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EXPANDING MEASURES FOR HOMEOMORPHISMS WITH EVENTUALLY SHADOWING PROPERTY

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Abstract. In this paper we present a measurable version of the Smale's spectral decomposition theorem for homeomorphisms on compact metric spaces. More precisely, we prove that if a homeomorphism f on a compact metric space X is invariantly measure expanding on its chain recurrent set $CR(f)$ and has the eventually shadowing property on $CR(f)$, then f has the spectral decomposition. Moreover we show that f is invariantly measure expanding on X if and only if its restriction on $CR(f)$ is invariantly measure expanding. Using this, we characterize the measure expanding diffeomorphisms on compact smooth manifolds via the notion of Ω-stability.

1. Introduction

Let f be a homeomorphism on a compact metric space X. For given $\delta > 0$, the dynamical δ-ball centered at $x \in X$ is defined by

$$\Gamma^f_\delta(x) = \{y \in X \mid d(f^n(x), f^n(y)) \le \delta \text{ for all } n \in \mathbb{Z}\}.$$

We say that a homeomorphism f is *expansive* if there is $\delta > 0$ such that $\Gamma^f_\delta(x) = \{x\}$ for all $x \in X$. Such constant δ is called an *expansive constant* of f. The notion of expansiveness has played an important role in the qualitative study of dynamical systems.

Another extension of expansiveness for a homeomorphism f was introduced by Morales and Sirvent [10] using the properties of Borel measures on X. A Borel probability measure μ on X is said to be *expansive* for f if there is $\delta > 0$ such that $\mu(\Gamma^f_\delta(x)) = 0$ for any $x \in X$. It is clear that if μ is expansive for f, then μ is non-atomic, i.e., $\mu(\{x\}) = 0$ for all $x \in X$. We say that a homeomorphism f is *measure expansive* if every non-atomic Borel probability measure on X is expansive for f. Clearly every expansive homeomorphism f is measure expansive, but the converse is not true in general.

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Recently, Cordeiro *et al.* [5] introduced another notion of expansiveness for Borel measures called strong expansiveness: a Borel probability measure μ on X is said to be *strongly expansive* for f if there is $\delta > 0$ such that $\mu(\Gamma^f_\delta(x)) = \mu(\{x\})$ for all $x \in X$. We say that a homeomorphism f is *strongly measure expansive* if every Borel probability measure μ is strongly expansive for f. Note that if f is strongly measure expansive, then it is measure expansive, but the converse is not true in general.

The Smale's spectral decomposition theorem [12] says that any Axiom A diffeomorphism f on a compact smooth manifold admits the spectral decomposition; i.e., the nonwandering set $\Omega(f)$ can be written by a finite union of disjoint compact invariant sets on which f is topologically transitive. Afterward, there are many works that generalize the Smale's spectral decomposition theorem to general settings (e.g., see [5], [6], [9]). The first topological version of Smale's spectral decomposition theorem for homeomorphisms on compact metric spaces was done by Aoki [1]. He claimed that if a homeomorphism f is expansive on its nonwandering set $\Omega(f)$ and has the shadowing property on $\Omega(f)$, then f has the spectral decomposition.

In this paper we introduce another notion of expansiveness for homeomorphisms using the Borel measures on X called expanding measures, and prove that any measure expanding homeomorphisms with the eventually shadowing property admits the spectral decomposition. It is clear that the shadowing property implies the eventually shadowing property, and any strongly measure expansive homeomorphism is measure expanding. We note here that measure expansive homeomorphisms with the shadowing property do not admit the spectral decomposition in general as we see in Section 4.

In Section 2, we characterize the measure expanding homeomorphisms on compact metric spaces by using the notion of geometrical expansiveness. We also introduce the notion of eventually shadowing property for homeomorphisms which is general than that of the shadowing property, and show that if a homeomorphism f has the eventually shadowing property on its chain recurrent set $CR(f)$, then its restriction on its nonwandering set $\Omega(f)$ has the shadowing property.

In Section 3, we show that if a homeomorphism f is invariantly measure expanding on its chain recurrent set $CR(f)$, then it is invariantly measure expanding on X. Moreover we construct a homeomorphism which is invariantly measure expanding on its nonwandering set but not invariantly measure expanding on X.

In Section 4, we show that if a homeomorphism f is invariantly measure expanding on its chain recurrent set $CR(f)$ and has the eventually shadowing property on $CR(f)$, then f has the spectral decomposition.

In Section 5, we characterize the measure expanding diffeomorphsims on compact smooth manifolds via the notion of Ω-stability. More precisely, we show that a diffeomorphism f on a compact smooth manifold is C^1 stably invariantly measure expanding if and only if it is Ω-stable; and prove that C^1-generically, f is invariantly measure expanding if and only if it is Ω-stable.

