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INTERPOLATION OF SURFACES WITH GEODESICS

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Abstract. In this paper, we introduce a new method to construct a parametric surface in terms of curves and points lying on Euclidean 3-space, called a C^0-Hermite surface interpolation. We also prove the existence of a C^0-Hermite interpolation of isoparametric surfaces with the so-called marching scale functions, and give some examples. Finally, we construct ruled surfaces and surfaces foliated by a circle as an isoparametric surface.

1. Introduction

A geodesic between two points on a surface is defined as the curve embedded in the surface that connects the points with the minimal distance. Also, a geodesic can be defined as a curve with zero geodesic curvature. The geodesic curvature of a curve on a surface at a point is equal to the curvature of the normal projection of the curve onto the tangent plane of the surface at the point. It is well known that a geodesic plays an important role in various applications, such as tent manufacturing, cutting an painting path, textile manufacturing and fiberglass tape windings in pipe manufacturing space [1, 2].

There are some recent works to construct surfaces using geodesics: in [6] the authors showed how to construct a general surface from a polynomial geodesic and a tangent ribbon. Paluszny [5] considered a 3D polynomial curve as a pregeodesic and constructed ruled cubic patches through pregeodesics and bicubic patches through pairs of pregeodesices. In [4] the authors investigated a developable surface which contains a given Bézier geodesic and studied G^1-connection of developable surfaces through abutting cubic Bézier geodesics.

On the other hand, parametric surfaces play an crucial role in the construction of different products, such as cars, ships, airplanes and shoes with basic theories and geometric properties [7]. In this sense, the study of constructions of a parametric surface that contain a given curve as a geodesic is an important in Computer Aided Design (CAD) or Computer Aided Geometric Design

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(CAGD). Wang et al. [7] was the first to handle the problem of finding a surface family possessing a given spatial curve. By utilizing the Frenet frame along a given curve, they defined parametric surfaces associated with geodesic and derived the sufficient condition on the marching scale functions given by a factor decomposition for which the curve is an isogeodesic on a parametric surface. After, in [3] Kasap et al. considered more general marching scale functions and they illustrated a method by finding the exact surface pencil formulation for some simple surfaces.

In this paper, we consider isoparametric surfaces which contains a geodesic and construct surfaces with prescribed given geodesics and C^0-Hermite data. To do this, we define a surface interpolation associated with a spatial curve passing through some m-points in Euclidean 3-space. Moreover, we present isoparametric surfaces in terms of polynomial marching scale functions and give some examples.

The outline of the paper is organized as follows: In Section 2 we give some geometric concepts for isoparametric surfaces in Euclidean 3-space $\mathbb{E}^3$. In Section 3 we introduce a C^0-Hermite surface interpolation to construct an isoparametric surface in terms of curves and points lying on $\mathbb{E}^3$, and prove the existence of a C^0-Hermite interpolation of isoparametric surfaces (see Theorem 3.1). As a result, we give some examples for such surfaces. In the last section, we classify isoparametric ruled surfaces (see Lemma 4.1) and provide an example of the surface passing through the two points lying on $\mathbb{E}^3$.

2. Preliminaries

Let C be a spatial parametric curve with arc-length s in Euclidean 3-space $\mathbb{E}^3$. Consider the orthonormal frame field $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along the curve $C(s)$ satisfying the relations:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $C(s)$, respectively. Using the curve and the frame field, we can define the parametric surface Σ as

$$\Sigma(s,t) = C(s) + (\alpha(s,t)\ \beta(s,t)\ \gamma(s,t))\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \tag{2.1}$$

$$0 \le s \le p, \quad 0 \le t \le q$$

where $\alpha(s,t)$, $\beta(s,t)$ and $\gamma(s,t)$ are smooth functions.

If the parameter t is seen as the time, $\alpha(s,t)$, $\beta(s,t)$ and $\gamma(s,t)$ can be viewed as directed marching distances of a point unit at the time t in the direction $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, respectively. In particular, the values of $\alpha(s,t)$, $\beta(s,t)$ and

$\gamma(s,t)$ indicate the extension-like, flexion-like and retortion-like effects, respectively, by the point unit through the time t, starting from $C(s)$. Sometimes, $\alpha(s,t)$, $\beta(s,t)$ and $\gamma(s,t)$ are said to be the marching-scale functions in the directions $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, respectively.

Definition. A curve $C(s)$ on a parametric surface $\Sigma(s,t)$ defined by (2.1) is said to be an isoparametric curve if there exists a time t_0 such that $C(s) = \Sigma(s,t_0)$.

Definition. A curve $C(s)$ on a parametric surface $\Sigma(s,t)$ defined by (2.1) is called isogeodesic of the surface $\Sigma(s,t)$ if it is both an isoparametric curve and a geodesic on the surface $\Sigma(s,t)$.

