}^{k_p + k_m + k_{-l}} \overline{G_{a_\nu}}\, ds
\end{aligned}$$

$$= \sqrt{-1}\sum_{i=0}^{j_p}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1} c_{i,j_p}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega} S_{a_i}S_{a_\mu}{f_a}^{k_p+k_m+k_{-l}}S_{a_\nu}\ dz$$

which equals zero because in this case the power $k_p + k_m + k_{-l} \geq 0$ and hence the integrand is holomorphic. □

**Lemma 3.2.** *For $p = -k_{-p}n \leq -1$, $m = k_m n + j_m \geq 0$ and $l = -k_{-l}n \leq -1$,*

$$\begin{aligned}
&\langle E_pE_m, E_l\rangle \\
&= \sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{n-1}\Bigg[\overline{c_{0,n-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega} S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}-1}\ dz \\
&\quad + \sum_{i=1}^{n-1}\overline{c_{i,n-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega} G_{a_i}S_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}}\ dz\Bigg].
\end{aligned}$$

*Proof.* It easily follows from (2) that

$$\begin{aligned}
&\langle E_pE_m, E_l\rangle \\
&= \sum_{i=0}^{n-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{n-1}\overline{c_{i,n-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega} G_{a_i}S_{a_\mu}{f_a}^{k_m+k_{-l}-k_{-p}}\ \overline{G_{a_\nu}}\ ds \\
&= \sqrt{-1}\ \sum_{i=0}^{n-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{n-1}\overline{c_{i,n-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega} G_{a_i}S_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}}\ dz.
\end{aligned}$$

Letting the summation with index $i$ expand, we obtain from the identity (6) the desired formula of the Lemma. □

In the several coming lemmas, we use the same method of proof as the previous Lemma 3.2 but we are willing to introduce proofs here for easy understanding of series for each case of $p$ and $l$.

**Lemma 3.3.** *For $p = -k_{-p}n \leq -1$, $m = k_m n + j_m \geq 0$ and $l = -k_{-l}n - j_{-l} \leq -1$ with $j_{-l} \geq 1$,*

$$\begin{aligned}
&\langle E_pE_m, E_l\rangle \\
&= \sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1}\Bigg[\overline{c_{0,n-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega} S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}}\ dz \\
&\quad + \sum_{i=1}^{n-1}\overline{c_{i,n-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega} G_{a_i}S_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}+1}\ dz\Bigg].
\end{aligned}$$

*Proof.* It easily follows from (2) that

$$\langle E_pE_m, E_l\rangle$$

$$= \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} \overline{c_{i,n-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} G_{a_i} S_{a_\mu} f_a{}^{k_m+k_{-l}-k_{-p}+1} \, \overline{G_{a_\nu}} \, ds$$

$$= \sqrt{-1} \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} \overline{c_{i,n-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}+1} \, dz.$$

Letting the summation with index $i$ expand, we obtain from the identity (6) the desired formula. □

**Lemma 3.4.** *For $p = -k_{-p}n - j_{-p} \leq -1$ with $j_{-p} \geq 1$, $m = k_m n + j_m \geq 0$ and $l = -k_{-l}n \leq -1$,*

$$\begin{aligned} &\langle E_p E_m, E_l \rangle \\ &= \sqrt{-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} \Bigg[ \overline{c_{0,j_{-p}-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}-2} \, dz \\ &\quad + \sum_{i=1}^{j_{-p}-1} \overline{c_{i,j_{-p}-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}-1} \, dz \Bigg]. \end{aligned}$$

*Proof.* It easily follows from (2) that

$$\begin{aligned} &\langle E_p E_m, E_l \rangle \\ &= \sum_{i=0}^{j_{-p}-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} \overline{c_{i,j_p-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} G_{a_i} S_{a_\mu} f_a{}^{k_m+k_{-l}-k_{-p}-1} \, \overline{G_{a_\nu}} \, ds \\ &= \sqrt{-1} \sum_{i=0}^{j_p-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} \overline{c_{i,j_p-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}-1} \, dz. \end{aligned}$$

Letting the summation with index $i$ expand, we obtain from the identity (6) the desired formula. □