2. Expanding measures and eventually shadowing property

Throughout this paper, we denote by X a compact metric space with a metric d and f a homeomorphism on X, and assume that Borel measure on X implies Borel probability measure on X. For $x \in X$, we write $\mathcal{O}_f(x) = \{f^i(x) \mid i \in \mathbb{Z}\}$ the orbit of $x \in X$ under f. We first introduce another notion of expansiveness for Borel measures.

Definition. A Borel measure μ on X is said to be *expanding* for f (or f is μ-expanding) if there is $\delta > 0$ (called an *expanding constant* of μ) such that $\mu(\Gamma^f_\delta(x) \setminus \mathcal{O}_f(x)) = 0$ for all $x \in X$. We say that f is *measure expanding* (resp. *invariantly measure expanding*) if every Borel measure (resp. invariant Borel measures) on X is expanding for f.

We denote by $\mathcal{M}(X)$ the collection of all Borel measures on X with the weak* topology. Then we can easily show that for any f, the collection $\mathcal{E}_f(X)$ of all expanding measures for f is dense in $\mathcal{M}(X)$. Clearly any strongly measure expansive homeomorphism is measure expanding, but the converse is not true as we see in the following example.

Example 2.1. For each $n \in \mathbb{N}$, let $A_n = \{a_{n0}, a_{n1}\} \subset \mathbb{R}^+$ be such that $A_n \cap A_m = \emptyset$ if $n \neq m$, and A_n converges to $\{0\}$ as $n \to \infty$ under the Hausdorff metric. Let

$$X = \{\infty\} \cup (\mathbb{Z} \times \{0\}) \cup \left(\bigcup_{n \in \mathbb{N}} \{-n, \ldots, n\} \times A_n \right)$$

be a subspace of the sphere $\mathbb{R}^2 \cup \{\infty\}$. We define a homeomorphism f on X by

$$f(x) = \begin{cases} \infty & \text{if } x = \infty, \\ (n+1, 0) & \text{if } x = (n, 0), \\ (i+1, a_{nj}) & \text{if } x = (i, a_{n_j}), \\ (-n, a_{n1}) & \text{if } x = (n, a_{n0}), \\ (-n, a_{n0}) & \text{if } x = (n, a_{n1}), \end{cases}$$

where $-n \leq i \leq n-1$, $j = 0, 1$, and $n \in \mathbb{N}$. For $\delta > 0$, we see that $\Gamma^f_\delta(x) \setminus \mathcal{O}_f(x) = \emptyset$ for any $x \in X$, and so any Borel measure on X is expanding for f.

On the other hand, we define an invariant Borel measure μ of f by

$$\mu(A) = \sum_{\substack{j=0,1 \\ n \in \mathbb{N}}} \frac{\delta_{(0, a_{nj})}(A)}{2^{n+1}}$$

for any Borel subset A of X, where δ_z is the Dirac measure centered at $z \in X$. Assume that μ is strongly expansive for f. Take $\delta > 0$ such that $\mu(\Gamma^f_\delta(x) \setminus$

$\{x\}) = 0$ for any $x \in X$. Let $n \in \mathbb{N}$ be such that $1/n < \delta$ and $(0, a_{n0}) \in \Gamma^f_\delta((0, a_{n1}))$. Then we have

$$\mu(\Gamma^f_\delta((0, a_{n1})) \setminus \{(0, a_{n1})\}) \geq \mu(\{(0, a_{n0})\}) > 0.$$

The contradiction shows that f is not strongly measure expansive.

Recall that a homeomorphism f is *geometrically expansive* if there is $\delta > 0$ such that $\Gamma^f_\delta(x) \subset \mathcal{O}_f(x)$ for all $x \in X$. Note that if a homeomorphism f is expansive, then it is geometrically expansive, but the converse is not true in general. In fact, the homeomorphism f in Example 2.1 is geometrically expansive, but it is not expansive. In the following theorem, we characterize measure expanding homeomorphisms on compact metric spaces via the notion of geometrical expansiveness.