Lemma 2.1 ([7]). *A curve $C(s)$ on a parametric surface $\Sigma(s,t)$ in* (2.1) *is an isogeodesic if and only if the following conditions are satisfied*

$$\begin{aligned}
&\alpha(s,t_0) = \beta(s,t_0) = \gamma(s,t_0) = 0,\\
&\frac{\partial\beta(s,t_0)}{\partial s}\frac{\partial\gamma(s,t_0)}{\partial t} - \frac{\partial\gamma(s,t_0)}{\partial s}\frac{\partial\beta(s,t_0)}{\partial t} = 0,\\
&\left(1 + \frac{\partial\alpha(s,t_0)}{\partial s}\right)\frac{\partial\gamma(s,t_0)}{\partial t} - \frac{\partial\gamma(s,t_0)}{\partial s}\frac{\partial\alpha(s,t_0)}{\partial t} \neq 0,\\
&\left(1 + \frac{\partial\alpha(s,t_0)}{\partial s}\right)\frac{\partial\beta(s,t_0)}{\partial t} - \frac{\partial\beta(s,t_0)}{\partial s}\frac{\partial\alpha(s,t_0)}{\partial t} = 0.
\end{aligned}\tag{2.2}$$

For the better analysis of a parametric surface with an isogeodesic, we now consider the marching-scale functions $\alpha(s,t)$, $\beta(s,t)$ and $\gamma(s,t)$ are expressed by as two factors, that is,

$$\begin{aligned}
\alpha(s,t) &= f(s)F(t),\\
\beta(s,t) &= g(s)G(t),\\
\gamma(s,t) &= h(s)H(t),
\end{aligned}\tag{2.3}$$

where $f(s)$, $g(s)$, $h(s)$, $F(t)$, $G(t)$ and $H(t)$ are smooth functions.

In this case, Lemma 2.3 can be rewritten as the following statement:

Corollary 2.2. *If the marching-scale functions are chosen as in* (2.3)*, the necessary and sufficient condition for the curve $C(s)$ to be an isogeodesic on a parametric surface $\Sigma(s,t)$ is*

$$\begin{aligned}
&F(t_0) = G(t_0) = H(t_0) = 0,\\
&\frac{dG(t_0)}{dt} = 0 \quad \mathit{or} \quad g(s) = 0,\\
&\frac{dH(t_0)}{dt} = \mathit{constant} \neq 0 \quad \mathit{and} \quad h(s) \neq 0.
\end{aligned}\tag{2.4}$$

In particular, take $f(s) = g(s) = h(s) = 1$ and consider $F(t)$, $G(t)$ and $H(t)$ as polynomials of the forms;

$$
\begin{aligned}
F(t) &= \sum_{i=1}^{n} a_i (t - t_0)^i, \\
G(t) &= \sum_{i=2}^{n} b_i (t - t_0)^i, \\
H(t) &= \sum_{i=1}^{n} c_i (t - t_0)^i, \ c_1 \neq 0,
\end{aligned} \tag{2.5}
$$

respectively, where a_i, b_i, c_i are constant. Then the polynomials $F(t), G(t)$ and $H(t)$ in (2.5) satisfy the isogeodesic condition (2.4).

3. Surface interpolations and examples

In this section, we construct an isogeodesic surface passing through finite control points lying on $\mathbb{E}^3$.

Now, we give a definition for surface interpolations passing through some control points on $\mathbb{E}^3$.

Definition. Let $P_1, P_2, \ldots, P_m$ be different points on $\mathbb{E}^3$ and $\Sigma : D \subset \mathbb{R}^2 \to \mathbb{E}^3$ be a parametric surface given by (2.1). For some different points $(s_i, t_i) \in D$ $(i = 1, \ldots, m)$, we can construct the surface $\Sigma(s, t)$ such that $\Sigma(s_i, t_i) = P_i$. It is called a surface interpolation associated with the given curve $C(s)$ passing through m-control points P_i $(i = 1, \ldots, m)$, simply, C^0*-Hermite surface interpolation*. In particular, $\{P_1, P_2, \ldots, P_m\}$ is called C^0*-Hermite data*.

Polynomials $F(t), G(t)$ and $H(t)$ with degree n in (2.5) have n, $n-1$ and n degrees of freedom in terms of coefficients a_i, b_i and c_i, respectively. For the case $n \geq m+1$, we can obtain various parametric surfaces passing through the given m-control points. In particular, if $n = m+1$, then

$$
\begin{aligned}
F(t) &= \sum_{i=1}^{m+1} a_i (t - t_0)^i, \\
G(t) &= \sum_{i=2}^{m+1} b_i (t - t_0)^i, \\
H(t) &= \sum_{i=1}^{m+1} c_i (t - t_0)^i, \quad c_1 \neq 0,
\end{aligned} \tag{3.1}
$$

have $3m + 2$ degrees of freedom. In this case, there are two extra degrees of freedom. To determine a unique parametric surface, we may assume $a_{m+1} = 0$ and $c_{m+1} = 0$.