**Lemma 3.5.** *For $p = -k_{-p}n - j_{-p} \leq -1$ with $j_{-p} \geq 1$, $m = k_m n + j_m \geq 0$ and $l = -k_{-l}n - j_{-l} \leq -1$ with $j_{-l} \geq 1$,*

$$\begin{aligned} &\langle E_p E_m, E_l \rangle \\ &= \sqrt{-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} \Bigg[ \overline{c_{0,j_{-p}-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}-1} \, dz \\ &\quad + \sum_{i=1}^{j_{-p}-1} \overline{c_{i,j_{-p}-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} f_a{}^{k_m+k_{-l}-k_{-p}} \, dz \Bigg]. \end{aligned}$$

*Proof.* It easily follows from (2) that

$$\langle E_p E_m, E_l \rangle$$

$$= \sum_{i=0}^{j_{-p}-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} \overline{c_{i,j_{-p}-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} G_{a_i} S_{a_\mu} {f_a}^{k_m+k_{-l}-k_{-p}} \, \overline{G_{a_\nu}} \, ds$$

$$= \sqrt{-1} \sum_{i=0}^{j_{-p}-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{j_{-l}-1} \overline{c_{i,j_{-p}-1}} c_{\mu,j_m} c_{\nu,j_{-l}-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}} \, dz.$$

Letting the summation with index $i$ expand, we obtain from the identity (6) the desired formula. $\square$

Now we collect all formulas obtained from Lemma 3.1 through Lemma 3.5 to compute the Hankel matrix.

**Theorem 3.6.** *Suppose that $\Omega$ is an $n$-connected bounded domain with $C^\infty$ smooth boundaries. Let $\mathcal{B}_+ := \{E_{kn+j} \mid k \geq 0, j = 0, 1, \ldots, n-1\}$ and $\mathcal{B}_- := \{E_{-kn-j} \mid k \geq 0, j = 0, 1, \ldots, n-1;\ kn + j \geq 1\}$ be the orthonormal bases for $H^2(b\Omega)$ and ${H^2}^\perp(b\Omega)$, respectively. Suppose that $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p \in L^\infty(b\Omega)$. Then the Hankel matrix $[H_\varphi]$ associated to the Hankel operator $H_\varphi$ with symbol $\varphi$ on the Hardy space $H^2(b\Omega)$ with respect to the bases $\mathcal{B}_+$ and $\mathcal{B}_-$ is given by*

$$[H_\varphi]_{lm} = \sum_{p=-\infty}^{-1} \alpha_p \cdot \Big[\delta_0^{j_{-p}} \delta_0^{j_{-l}} A_p + \delta_0^{j_{-p}} (1 - \delta_0^{j_{-l}}) B_p + (1 - \delta_0^{j_{-p}}) \delta_0^{j_{-l}} C_p$$
$$+ (1 - \delta_0^{j_{-p}})(1 - \delta_0^{j_{-l}}) D_p\Big]$$

*for $l \leq -1, m \geq 0$ where $A_p, B_p, C_p$ and $D_p$ are the numbers depending on the remainders of $p$, $l$ and $m$ in the Euclidean division modulo $n$ as follows*:

$$A_p = 2\pi \chi_{[k_m+k_{-l},\infty)}(k_{-p}) \overline{c_{0,n-1}} c_{0,j_m} c_{0,n-1} \frac{1}{(k_{-p} - k_m - k_{-l})!}$$
$$\cdot \left( \frac{\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+1}}{S_a(z)^{k_{-p}-k_m-k_{-l}-2}} \right)^{(k_{-p}-k_m-k_{-l})} \qquad (a)$$
$$+ 2\pi \chi_{[k_m+k_{-l}+3,\infty)}(k_{-p}) \sum_{j=1}^{n-1} \overline{c_{0,n-1}} c_{0,j_m} c_{0,n-1} \frac{1}{(k_{-p} - k_m - k_{-l} - 3)!}$$
$$\cdot \left( \frac{(z - a_j)^{k_{-p}-k_m-k_{-l}-2} \widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+1}}{(z - a)^{k_{-p}-k_m-k_{-l}+1} S_a(z)^{k_{-p}-k_m-k_{-l}-2}} \right)^{(k_{-p}-k_m-k_{-l}-3)} \qquad (a_j)$$
$$\sqrt{-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} (1 - \delta_0^{\mu+\nu}) \overline{c_{0,n-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-1} \, dz$$
$$\sqrt{-1} \sum_{\mu=0}^{j_m} \sum_{\nu=0}^{n-1} \sum_{i=1}^{n-1} \overline{c_{i,n-1}} c_{\mu,j_m} c_{\nu,n-1} \int_{b\Omega} G_{a_i} S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}} \, dz,$$