Theorem 2.2. *A homeomorphism f is measure expanding if and only if it is geometrically expansive.*

Proof. Suppose that a homeomorphism f on a compact metric space X is measure expanding, but not geometrically expansive. Then for each $n \in \mathbb{N}$, there is $x_n \in X$ such that $\Gamma^f_{1/n}(x_n) \setminus \mathcal{O}_f(x_n) \neq \emptyset$. Take a point $y_n \in \Gamma^f_{1/n}(x_n) \setminus \mathcal{O}_f(x_n)$ for each $n \in \mathbb{N}$, and define a measure $\mu \in \mathcal{M}(X)$ by

$$\mu(A) = \sum_{i=1}^{\infty} \frac{\delta_{y_n}(A)}{2^n}, \ \forall A \in \beta(X),$$

where $\beta(X)$ denotes the Borel σ-algebra on X, and δ_z is the Dirac measure centered at $z \in X$. Let $\varepsilon > 0$ be a measure expanding constant of f, and choose $n \in \mathbb{N}$ with $1/n < \varepsilon$. Since

$$\mu(\Gamma^f_\varepsilon(x_n) \setminus \mathcal{O}_f(x_n)) \geq \frac{\delta_{y_n}(\Gamma^f_\varepsilon(x_n) \setminus \mathcal{O}_f(x_n))}{2^n} > 0,$$

we derive a contradiction. Consequently we have that f is geometrically expansive. The converse is clear, and so completes the proof. □

Remark 2.3. If a homeomorphism f on X is invariantly measure expanding, then there exists $\delta > 0$ which is independent of the choice of invariant measures on X such that $\mu(\Gamma^f_\delta(x) \setminus \mathcal{O}_f(x)) = 0$ for all $x \in X$ and invariant Borel measures μ on X. Indeed, by contradiction, for each $n \in \mathbb{N}$ there are $x_n \in X$ and an invariant Borel measure μ_n such that $\mu_n(\Gamma^f_{1/n}(x_n) \setminus \mathcal{O}_f(x_n)) > 0$. Define a measure μ on X by

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n}, \ \forall A \in \beta(X).$$

Then it is an invariant measure on X. Since μ is expanding for f, take an expanding constant δ of μ. Choose $n \in \mathbb{N}$ such that $1/n < \delta$. Then we have

$$\mu(\Gamma^f_\delta(x_n) \setminus \mathcal{O}_f(x_n)) \geq \frac{\mu_n(\Gamma^f_\delta(x_n) \setminus \mathcal{O}_f(x_n))}{2^n} > 0.$$

This contradicts the assumption that f is invariantly measure expanding.

In the rest of this section, we introduce the notion of eventually shadowing property for homeomorphisms on compact metric spaces. First we recall the notion of shadowing property for homeomorphisms. For given $\delta > 0$, a sequence $\{x_i\}_{i=a}^{b}$ $(-\infty \le a < b \le \infty)$ in X is called a *δ-pseudo orbit* (δ-chain) of f if $d(f^i(x_i), x_{i+1}) \le \delta$ for all $a \le i \le b-1$. We say that a homeomorphism f has the *shadowing property* on an invariant subset Λ of X if for any $\varepsilon > 0$, there is $\delta > 0$ such that any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ can be ε-shadowed by a point $x \in X$; that is, $d(f^i(x), x_i) \le \varepsilon$ for all $i \in \mathbb{Z}$.

Recently, Good and Meddaugh [8] introduced another notion of shadowing property for continuous maps called eventual shadowing. We say that a continuous map f on X has the *eventually shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ can be eventually ε-shadowed by a point $x \in X$; that is, there is $N > 0$ such that $d(f^i(x), x_i) < \varepsilon$ for all $i \ge N$. This concept was also called by $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing property in [11]. In the sequel, we introduce the notion of eventually shadowing property for homeomorphisms on compact metric spaces as follows.

Definition. We say that a homeomorphism f has the eventually shadowing property on an invariant subset Λ of X if for any $\varepsilon > 0$, there is $\delta > 0$ such that any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ can be eventually ε-shadowed by a point $x \in X$; that is, there is $N > 0$ such that $d(f^i(x), x_i) \le \varepsilon$ for all $|i| \ge N$.

It is clear that if a homeomorphism f has the shadowing property, then it has the eventually shadowing property. However the converse is not true in general. To show this, we need some definitions and a lemma.

We say that a sequence $\{x_i\}_{i=0}^{n}$ in X is a δ-chain of f from x to y $(x, y \in X)$ if $x_0 = x$ and $d(f(x_n), y) \le \delta$. We write $x \sim^{\delta} y$ if there are δ-chains from x to y and from y to x; and write $x \sim y$ if $x \sim^{\delta} y$ for any $\delta > 0$. The chain recurrent set of f, denoted by $CR(f)$, is the collection of all $x \in X$ such that $x \sim x$. Note that $\sim$ (or $\sim^{\delta}$) is an equivalence relation on $CR(f)$. Every equivalent class of $\sim$ (resp. $\sim^{\delta}$) is called a chain component (resp. δ-chain component) of f, respectively.