Now, we consider an isogeodesic surface parametrization

$$\Sigma(s,t) = C(s) + (F(t)\ G(t)\ H(t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \quad 0 \le s \le p, \quad 0 \le t \le q \tag{3.2}$$

with the marching-scale functions (3.1).

Theorem 3.1. *Let $P_1, P_2, \ldots, P_m$ be different points on a parametric surface $\Sigma(s,t)$ given in* (3.2). *For $\Sigma(s_i,t_i) = P_i$, $i = 1,\ldots,m$, there exists a unique C^0-Hermite surface interpolation such that the marching-scale functions are given by*

$$\begin{aligned} F(t) &= \sum_{i=1}^{m} a_i (t-t_0)^i, \\ G(t) &= \sum_{i=2}^{m+1} b_i (t-t_0)^i, \\ H(t) &= \sum_{i=1}^{m} c_i (t-t_0)^i, \quad c_1 \ne 0, \end{aligned} \tag{3.3}$$

where a_i, b_i, c_i are constant.

Proof. We define m-points by:

$$\Sigma(s_i,t_i) = P_i \text{ for } 0 \le t_0 < t_1 < t_2 < \cdots < t_m \le q.$$

Then, $\Sigma(s_i,t_i) = P_i = C(s_i) + F(t_i)\mathbf{t}(s_i) + G(t_i)\mathbf{n}(s_i) + H(t_i)\mathbf{b}(s_i)$ implies

$$\begin{aligned} F(t_i) &= \langle P_i - C(s_i),\ \mathbf{t}(s_i)\rangle, \\ G(t_i) &= \langle P_i - C(s_i),\ \mathbf{n}(s_i)\rangle, \\ H(t_i) &= \langle P_i - C(s_i),\ \mathbf{b}(s_i)\rangle, \end{aligned} \tag{3.4}$$

where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbb{E}^3$. We put

$$F(t_i) = d_{i1},\ G(t_i) = d_{i2},\ H(t_i) = d_{i3}, \tag{3.5}$$

where d_{i1}, d_{i2} and d_{i3} are constant.

From (3.1), we have

$$\begin{aligned} F(t_i) &= a_1(t_i-t_0) + a_2(t_i-t_0)^2 + \cdots + a_m(t_i-t_0)^m = d_{i1}, \\ G(t_i) &= b_2(t_i-t_0)^2 + b_3(t_i-t_0) + \cdots + b_{m+1}(t_i-t_0)^{m+1} = d_{i2}, \\ H(t_i) &= c_1(t_i-t_0) + c_2(t_i-t_0)^2 + \cdots + c_m(t_i-t_0)^m = d_{i3}, \end{aligned} \tag{3.6}$$

equivalently,

$$\begin{pmatrix} \widetilde{t}_1 & \widetilde{t}_1^2 & \cdots & \widetilde{t}_1^m \\ \widetilde{t}_2 & \widetilde{t}_2^2 & \cdots & \widetilde{t}_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_m & \widetilde{t}_m^2 & \cdots & \widetilde{t}_m^m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{m1} \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{t}_1^2 & \widetilde{t}_1^3 & \cdots & \widetilde{t}_1^{m+1} \\ \widetilde{t}_2^2 & \widetilde{t}_2^3 & \cdots & \widetilde{t}_2^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_m^2 & \widetilde{t}_m^3 & \cdots & \widetilde{t}_m^{m+1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} d_{12} \\ d_{22} \\ \vdots \\ d_{m2} \end{pmatrix}, \tag{3.7}$$

$$\begin{pmatrix} \widetilde{t}_1 & \widetilde{t}_1^2 & \cdots & \widetilde{t}_1^m \\ \widetilde{t}_2 & \widetilde{t}_2^2 & \cdots & \widetilde{t}_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_m & \widetilde{t}_m^2 & \cdots & \widetilde{t}_m^m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} d_{13} \\ d_{23} \\ \vdots \\ d_{m3} \end{pmatrix}.$$