$$B_p = 2\pi\chi_{[k_m+k_{-l}+1,\infty)}(k_{-p})\overline{c_{0,n-1}}c_{0,j_m}c_{0,j_{-l}-1}\frac{1}{(k_{-p}-k_m-k_{-l}-1)!}$$
$$\cdot\left(\frac{\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}}}{S_a(z)^{k_{-p}-k_m-k_{-l}-3}}\right)^{(k_{-p}-k_m-k_{-l}-1)}(a)$$
$$+2\pi\chi_{[k_m+k_{-l}+4,\infty)}(k_{-p})\sum_{j=1}^{n-1}\overline{c_{0,n-1}}c_{0,j_m}c_{0,j_{-l}-1}\frac{1}{(k_{-p}-k_m-k_{-l}-4)!}$$
$$\cdot\left(\frac{(z-a_j)^{k_{-p}-k_m-k_{-l}-3}\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}}}{(z-a)^{k_{-p}-k_m-k_{-l}}S_a(z)^{k_{-p}-k_m-k_{-l}-3}}\right)^{(k_{-p}-k_m-k_{-l}-4)}(a_j)$$
$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1}(1-\delta_0^{\mu+\nu})\overline{c_{0,n-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega}S_aS_{a_\mu}S_{a_\nu}f_a{}^{k_m+k_{-l}-k_{-p}}\,dz$$
$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1}\sum_{i=1}^{n-1}\overline{c_{i,n-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega}G_{a_i}S_{a_\mu}S_{a_\nu}f_a{}^{k_m+k_{-l}-k_{-p}+1}\,dz,$$

$$C_p = 2\pi\chi_{[k_m+k_{-l}-1,\infty)}(k_{-p})\overline{c_{0,j_{-p}-1}}c_{0,j_m}c_{0,n-1}\frac{1}{(k_{-p}-k_m-k_{-l}+1)!}$$
$$\cdot\left(\frac{\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+2}}{S_a(z)^{k_{-p}-k_m-k_{-l}-1}}\right)^{(k_{-p}-k_m-k_{-l}+1)}(a)$$
$$+\ 2\pi\chi_{[k_m+k_{-l}+2,\infty)}(k_{-p})\sum_{j=1}^{n-1}\overline{c_{0,j_{-p}-1}}c_{0,j_m}c_{0,n-1}\frac{1}{(k_{-p}-k_m-k_{-l}-2)!}$$
$$\cdot\left(\frac{(z-a_j)^{k_{-p}-k_m-k_{-l}-1}\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+2}}{(z-a)^{k_{-p}-k_m-k_{-l}+2}S_a(z)^{k_{-p}-k_m-k_{-l}-1}}\right)^{(k_{-p}-k_m-k_{-l}-2)}(a_j)$$
$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{n-1}(1-\delta_0^{\mu+\nu})\overline{c_{0,j_{-p}-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega}S_aS_{a_\mu}S_{a_\nu}f_a{}^{k_m+k_{-l}-k_{-p}}\,dz$$
$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{n-1}\sum_{i=1}^{j_{-p}-1}\overline{c_{i,j_{-p}-1}}c_{\mu,j_m}c_{\nu,n-1}\int_{b\Omega}G_{a_i}S_{a_\mu}S_{a_\nu}f_a{}^{k_m+k_{-l}-k_{-p}-1}\,dz,$$

$$D_p = 2\pi\chi_{[k_m+k_{-l},\infty)}(k_{-p})\overline{c_{0,j_{-p}-1}}c_{0,j_m}c_{0,j_{-l}-1}\frac{1}{(k_{-p}-k_m-k_{-l})!}$$
$$\cdot\left(\frac{\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+1}}{S_a(z)^{k_{-p}-k_m-k_{-l}-2}}\right)^{(k_{-p}-k_m-k_{-l})}(a)$$

$$+\chi_{[k_m+k_{-l}+3,\infty)}(k_{-p})\sum_{j=1}^{n-1}\overline{c_{0,j_{-p}-1}}c_{0,j_m}c_{0,j_{-l}-1}\frac{1}{(k_{-p}-k_m-k_{-l}-3)!}$$

$$\cdot\left(\frac{(z-a_j)^{k_{-p}-k_m-k_{-l}-2}\widetilde{G_a}(z)^{k_{-p}-k_m-k_{-l}+1}}{(z-a)^{k_{-p}-k_m-k_{-l}+1}S_a(z)^{k_{-p}-k_m-k_{-l}-2}}\right)^{(k_{-p}-k_m-k_{-l}-3)}(a_j)$$