Lemma 2.4. *Let f be a homeomorphism on a compact metric space X. For any $\varepsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$, there is $N > 0$ such that $d(x_i, CR(f)) \le \varepsilon$ for all $|i| \ge N$.*

Proof. By contradiction, suppose that there is $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there are a $\frac{1}{n}$-pseudo orbit $\{x_i^n\}_{i\in\mathbb{N}}$ and a subsequence $\{x_{i_k}^n\}_{k\in\mathbb{N}}$ $(i_k \to \infty)$ such that $d(x_{i_k}^n, CR(f)) > \varepsilon$. Since X is compact, taking a subsequence if necessary, we may assume that $x_{i_k}^n$ converges to a point, say p^n, in X as $k \to \infty$, and p^n converges to $p \in X$ as $n \to \infty$. It is clear that $d(p, CR(f)) \ge \varepsilon$.

We observe that $p \in CR(f)$. Indeed, for any $\delta > 0$, let $0 < \gamma < \delta/3$ be such that $d(x, y) < \gamma$ $(x, y \in X)$ implies $d(f(x), f(y)) < \delta/3$. Take $n \in \mathbb{N}$ such that

$1/n < \gamma$, $d(p^n, p) < \gamma$ and $k \in \mathbb{N}$ with $d(x^n_{i_k}, p), d(x^n_{i_{k+1}}, p) < \gamma$. Then, we have

$$d(f(p), x^n_{i_k+1}) \leq d(f(p), f(p^n)) + d(f(p^n), f(x^n_{i_k})) + d(f(x^n_{i_k}), x^n_{i_k+1}) < \delta,$$

and

$$d(f(x^n_{i_{k+1}-1}), p) \leq d(f(x^n_{i_{k+1}-1}), x^n_{i_{k+1}}) + d(x^n_{i_{k+1}}, p^n) + d(p_n, p) < \delta.$$

It implies that $\{p, x^n_{i_k+1}, \ldots, x^n_{i_{k+1}-1}\}$ is a δ-chain from p to itself, and so $p \in CR(f)$. The contradiction completes the proof. $\square$

Denote by $Fix(f)$ and $Per(f)$ the collection of all fixed and periodic points of f, respectively. For given $x \in X$, we denote by $\omega(x)$ (resp. $\alpha(x)$) the collection of $y \in X$ satisfying that there is a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $f^{n_k}(x) \to y$ as $n_k \to \infty$ (resp. $n_k \to -\infty$).

Now we give an example to show that it has the eventually shadowing property, but does not have the shadowing property.

Example 2.5. Let f be a homeomorphism on the unit circle $S^1 = \{(1, \theta) \mid \theta \in [0, 2\pi)\}$ with $Fix(f) = \{(1, \pi), (1, \pi/2), (1, 0)\}$. For each $x = (1, \theta) \notin Fix(f)$, we assume that

- if $\theta \in (0, \pi/2)$, then $\alpha(x) = \{(1, \pi/2)\}$ and $\omega(x) = \{(1, 0)\}$;
- if $\theta \in (\pi/2, \pi)$, then $\alpha(x) = \{(1, \pi)\}$ and $\omega(x) = \{(1, \pi/2)\}$;
- if $\theta \in (\pi, 2\pi)$, then $\alpha(x) = \{(1, \pi)\}$ and $\omega(x) = \{(1, 0)\}$.

It is clear that f does not have the shadowing property. We show that f has the eventually shadowing property. Indeed, for any $\varepsilon > 0$, take $\delta > 0$ corresponding to $\varepsilon/2$ by Lemma 2.4. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a δ-pseudo orbit of f. Then there is $N > 0$ such that $x_i \in B(p, \varepsilon/2)$ for all $i \geq N$ and $x_i \in B(q, \varepsilon/2)$ for all $-i \geq N$ for some $p, q \in CR(f) = Fix(f)$. It is clear that $\{x_i\}_{i\in\mathbb{Z}}$ is eventually ε-shadowed by some point z such that $\omega(z) = p$ and $\alpha(z) = q$.

A point x is said to be *nonwandering* for f if for any neighborhood U of x, there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The collection of all nonwandering points of f is called the nonwandering set of f, denoted by $\Omega(f)$. It is well known that $\Omega(f) \subset CR(f)$. In the following theorem, we show that if f has the eventually shadowing property on its chain recurrent set $CR(f)$, then its restriction $f|_{\Omega(f)}$ on nonwandering set has the shadowing property.

Theorem 2.6. *Suppose that a homeomorphism f on a compact metric space X has the eventually shadowing property on its chain recurrent set $CR(f)$. Then the restriction $f|_{\Omega(X)}$ on its nonwandering set has the shadowing property.*

To prove the above theorem, we need the following lemma.