Let

$$M_1 = \begin{pmatrix} \widetilde{t}_1 & \widetilde{t}_1^2 & \cdots & \widetilde{t}_1^m \\ \widetilde{t}_2 & \widetilde{t}_2^2 & \cdots & \widetilde{t}_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_m & \widetilde{t}_m^2 & \cdots & \widetilde{t}_m^m \end{pmatrix}, \quad M_2 = \begin{pmatrix} \widetilde{t}_1^2 & \widetilde{t}_1^3 & \cdots & \widetilde{t}_1^{m+1} \\ \widetilde{t}_2^2 & \widetilde{t}_2^3 & \cdots & \widetilde{t}_2^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_m^2 & \widetilde{t}_m^3 & \cdots & \widetilde{t}_m^{m+1} \end{pmatrix},$$

where $\widetilde{t}_i = t_i - t_0$ for $i = 1, \ldots, m$. Then we have

$$\det(M_1) = (-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} (t_i - t_0) \prod_{1 \le i < j \le m} (t_i - t_j),$$

$$\det(M_2) = (-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} (t_i - t_0)^2 \prod_{1 \le i < j \le m} (t_i - t_j).$$

Since t_i and t_j are non-zero and distinct each other, for $1 \le i < j \le m$, we get $\det(M_1) \ne 0$ and $\det(M_2) \ne 0$, that is, $a_1, a_2, \ldots, a_m$, $b_2, b_3, \ldots, b_{m+1}$ and $c_1, c_2, \ldots, c_m$ have unique solutions. Thus, there exists a uniquely C^0-Hermite surface interpolation with the C^0-Hermite data $\{P_1, P_2, \ldots, P_m\}$. □

Using points with known sample points, we will construct isogeodesic surfaces with given curves. First, we solve an isogeodesic surface passing through one point. Next, we observe the change of the isogeodesic surface passing through the additional points. Also, for the isogeodesic surface passing through the given points, we observe the change of the isogeodesic surface by moving the point.

Example 3.2. Consider a circular helix parametrized by

$$C(s) = (\frac{\sqrt{2}}{2} \cos s, \frac{\sqrt{2}}{2} \sin s, \frac{\sqrt{2}}{2} s), \quad 0 \le s \le 2\pi.$$

By a direct computation, we have

$$\begin{aligned}
\mathbf{t}(s) &= (-\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}),\\
\mathbf{n}(s) &= (-\cos s, -\sin s, 0),\\
\mathbf{b}(s) &= (\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}),\\
\kappa(s) &= \frac{\sqrt{2}}{2}, \quad \tau(s) = \frac{\sqrt{2}}{2}.
\end{aligned}$$

For $P_1(3,2,1)$, the point P_1 lies on the isogeodesic surface pencil given by (3.2). If we take

$$F(t) = a_1 t,\ G(t) = b_2 t^2,\ H(t) = c_1 t,$$

then there is only one isogeodesic surface passing the point P_1. For the convenience of calculations, take $s_1 = 0$ and $t_1 = 1$, i.e., $\Sigma(0,1) = P_1(3,2,1)$. We obtain the equations:

$$\begin{aligned}
-b_2 + \frac{\sqrt{2}}{2} &= 3,\\
\frac{\sqrt{2}}{2}a_1 - \frac{\sqrt{2}}{2}c_1 &= 2,\\
\frac{\sqrt{2}}{2}a_1 + \frac{\sqrt{2}}{2}c_1 &= 1,
\end{aligned}$$

which imply

$$a_1 = \frac{3\sqrt{2}}{2}, \quad b_2 = -3 + \frac{\sqrt{2}}{2}, \quad c_1 = -\frac{\sqrt{2}}{2}.$$

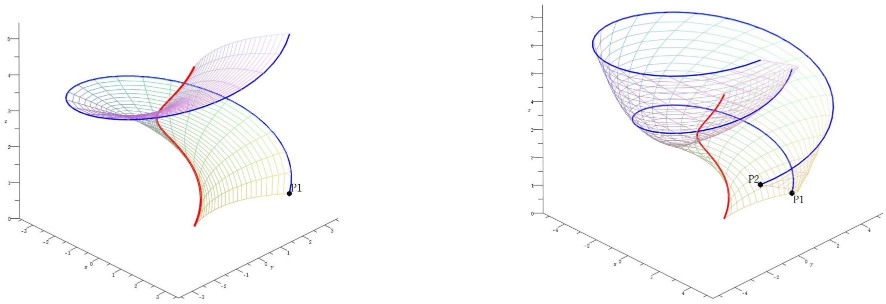
Thus, we can construct the isogeodesic surface passing the one point $P_1(3,2,1)$ given by

$$\Sigma(s,t) = C(s) + (F(t)\ G(t)\ H(t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}$$
$$0 \le s \le p, \quad 0 \le t \le q$$

consisting of

$$\begin{aligned}
C(s) &= (\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s),\\
F(t) &= \frac{3\sqrt{2}}{2},\\
G(t) &= (-3 + \frac{\sqrt{2}}{2})t^2,\\
H(t) &= -\frac{\sqrt{2}}{2}t.
\end{aligned}$$

The surface is shown in Figure 1(a).