$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1}(1-\delta_0^{\mu+\nu})\overline{c_{0,j_{-p}-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}1\int_{b\Omega}S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}-1}\,dz$$

$$\sqrt{-1}\sum_{\mu=0}^{j_m}\sum_{\nu=0}^{j_{-l}-1}\sum_{i=1}^{j_{-p}-1}\overline{c_{i,j_{-p}-1}}c_{\mu,j_m}c_{\nu,j_{-l}-1}\int_{b\Omega}G_{a_i}S_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}}\,dz,$$

*where $\chi_A$ is the characteristic function of the subset $A$ of $\mathbb{R}$ defined on $\mathbb{R}$ having the value 1 for all elements of $A$ and the value 0 for all elements of $\mathbb{R}$ not in $A$ and the function $\widetilde{G_a}$ is defined by $2\pi G_a(z)=\widetilde{G_a}(z)/(z-a)$.*

*Remark* 3.7. Note that for each index $p$ in the summation, exactly one of four terms inside of the parentheses occurs.

*Proof.* We fix integers $p,m,l$ with $m\geq 0$ and $l\leq -1$. We let $m=k_mn+j_m$ and $l=-k_{-l}n-j_{-l}$ and may even assume from Lemma 3.1 that the number $p$ is of the form $p=-k_{-p}n-j_{-p}$ with $p\leq -l$ in the Euclidean division.

It follows from the identity (9) and Lemma 3.1 that

$$[H_\varphi]_{lm}=\sum_{p=-\infty}^{-1}\alpha_p\langle E_pE_m,E_l\rangle$$

and hence we need to compute the inner product $\langle E_pE_m,E_l\rangle$ for $p\leq -l, m\geq 0$ and $l\leq -1$.

Now we consider several cases depending on whether the remainders $j_{-p}$ and $j_{-l}$ equal to zero or not.

(1) If $p=-k_{-p}n\leq -1, j_{-p}=0, l=-k_{-l}n\leq -1, j_{-l}=0; 1\leq k_{-p}\leq k_m+k_{-l}-1$, then

$$\int_{b\Omega}S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}-1}\,dz=0.$$

And if $p=-k_{-p}n\leq -1, j_{-p}=0, l=-k_{-l}n\leq -1, j_{-l}=0; k_{-p}\geq k_m+k_{-l}$, then

$$\begin{aligned}&\int_{b\Omega}S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}-1}\,dz\\&=2\pi i\sum_{j=0}^{n-1}\operatorname{Res}(S_aS_{a_\mu}S_{a_\nu}{f_a}^{k_m+k_{-l}-k_{-p}-1};a_j)\end{aligned}$$

which is equal to the number $A_p$ from Lemma 3.2 for the case of $j_{-p}=0$ and $j_{-l}=0$.

(2) If $p=-k_{-p}n\le -1, j_{-p}=0, l=-k_{-l}n-j_{-l}\le -1, j_{-l}\ge 1; 1\le k_{-p}\le k_m+k_{-l}$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}}\ dz=0.$$

And if $p=-k_{-p}n\le -1, j_{-p}=0, l=-k_{-l}n-j_{-l}\le -1, j_{-l}\ge 1; k_{-p}\ge k_m+k_{-l}+1$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}}\ dz$$
$$=2\pi i\sum_{j=0}^{n-1}\operatorname{Res}(S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}}; a_j)$$

which is equal to the number $B_p$ from Lemma 3.3 for the case of $j_{-p}=0$ and $j_{-l}\ge 1$.

(3) If $p=-k_{-p}n-j_{-p}\le -1, j_{-p}\ge 1, l=-k_{-l}n\le -1, j_{-l}=0; k_{-p}\le k_m+k_{-l}-2$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-2}\ dz=0.$$

And if $p=-k_{-p}n-j_{-p}\le -1, j_{-p}\ge 1, l=-k_{-l}n\le -1, j_{-l}=0; k_{-p}\ge k_m+k_{-l}-1$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-2}\ dz$$
$$=2\pi i\sum_{j=0}^{n-1}\operatorname{Res}(S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-2}; a_j)$$

which is equal to the number $C_p$ from Lemma 3.4 for the case of $j_{-p}\ge 1$ and $j_{-l}=0$.