Lemma 2.7. *Suppose that a homeomorphism f on a compact metric space X has the eventually shadowing property on its chain recurrent set $CR(f)$. Then we have $\Omega(f) = CR(f)$.*

Proof. For any $\varepsilon > 0$, take a constant $\delta > 0$ corresponding to ε by the eventually shadowing property of f on $CR(f)$. For any $x \in CR(f)$, let $\{x_i\}_{i=0}^k$ be a finite δ-chain from x to itself. We extend it to an infinite δ-chain $\{x_i\}_{i\in\mathbb{Z}}$ in $CR(f)$ by defining $x_{n(k+1)+i} = x_i$ for all $0 \le i \le k$ and $n \in \mathbb{Z}$. By the eventually shadowing property of f on $CR(f)$, there are $y \in X$ and $N > 0$ such that $d(f^i(y), x_i) < \varepsilon$ for all $|i| \ge N$. Then, we have

$$d(f^{N(k+1)}(y), x_{N(k+1)}) = d(f^{N(k+1)}(y), x) < \varepsilon,$$

and

$$d(f^{-N(k+1)}(y), x_{-N(k+1)}) = d(f^{-N(k+1)}(y), x) < \varepsilon.$$

Hence we derive that

$$f^{2N(k+1)}(f^{-N(k+1)}(y)) \in f^{2N(k+1)}(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \emptyset.$$

Since ε is arbitrary, we see that x is a nonwandering point of f, and so $CR(f) = \Omega(f)$. □

To prove Theorem 2.6, we use the following well known fact: f has the shadowing property if and only if it has the *finite shadowing property*, i.e., for any $\varepsilon > 0$, there is $\delta > 0$ such that any finite δ-pseudo orbit $\{x_i\}_{i=0}^n$ can be ε-shadowed by a point $x \in X$.

End of Proof of Theorem 2.6. Suppose that a homeomorphism f has the eventually shadowing property on $CR(f)$. For any $\varepsilon > 0$, take a constant $\delta > 0$ corresponding to ε by the eventually shadowing property of f on $CR(f)$. By Lemma 2.7, we have

$$\Omega(f) = CR(f) = \bigcup_{\lambda\in\Lambda} B_\lambda,$$

where B_λ's are equivalent classes under δ-relation.

Let $\{x_i\}_{i=0}^k$ be a δ-pseudo orbit of f in $\Omega(f)$, and $\lambda \in \Lambda$ be such that $x_0 \in B_\lambda$. It is easy to see that $\{x_i\}_{i=0}^k \subset B_\lambda$. Since $x_0 \sim^\delta x_k$, we can take $\{x_{k+i}\}_{i=0}^l$ as a δ-chain from x_k to x_0. Then $\{x_i\}_{i=0}^{k+l}$ is a δ-chain from x to itself. We extend it to be a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ by $x_{n(k+l+1)+i} = x_i$ for all $0 \le i \le k+l$ and $n \in \mathbb{Z}$. By the eventually shadowing property of f on $CR(f)$, there are $y \in X$ and $N > 0$ such that

$$d(f^i(y), x_i) < \varepsilon \text{ for all } |i| \ge N.$$

Since X is compact, we assume that $f^{(k+l+1)n_j}(y)$ converges to a point, say $z \in X$, as $n_j \to \infty$. We note that $z \in \Omega(f)$. Moreover we have

$$d(f^i(z), x_i) = \lim_{j\to\infty} d(f^i(f^{(k+l+1)n_j}(y)), x_{(k+l+1)n_j+i}) \le \varepsilon, \ \forall 0 \le i \le k.$$

It means that $\{x_i\}_{i=0}^k$ is ε-shadowed by $z \in \Omega(f)$. Therefore $f|_{\Omega(f)}$ has the finite shadowing property, and so completes the proof. □

Remark 2.8. By Theorem 4.5 in [11], we see that a continuous map f on a compact metric space X has the eventually shadowing property on its chain recurrent set $CR(f)$ if and only if its restriction $f|_{\Omega(f)}$ has the shadowing property. However the result is not true for the case of homeomorphisms, that is, the converse of Theorem 2.6 does not hold in general. Indeed, let f be a homeomorphism on the interval $X = [0, 2]$ such that $\Omega(f) = Fix(f) = \{0, 1, 2\}$; for any $x \in (0, 1)$, $\omega(x) = \{1\}$ and $\alpha(x) = \{0\}$; for any $x \in (1, 2)$, $\omega(x) = \{2\}$ and $\alpha(x) = \{1\}$. Clearly, we see that $f|_{\Omega(f)}$ has the shadowing property but f does not have the eventually shadowing property on X.

3. Extension of expanding measures

We say that a homeomorphism f on a compact metric space X is expansive (resp. invariantly measure expanding) on an invariant subset Λ of X if its restriction $f|_\Lambda$ on Λ is expansive (resp. invariantly measure expanding). It is clear that if a homeomorphism f is expansive, then it is expansive on every invariant subset Λ of X. However, the converse is not true in general.