(a) (b)

FIGURE 1. (a) the isogeodesic surface passing through the point $P_1(3,2,1)$ in Example 3.2; (b) the isogeodesic surface given by Example 3.2 passing through additional point $P_2(5,-2,3)$ in Example 3.3. The red curve is a helix that is a geodesic of the surface and the blue curves passing through P_1 and P_2 are helices, respectively.

Example 3.3. Let the isogeodesic surface of Example 3.2 pass through the additional point $P_2(5,-2,3)$. For the convenience of calculations, taking $s_2 = 0$ and $t_2 = 2$, we have $\Sigma(0,2) = P_2$ and obtain the system of linear equations as follows:

$$\begin{aligned} -b_2 - b_3 + \frac{\sqrt{2}}{2} &= 3, \\ \frac{\sqrt{2}}{2}a_1 + \frac{\sqrt{2}}{2}a_2 - \frac{\sqrt{2}}{2}c_1 - \frac{\sqrt{2}}{2}c_2 &= 2, \\ \frac{\sqrt{2}}{2}a_1 + \frac{\sqrt{2}}{2}a_2 + \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 &= 1, \\ -4b_2 - 8b_3 + \frac{\sqrt{2}}{2} &= 5, \\ \sqrt{2}a_1 + 2\sqrt{2}a_2 - \sqrt{2}c_1 - 2\sqrt{2}c_2 &= -2, \\ \sqrt{2}a_1 + 2\sqrt{2}a_2 + \sqrt{2}c_1 + 2\sqrt{2}c_2 &= 3. \end{aligned} \tag{3.8}$$

From (3.8), we obtain

$$\begin{aligned} &a_1 = \frac{11\sqrt{2}}{4}, \quad a_2 = -\frac{5\sqrt{2}}{4}, \\ &b_2 = -\frac{19}{4} + \frac{7\sqrt{2}}{8}, \quad b_3 = \frac{7}{4} - \frac{3\sqrt{2}}{8}, \\ &c_1 = -\frac{9\sqrt{2}}{4}, \quad c_2 = \frac{7\sqrt{2}}{4}. \end{aligned}$$

Thus the isogeodesic surface passing the two points $P_1(3,2,1)$ and $P_2(5,-2,3)$ is uniquely given by

$$\Sigma(s,t) = C(s) + (F(t)\ G(t)\ H(t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}$$

$$0 \le s \le p, \quad 0 \le t \le q,$$

consisting of

$$\begin{aligned} C(s) &= (\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s), \\ F(t) &= \frac{11\sqrt{2}}{4}t - \frac{5\sqrt{2}}{4}t^2, \\ G(t) &= (-\frac{19}{4} + \frac{7\sqrt{2}}{8})t^2 + (\frac{7}{4} - \frac{3\sqrt{2}}{8})t^3, \\ H(t) &= -\frac{9\sqrt{2}}{4}t + \frac{7\sqrt{2}}{4}t^2. \end{aligned}$$

The surface is shown in Figure 1(b).

Let an isogeodesic $C(s)$ of a parametric surface $\Sigma(s,t)$ given by (3.2) be a circular helix. For a fixed value $\tilde{t_0}$, we put $\tilde{C}(s) = \Sigma(s,\tilde{t_0})$. That is, $\tilde{C}(s)$ is an isoparametric curve of the isogeodesic surface $\Sigma(s,t)$. To find the curvature and the torsion of $\tilde{C}(s)$, we have

$$\begin{aligned} \tilde{C}'(s) &= C'(s) + F(\tilde{t_0})\mathbf{t}'(s) + G(\tilde{t_0})\mathbf{n}'(s) + G(\tilde{t_0})\mathbf{b}'(s) \\ &= (1 - a\ G(\tilde{t_0}))\mathbf{t}(s) + (a\ F(\tilde{t_0}) - b\ H(\tilde{t_0}))\mathbf{n}(s) + b\ G(\tilde{t_0})\mathbf{b}(s). \end{aligned}$$

Since $F(\tilde{t_0})$, $G(\tilde{t_0})$ and $H(\tilde{t_0})$ are constant for s only, $\tilde{C}'$ is rewritten as $\tilde{C}'(s) = c_1\mathbf{t}(s) + c_2\mathbf{n}(s) + c_3\mathbf{b}(s)$ for some constants c_1, c_2 and c_3. In a similar way, we have

$$\begin{aligned} \tilde{C}''(s) &= d_1\mathbf{t}(s) + d_2\mathbf{n}(s) + d_3\mathbf{b}(s), \\ \tilde{C}'''(s) &= e_1\mathbf{t}(s) + e_2\mathbf{n}(s) + e_3\mathbf{b}(s) \end{aligned}$$

for some constants d_i and e_i $(i = 1,2,3)$.

Thus, the curvature and the torsion of $\tilde{C}$ are constant, and hence the isoparametric curve $\tilde{C}$ of the isogeodesic surface $\Sigma(s,t)$ is a circular helix.