(4) If $p=-k_{-p}n-j_{-p}\le -1, j_{-p}\ge 1, l=-k_{-1}n-j_{-1}\le -1, j_{-1}\ge 1; k_{-p}\le k_m+k_{-1}-1$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-1}-k_{-p}-1}\ dz=0.$$

And if $p=-k_{-p}n-j_{-p}\le -1, j_{-p}\ge 1, l=-k_{-l}n-j_{-l}\le -1, j_{-l}\ge 1; k_{-p}\ge k_m+k_{-l}$, then

$$\int_{b\Omega} S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-1}\ dz$$
$$=2\pi i\sum_{j=0}^{n-1}\operatorname{Res}(S_a S_{a_\mu} S_{a_\nu} {f_a}^{k_m+k_{-l}-k_{-p}-1}; a_j)$$

which is equal to the number $D_p$ from Lemma 3.5 for the case of $j_{-p}\ge 1$ and $j_{-1}\ge 1$. $\square$

As a special simple case, we consider the Hankel matrices for simply connected domains. It easily follows from the identities (7) and (8) (or see [5]) that when $\Omega$ is simply connected, the classes

$$\mathcal{B}_+ = \{E_m = \frac{1}{\sqrt{S(a,a)}} S(z,a) {f_a}^m \mid m \geq 0\} \tag{10}$$

and

$$\mathcal{B}_- = \{E_l = \frac{1}{\sqrt{S(a,a)}} G(z,a_i) {f_a}^{l+1} \mid l \leq -1\} \tag{11}$$

are orthonormal bases for $H^2(b\Omega)$ and $H^2(b\Omega)^\perp$, respectively. On the other hand, in this case, since the remainder is always zero in the Euclidean division, $k_{-p} = -p, k_m = m, k_{-l} = -l$ for $p \leq -1, m \geq 0, l \leq -1$. Thus it follows from Theorem 3.6 that

$$\begin{aligned}
[H_\varphi]_{lm} &= A \\
&= 2\pi \delta_0^{j-p} \delta_0^{j-l} \sum_{\substack{p=-\infty \\ k_{-p} \geq k_m + k_{-l}}}^{-1} \alpha_p \overline{c_{0,n-1}} c_{0,j_m} c_{0,n-1} \frac{1}{(k_{-p} - k_m - k_{-l})!} \\
&\quad \cdot \left( \frac{\widetilde{G_a}(z)^{k_{-p} - k_m - k_{-l} + 1}}{S_a(z)^{k_{-p} - k_m - k_{-l} - 2}} \right)^{(k_{-p} - k_m - k_{-l})} (a) \\
&= 2\pi \sum_{\substack{p=-\infty \\ -p \geq m-l}}^{-1} \alpha_p c_{00}^3 \frac{1}{(-p-m+l)!} \left( \frac{\widetilde{G_a}(z)^{-p-m+l+1}}{S_a(z)^{-p-m+l-2}} \right)^{(-p-m+l)} (a).
\end{aligned}$$

Observe that the expression in the above summation

$$\left( \frac{\widetilde{G_a}(z)^{-p-m+l+1}}{S_a(z)^{-p-m+l-2}} \right)^{(-p-m+l)} (a)$$

equals $\widetilde{G_a}(a) S_a^2(a)$ for $p = l - m$ and equals $(\widetilde{G_a}^2 S_a)'(a)$ for $p = l - m - 1$.

Hence we have obtained the following result on the Hankel matrix for the case of simply connected domains.

**Corollary 3.8.** *Suppose that $\Omega$ is a simply connected bounded domain with $C^\infty$ smooth boundaries. Let $\mathcal{B}_+ = \{E_m = \frac{1}{\sqrt{S(a,a)}} S(z,a) {f_a}^m \mid m \geq 0\}$ and $\mathcal{B}_- = \{E_l = \frac{1}{\sqrt{S(a,a)}} G(z,a_i) {f_a}^{l+1} \mid l \leq -1\}$ be the orthonormal bases for $H^2(b\Omega)$ and ${H^2}^\perp(b\Omega)$, respectively. Suppose that $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p \in L^\infty(b\Omega)$. Then the Hankel matrix $[H_\varphi]$ associated to the Hankel operator $H_\varphi$ with symbol $\varphi$ on the Hardy space $H^2(b\Omega)$ with respect to the bases $\mathcal{B}_+$ and $\mathcal{B}_-$ is given by*

$$[H_\varphi]_{lm} = 2\pi c^3 \alpha_{l-m} \widetilde{G_a}(a) S_a^2(a) \alpha_{l-m-1} (\widetilde{G_a}^2 S_a)'(a) \tag{12}$$

$$+\ 2\pi c^3 \sum_{p=-\infty}^{l-m-2} \alpha_p \left( \frac{\widetilde{G_a}(z)^{-p-m+l+1}}{S_a(z)^{-p-m+l-2}} \right)^{(-p-m+l)} (a) \tag{13}$$

*where* $l \leq -1, m \geq 0$ *and* $c = \sqrt{2\pi(1-|a|^2)}$.