In this section, we prove that if a homeomorphism f is invariantly measure expanding on its chain recurrent set $CR(f)$, then it is invariantly measure expanding. Moreover, we construct a homeomorphism f which is invariantly measure expanding on its nonwandering set $\Omega(f)$, but not invariantly measure expanding.

For an invariant set $\Lambda \subset X$ and $\delta > 0$, we observe that $\Gamma_\delta^{f|_\Lambda}(x) = \Gamma_\delta^f(x) \cap \Lambda$ for all $x \in X$. We denote by $\mathcal{M}(\Lambda, f)$ the collection of all invariant Borel probability measures on $\Lambda \subset X$. For any $\mu \in \mathcal{M}(X, f)$ and $p \notin Per(f)$, we note that

$$\mu(\mathcal{O}_f(p)) = \sum_{i\in\mathbb{Z}} \mu(f^i(p)) = \sum_{i\in\mathbb{Z}} \mu(p).$$

Hence if $\mu(\mathcal{O}_f(p)) > 0$, then p is a periodic point of f.

Theorem 3.1. *A homeomorphism f on a compact metric space X is invariantly measure expanding if and only if it is invariantly measure expanding on its chain recurrent set $CR(f)$.*

Proof. Suppose that f is invariantly measure expanding on $CR(f)$. Then there exists $\delta > 0$ such that

$$\mu\big(\Gamma_{2\delta}^f(x) \cap CR(f) \setminus \mathcal{O}_f(x)\big) = 0, \ \forall x \in CR(f), \ \forall \mu \in \mathcal{M}(CR(f), f).$$

We first show that $\mu(\Gamma_{2\delta}^f(x) \setminus \mathcal{O}_f(x)) = 0$ for all $x \in CR(f)$ and $\mu \in \mathcal{M}(X, f)$. For any $\mu \in \mathcal{M}(X, f)$, we define an invariant measure $\nu \in \mathcal{M}(CR(f), f)$ by $\nu(A) = \mu(A)$ for any Borel set $A \subset CR(f)$. Since f is invariantly measure expanding on $CR(f)$, for any $x \in CR(f)$ we have

$$\begin{aligned}\mu(\Gamma_{2\delta}^f(x) \setminus \mathcal{O}_f(x)) &= \mu((\Gamma_{2\delta}^f(x) \cap CR(f)) \setminus \mathcal{O}_f(x)) + \mu((\Gamma_{2\delta}^f \cap CR(f)^c) \setminus \mathcal{O}_f(x)) \\ &= \nu((\Gamma_{2\delta}^f(x) \cap CR(f)) \setminus \mathcal{O}_f(x)) = 0.\end{aligned}$$

Next we claim that f is invariantly measure expanding, i.e., $\mu(\Gamma_\delta^f(x) \setminus \mathcal{O}_f(x)) = 0$ for all $x \in X$ and $\mu \in \mathcal{M}(X, f)$. By contradiction, we suppose that there are an invariant measure $\mu \in \mathcal{M}(X, f)$ and $x \in X$ such that $\mu(\Gamma_\delta^f(x) \setminus \mathcal{O}_f(x)) > 0$. By the above result, we observe that $x \notin CR(f)$. Since $\mu(CR(f)) = 1$, there is $p \in (\Gamma_\delta^f(x) \cap CR(f)) \setminus \mathcal{O}_f(x)$. Note that $\mu(\Gamma_{2\delta}^f(p) \setminus \mathcal{O}_f(p)) = 0$ since $p \in CR(f)$. Hence we have $\mu(\Gamma_\delta^f(x) \setminus \mathcal{O}_f(p)) = 0$. It derives that $\mu(\mathcal{O}_f(p)) > 0$, and so p is a periodic point of f. Let m be the period of p. Since

$$d(f^{nm+k}(x), f^k(p)) = d(f^{nm+k}(x), f^{nm+k}(p)) < \delta, \ \forall n \in \mathbb{Z}, \ \forall k \in \mathbb{Z},$$

we have $f^{nm}(x) \in \Gamma_\delta^f(p)$ for all $n \in \mathbb{Z}$. We consider the following two cases.

Case 1: There is a limit point y of $\{f^{nm}(x)\}_{n\in\mathbb{Z}}$ such that $y \notin \mathcal{O}_f(x)$. Note that $y \in \Gamma_\delta^f(p)$ and $y \in CR(f)$. Define an invariant measure $\mu_p \in \mathcal{M}(X, f)$ by

$$\mu_p(A) = \frac{1}{m} \sum_{i=0}^{m-1} \delta_{f^i(p)}(A), \ \forall A \in \beta(X),$$

where δ_z is the Dirac measure centered at $z \in X$. Since $p \in \Gamma_\delta^f(y)$, we have

$$\mu_p\big(\Gamma_\delta^f(y) \setminus \mathcal{O}_f(y)\big) \geq \mu_p(\{p\}) > 0,$$

which is a contradiction.