Thus, we have:

Theorem 3.4. *Let $C(s)$ be a circular helix on an isogeodesic parametric surface $\Sigma(s,t)$ given by* (3.2)*. Then, for a fixed value $\tilde{t_0}$, a curve $\tilde{C}(s) = \Sigma(s,\tilde{t_0})$ on the surface $\Sigma(s,t)$ is also a circular helix.*

Next, we consider a circle $C(s) = (\cos s, \sin s, 0)$ as a geodesic. It is easy to show that

$$\mathbf{t}(s) = (-\sin s, \cos s, 0),$$

$$\mathbf{n}(s) = (-\cos s, -\sin s, 0),$$
$$\mathbf{b}(s) = (0, 0, 1).$$

Thus, an isogeodesic parametric surface $\Sigma(s,t)$ given by (3.2) is parametrized by

$$\Sigma(s,t) = ((1-G(t))\cos s - F(t)\sin s, (1-G(t))\sin s + F(t)\cos s, H(t)). \tag{3.9}$$

Consider the isoparameter curve $\tilde{C}(s) = \Sigma(s, \tilde{t_0})$ at $t = \tilde{t_0}$ on $\Sigma(s,t)$. Let

$$X_1 = (1 - G(\tilde{t_0}))\cos s - F(\tilde{t_0})\sin s,$$
$$X_2 = (1 - G(\tilde{t_0}))\sin s + F(\tilde{t_0})\cos s,$$
$$X_3 = H(\tilde{t_0}).$$

Then we obtain

$$X_1^2 + X_2^2 = (1 - G(\tilde{t_0}))^2 + F(\tilde{t_0})^2.$$

Since $F(\tilde{t_0})$, $G(\tilde{t_0})$ and $H(\tilde{t_0})$ are constant, the isoparameter curve $\tilde{C}(s)$ is a circle with center $(0, 0, H(\tilde{t_0}))$ and radius $\sqrt{(1 - G(\tilde{t_0}))^2 + F(\tilde{t_0})^2}$ at $t = \tilde{t_0}$.

Thus, we have:

Theorem 3.5. *The isogeodesic parametric surface* (3.9) *with the circle* $C(s)$ *as a geodesic is a surface foliated by the circle* $\tilde{C}(s)$*.*

In particular, if $F(t) = 0$*, the isogeodesic parametric surface* $\Sigma(s,t)$ *is a surface of revolution with a profile curve* $R(t) = (1 - G(t), 0, H(t))$*.*

Example 3.6. Let $C(s) = (\cos s, \sin s, 0)$ be a curve, where $0 \leq s \leq 2\pi$. For $P_1(1,2,1)$, the point P_1 lies on an isogeodesic parametric surface given by (3.2). If we take $F(t) = u_1 t, G(t) = v_2 t^2$ and $H(t) = w_1 t$, then there is the only one isogeodesic parametric surface passing through the point P_1. For the convenience of calculations, if we take $s_1 = 0$ and $t_1 = 1$, then we have $\Sigma(0,1) = P_1(1,2,1)$ and the equations:

$$-b_2 + 1 = 1,$$
$$a_1 = 2,$$
$$c_1 = 1.$$

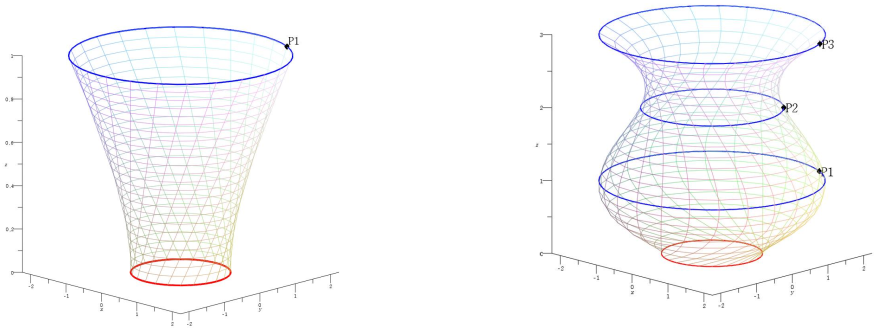
That is,

$$a_1 = 2, \quad b_2 = 0, \quad c_1 = 1.$$

Thus, the isogeodesic parametric surface passing through the one point $P_1(1,2,1)$ is given by

$$\Sigma(s,t) = C(s) + (F(t)\ G(t)\ H(t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}$$
$$0 \leq s \leq p, \quad 0 \leq t \leq q$$

consisting of $C(s) = (\cos s, \sin s, 0)$, $F(t) = 2t$, $G(t) = 0$ and $H(t) = t$ (see Figure 2(a)).