In particular, when the domain is the unit disc, we have much simpler form for the Hankel matrix. Notice that if $\Omega = U$ is the unit disc, then

$$S(z,a) = \frac{1}{2\pi(1-\overline{a}z)},\ G(z,a) = \frac{1}{2\pi(z-a)},\ \widetilde{G_a}(z) = 1.$$

It thus follows from (10) and (11) that

$$\mathcal{B} = \{E_p = \sqrt{\frac{1-|a|^2}{2\pi}} \frac{(z-a)^p}{(1-\overline{a}z)^{p+1}} \mid p \in \mathbb{Z}\} \tag{14}$$

is an orthonormal basis for $L^2(bU)$.

Observing that for $k \geq 2$

$$\left[(1-\overline{a}z)^{k-2}\right]^{(k)}\Big|_{z=a} = 0,$$

the second term (13) of the entry $[H_\varphi]_{lm}$ above vanishes. Therefore we have obtained the compact form of the Hankel matrix for the unit disc.

**Corollary 3.9.** *Suppose that* $U$ *is the unit disc and let* $\mathcal{B}_+ = \{E_m \mid m \geq 0\}$ *and* $\mathcal{B}_- = \{E_l \mid l \leq -1\}$ *be the orthonormal bases for* $H^2(bU)$ *and* $H^{2\perp}(bU)$*, respectively where*

$$E_p = \sqrt{\frac{1-|a|^2}{2\pi}} \frac{(z-a)^p}{(1-\overline{a}z)^{p+1}}$$

*for* $p \in \mathbb{Z}$*. Suppose that* $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p \in L^\infty(bU)$*. Then the Hankel matrix* $[H_\varphi]$ *associated to the Hankel operator* $H_\varphi$ *with symbol* $\varphi$ *on the Hardy space* $H^2(bU)$ *with respect to the bases* $\mathcal{B}_+$ *and* $\mathcal{B}_-$ *is given by*

$$[H_\varphi]_{lm} = \alpha_{l-m} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-|a|^2}} + \alpha_{l-m-1} \frac{1}{\sqrt{2\pi}} \frac{\overline{a}}{\sqrt{1-|a|^2}}. \tag{15}$$

*Remark* 3.10. Using the formula (15), we can even write the Hankel matrix for the unit disc as a compact form via

$$[H_\varphi] = \frac{1}{\sqrt{2\pi(1-|a|^2)}}(A_- + \overline{a}A_- L),$$

where

$$A_- = \begin{bmatrix} \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \cdots \\ \alpha_{-2} & \alpha_{-3} & \alpha_{-1} & \cdots \\ \alpha_{-3} & \alpha_{-4} & \alpha_{-5} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

is the essential part of the Hankel matrix and

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

is the one-way infinite lower shift matrix.

## References

[1] S. Bell, *Solving the Dirichlet problem in the plane by means of the Cauchy integral*, Indiana Univ. Math. J. **39** (1990), no. 4, 1355–1371. https://doi.org/10.1512/iumj.1990.39.39060

[2] ______, *The Szegő projection and the classical objects of potential theory in the plane*, Duke Math. J. **64** (1991), no. 1, 1–26. https://doi.org/10.1215/S0012-7094-91-06401-X

[3] ______, *The Cauchy transform, potential theory, and conformal mapping*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[4] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/64), 89–102. https://doi.org/10.1007/978-1-4613-8208-9_19

[5] Y.-B. Chung, *Classification of Toeplitz operators on Hardy spaces of bounded domains in the plane*, Math. Notes **101** (2017), no. 3-4, 529–541. https://doi.org/10.1134/S0001434617030142

[6] P. R. Garabedian, *Schwarz's lemma and the Szegö kernel function*, Trans. Amer. Math. Soc. **67** (1949), 1–35. https://doi.org/10.2307/1990414

[7] K. Zhu, *Operator theory in function spaces*, second edition, Mathematical Surveys and Monographs, **138**, American Mathematical Society, Providence, RI, 2007. https://doi.org/10.1090/surv/138

Young-Bok Chung
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
*Email address*: ybchung@chonnam.ac.kr