Case 2: Any limit point of $\{f^{nm}(x)\}_{n\in\mathbb{Z}}$ belongs to the orbit $\mathcal{O}_f(p)$. We may assume that there are subsequences $\{f^{mn_k}(x)\}_{k\in\mathbb{N}}$ with $n_k \to \infty$ and $\{f^{mn_l}\}_{k\in\mathbb{N}}$ with $n_l \to -\infty$ such that

$$f^{mn_k}(x) \to f^a(p) \text{ as } k \to \infty, \text{ and } f^{mn_l}(x) \to f^b(p) \text{ as } l \to \infty$$

for some $a \leq b \in \mathbb{N}$. For any $\varepsilon > 0$, there are $k > 0$ and $l < 0$ such that

$$d(f^{mn_k}(x), f^a(p)) < \varepsilon \text{ and } d(f^{mn_l}(x), f^b(p)) < \varepsilon.$$

We can check that

$$\{x, \ldots, f^{mn_k-1}(x), f^a(p), f^{a+1}(p), \ldots, f^{b-1}(p), f^{mn_l}(x), \ldots, f^{-1}(x)\}$$

is an ε-chain from x to itself. Since ε is arbitrary, $x \in CR(f)$ which is a contradiction. Therefore f is invariantly measure expanding. □

In the following example, we construct a homeomorphism f which is invariantly measure expanding on its nonwandering set $\Omega(f)$, but not invariantly measure expanding.

Example 3.2. For each $n \in \mathbb{N}$, let S_n be the circle in $\mathbb{R}^2$ centered at $(0, \frac{1}{n})$ with radius $\frac{1}{n}$, and $X = \bigcup_{n\in\mathbb{N}} S_n$. Let f be a homeomorphism on X such that the origin $\mathbf{0} = (0, 0)$ is the unique fixed point of f and $\alpha(p) = \omega(p) = \{\mathbf{0}\}$ for all $p \in X$.

Since $\Omega(f) = \{\mathbf{0}\}$, it is clear that f is invariantly measure expanding on $\Omega(f)$. For any $\delta > 0$, there is $p_n = (0, \frac{1}{n})$ $(n \in \mathbb{N})$ such that $\mathbf{0} \in \Gamma_\delta^f(p_n)$. Denote by $\delta_{\mathbf{0}}$ the Dirac measure centered at $\mathbf{0}$. We see that $\delta_{\mathbf{0}} \in \mathcal{M}(X, f)$ and

$\delta_{\mathbf{0}}(\Gamma^f_\delta(p_n)\setminus\mathcal{O}_f(p_n)) > 0$. It implies that f is not invariantly measure expanding on X.

4. Spectral decomposition

The classical spectral decomposition theorem by Smale [12] and Bowen [3] says that if a diffeomorphism f on a compact smooth manifold M satisfies Axiom A, then the nonwandering set $\Omega(f)$ can be decomposed into a disjoint union of finitely many closed invariant sets (called basic sets) on which f is topologically transitive. Moreover, any basic set can be expressed by a finite disjoint union of mixing sets.

There are many works that generalize the spectral decomposition theorem to more general settings (e.g, see [1], [5], [6] and [9]). In particular, Aoki [1] proved that if a homeomorphism f on a compact metric space is expansive and has the shadowing property on its nonwandering set $\Omega(f)$, then the nonwandering set has the spectral decomposition.

In this section we present a measurable version of spectral decomposition theorem by Smale [12] and Bowen [3] for invariantly measure expanding homeomorphisms with the eventually shadowing property on compact metric spaces. More precisely, we show that if a homeomorphism is invariantly measure expanding on its chain recurrent set $CR(f)$ and has the eventually shadowing property on $CR(f)$, then f has the spectral decomposition which is slightly different with the Bowen's decomposition as we can see in the following theorem.

Theorem 4.1. *Let f be a homeomorphism on a compact metric space. Suppose that f is invariant measure expanding on its chain recurrent set $CR(f)$ and has the eventually shadowing property on $CR(f)$. Then there is a decomposition of the nonwandering set $\Omega(f)$ into disjoint closed sets, $\Omega(f) = B_1 \cup \cdots \cup B_n$ such that*

(i) *each B_i is invariant and f is topologically transitive on each B_i;*

(ii) *for each $1 \leq i \leq n$, there is $a_i \in \mathbb{N}$ such that $B_i = \bigcup_{k=1}^{a_i-1} C_{i,k}$. Here each $C_{i,k}$ is a closed set with nonempty interior such that $f(C_{i,k}) = C_{i,k+1}$ for each $0 \leq k \leq a_i - 1$, $f^{a_i}(C_{i,k}) = C_{i,k}$, $C^o_{i,k} \cap C^o_{i,l} = \emptyset$ for $0 \leq k \neq l \leq a_i - 1$, and f^{a_i} is topologically mixing on $C^o_{i,k}$, where $C^o_{i,k}$ denotes the interior of $C_{i,k}$ in B_i.*

We recall that a homeomorphism f is *topologically transitive* on an invariant set Λ if for any nonempty open subset U, V of f, there is $N > 0$ such that $f^N(U) \cap V \neq \emptyset$.