(a) (b)

FIGURE 2. (a) the isogeodesic surface passing through the point $P_1(1,2,1)$ in Example 3.6; (b) the isogeodesic surface of Example 3.6 passing through additional points $P_2(1,1,2)$ and $P_3(2,1,3)$ in Example 3.7. The red curve is a circle that is a geodesic of the surface and the blue curves passing through P_1, P_2 and P_3 are circles, respectively.

Example 3.7. Let the isogeodesic surface of Example 3.6 pass through the additional points $P_2(1,1,2)$ and $P_3(2,1,3)$. For the convenience of calculations, choose $s_2 = 0$, $t_2 = 2$, $s_3 = 0$, $t_3 = 3$. Then, $\Sigma(0,2) = P_2$ and $\Sigma(0,3) = P_3$. Moreover, we obtain the following system of linear equations:

$$\begin{aligned}
-b_2 - b_3 - b_4 + 1 &= 1,\\
a_1 + a_2 + a_3 + a_3 &= 2,\\
c_1 + c_2 + c_3 &= 1,\\
-4b_2 - 8b_3 - 16b_4 + 1 &= 1,\\
2a_1 + 4a_2 + 8a_3 &= 1,\\
2c_1 + 4c_2 + 8c_3 &= 2,\\
-9b_2 - 27b_3 - 81b_4 + 1 &= 2,\\
3a_1 + 9a_2 + 27a - 3 &= 1,\\
3c_1 + 9c_2 + 27c_3 &= 3,
\end{aligned}$$

and its solutions become

$$\begin{aligned}
&a_1 = \frac{29}{6}, \quad a_2 = -\frac{7}{2}, \quad a_3 = \frac{2}{3},\\
&b_2 = -\frac{1}{9}, \quad b_3 = \frac{1}{6}, \quad b_4 = -\frac{1}{18},\\
&c_1 = 1, \quad c_2 = 0, \quad c_3 = 0.
\end{aligned}$$

Therefore, the surface foliated by the circle $\tilde{C}(s)$ is expressed as

$$\Sigma(s,t) = C(s) + (F(t)\ G(t)\ H(t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},$$
$$0 \le s \le p, \quad 0 \le t \le q$$

consisting of

$$\begin{aligned} C(s) &= (\cos s, \sin s, 0), \\ F(t) &= \frac{29}{6}t + -\frac{7}{2}t^2 + \frac{2}{3}t^3, \\ G(t) &= -\frac{1}{9}t^2 + \frac{1}{6}t^3 - \frac{1}{18}t^4, \\ H(t) &= t \end{aligned}$$

passing through the three points $P_1(1,2,1)$, $P_2(1,1,2)$ and $P_3(2,1,3)$ (see Figure 2(b)).

4. Interpolation of ruled surfaces

A ruled surface in $\mathbb{E}^3$ is a surface that admits a parametrization

$$\Sigma(s,t) = C(s) + t\mathcal{B}(s), \tag{4.10}$$

where C is the directrix and $\mathcal{B}$ is a nowhere vanishing vector field (field of generators) along the curve C.

Consider a ruled surface $\Sigma(s,t)$ as an isoparametric surface. In this case, there exists t_0 such that $\Sigma(s,t_0) = C(s)$. Then the surface can be expressed as

$$\Sigma(s,t) - \Sigma(s,t_0) = (t-t_0)\mathcal{B}(s).$$

Also, we get

$$(t-t_0)\mathcal{B}(s) = \alpha(s,t)\mathbf{t}(s) + \beta(s,t)\mathbf{n}(s) + \gamma(s,t)\mathbf{b}(s),$$

it follows that

$$\begin{aligned} \alpha(s,t) &= (t-t_0)\langle \mathcal{B}(s), \mathbf{t}(s)\rangle, \\ \beta(s,t) &= (t-t_0)\langle \mathcal{B}(s), \mathbf{n}(s)\rangle, \\ \gamma(s,t) &= (t-t_0)\langle \mathcal{B}(s), \mathbf{b}(s)\rangle. \end{aligned}$$

By applying isogeodesic conditions (2.4), we get $\beta(s,t) = 0$ and $\langle \mathcal{B}(s), \mathbf{b}(s)\rangle \neq 0$. Thus, for some real valued functions $\phi(s)$ and $\psi(s)$, we can write

$$\mathcal{B}(s) = \phi(s)\mathbf{t}(s) + \psi(s)\mathbf{b}(s), \tag{4.11}$$

where $\psi(s) \neq 0$ for all $s \in [0,p]$.

Thus, we have the following result:

Lemma 4.1. *A ruled surface with a geodesic directrix $C(s)$ in $\mathbb{E}^3$ is parametrized as the form*:

$$\begin{gathered} \Sigma(s,t) = C(s) + t\left(\phi(s)\mathbf{t}(s) + \psi(s)\mathbf{b}(s)\right) \\ 0 \le s \le p, \quad 0 \le t \le q \end{gathered} \tag{4.12}$$

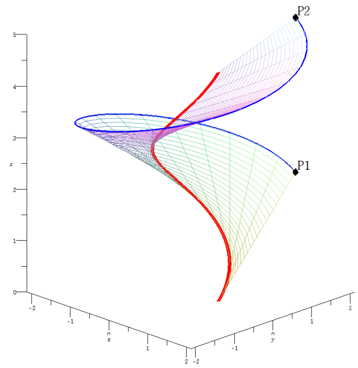


FIGURE 3. Ruled surface passing through the two points P_1 and P_2 in Example 4.4.

for some smooth functions ϕ *and* ψ *with* $\psi(s) \neq 0$ *on* $s \in [0, p]$.

Proposition 4.2. *For any point* P *in the given ruled surface* (4.12)*, the necessary condition for the existence of* C^0*-Hermite surface interpolation is*

$$\langle P - C(s), \mathbf{n}(s)\rangle = 0. \tag{4.13}$$

Proof. Let P be a point on the ruled surface, that is, $P = \Sigma(s,t)$. (4.12) implies

$$\langle \Sigma(s,t) - C(s), \mathbf{n}(s)\rangle = 0.$$

Thus, (4.13) is obtained. □

Remark 4.3. When s_0 is fixed, $\Sigma(s_0,t) = C(s_0) + t(\phi(s_0)\mathbf{t}(s_0) + \psi(s_0)\mathbf{b}(s_0))$ and $C(s_0)$ are linear dependent for $0 \le t \le q$. Therefore, it does not make sense to choose a C^0-Hermite data in the same line generated by $\mathcal{B}$ for a C^0-Hermite surface interpolation.

Example 4.4. Let $C(s) = (\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s)$ be a curve, where $0 \le s \le 2\pi$. Calculating the Frenet frame of $C(s)$, we have

$$\begin{aligned}
\mathbf{t}(s) &= (-\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}),\\
\mathbf{n}(s) &= (-\cos s, -\sin s, 0),\\
\mathbf{b}(s) &= (\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}).
\end{aligned}$$

For $P_1(\frac{\sqrt{2}}{2}, 2, 2)$ and $P_2(\frac{\sqrt{2}}{2}, 2, 5)$, P_1 and P_2 lies on an isogeodesic parametric surface given by (4.12). We may assume that $\phi(s) = a_0 + a_1 s$ and $\psi(s) = b_0 + b_1 s$. For the convenience of calculations, take $s_1 = 0$, $t_1 = 1$, $s_2 = 2\pi$ and $t_2 = 1$. Then, we get $\Sigma(0,1) = P_1(\frac{\sqrt{2}}{2}, 2, 1)$ and $\Sigma(2\pi, 1) = P_2(\frac{\sqrt{2}}{2}, 2, 5)$. Since $C(0) =$

$(\frac{\sqrt{2}}{2}, 0, 0)$, $\mathbf{n}(0) = (-1, 0, 0)$, $C(2\pi) = (\frac{\sqrt{2}}{2}, 0, \sqrt{2}\pi)$ and $\mathbf{n}(2\pi) = (-1, 0, 0)$, there exists the only one ruled surface (4.12) satisfying (4.13), passing through the mutually distinct points P_1 and P_2. In fact, we obtain the equations:

$$\begin{aligned} a_0 &= 2\sqrt{2}, \\ a_0 + 2\pi a_1 &= \frac{7\sqrt{2}}{2} - \pi, \\ b_0 &= 0, \\ b_0 + 2\pi b_1 &= \frac{3\sqrt{2}}{2} - \pi. \end{aligned}$$

That is,

$$a_0 = 2\sqrt{2}, \quad a_1 = \frac{3\sqrt{2} - 2\pi}{4\pi}, \quad b_0 = 0, \quad b_1 = \frac{3\sqrt{2} - 2\pi}{4\pi}.$$

Thus, the ruled surface passing through $P_1(\frac{\sqrt{2}}{2}, 2, 2)$ and $P_2(\frac{\sqrt{2}}{2}, 2, 5)$ is given by

$$\begin{aligned} \Sigma(s,t) = &C(s) + t\left(\phi(s)\mathbf{t}(s) + \psi(s)\mathbf{b}(s)\right) \\ &0 \le s \le 2\pi, \quad 0 \le t \le 1, \end{aligned}$$

consisting of (see Figure 3)

$$\begin{aligned} C(s) &= (\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s), \\ \phi(s) &= 2\sqrt{2} + \frac{3\sqrt{2} - 2\pi}{4\pi}s, \\ \psi(s) &= \frac{3\sqrt{2} - 2\pi}{4\pi}s. \end{aligned}$$

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