Before proving the above theorem, we observe that spectral decomposition theorem does not hold for the measure expansive homeomorphisms with the shadowing property (see Theorem A in [4]). For completeness we present an example as follows.

Example 4.2. Let Σ_2 be the sequence space on two symbols 0 and 1 with the metric

$$d_0(x,y) = \begin{cases} \frac{1}{2^n} & \text{if} \quad n = \max\{k \in \mathbb{N} \,|\, x_i = y_i \ \text{ for all } |i| < k\}, \\ 0 & \text{if} \quad x = y, \end{cases}$$

where $x = (x_i)_{i\in\mathbb{Z}},\ y = (y_i)_{i\in\mathbb{Z}} \in \Sigma_2$. Consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$ given by

$$(\sigma(x))_i = x_{i+1}, \quad \forall x = (x)_{i\in\mathbb{Z}} \in \Sigma_2.$$

Then σ is an expansive homeomorphism of Σ_2 and has the shadowing property.

For each $n \in \mathbb{N}$, choose a point $p_n \in \Sigma_2$ with the period n, and let $E = \bigcup_{n\in\mathbb{N}} O_\sigma(p_n)$. Take a copy F of E such that $\Sigma_2 \cap F = \emptyset$, and let $X = \Sigma_2 \cup F$. Then F can be enumerated by a bijection $b : E \to F$ which assign an element $\sigma^k(p_n)$ of E to an element $b(\sigma^k(p_n)) := q_{nk}$ in F, where $n \in \mathbb{N}$ and $0 \le k < n$. Consider a metric D on X defined by

$$D(x,y) = \begin{cases} d_0(x,y) & \text{if} \quad x, y \in \Sigma_2, \\ \frac{1}{n} + d_0(x, \sigma^k(p_n)) & \text{if} \quad x \in \Sigma_2,\ y = q_{nk} \in F, \\ \frac{1}{n} + \frac{1}{m} + d_0(\sigma^k(p_n)), \sigma^l(p_m)) & \text{if} \quad x = q_{nk},\ y = q_{ml} \in F. \end{cases}$$

Consider a homeomorphism f of X given by

$$f(x) = \begin{cases} \sigma(x) & \text{if } x \in \Sigma_2, \\ q_{n(k+1)} & \text{if } x = q_{nk},\ 0 \le k < n-1, \\ q_{n0} & \text{if } x = q_{n(n-1)}. \end{cases}$$

Then we can see that f is 2-expansive, and has the shadowing property.

Suppose that f has the spectral decomposition, i.e., the nonwandering set $\Omega(f)$ can be decomposed by a disjoint union of finitely many invariant closed sets

$$\Omega(f) = B_1 \cup \cdots \cup B_l,$$

on each of which f is topologically transitive. Then there is B_i $(1 \le i \le l)$ contains infinitely many periodic points q_{n_j0} for $j \in \mathbb{N}$. Since each of $\mathcal{O}_f(q_{n_j0})$ is open and f is topologically transitive on B_i, for any $j \ne k$ there is $N > 0$ such that

$$f^N(\mathcal{O}_f(q_{n_j0})) \cap \mathcal{O}_f(q_{n_k0}) \ne \emptyset.$$

This contradicts to the fact that each $\mathcal{O}_f(q_{n_j0})$ is invariant. Therefore f does not admit the spectral decomposition.

We recall the concepts of local stable sets (or local unstable sets) and stable sets (or unstable sets), respectively, as follows. For given $\varepsilon > 0$ and $p \in X$, we define

$$W^s_\varepsilon(p) = \{x \in X \,|\, d(f^n(x), f^n(p)) < \varepsilon, \forall n \in \mathbb{N}\},$$
$$W^u_\varepsilon(p) = \{x \in X \,|\, d(f^{-n}(x), f^{-n}(p)) < \varepsilon, \forall n \in \mathbb{N}\},$$
$$W^s(p) = \{x \in X \,|\, d(f^n(x), f^n(p)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(p) = \{x \in X \,|\, d(f^{-n}(x), f^{-n}(p)) \to 0 \text{ as } n \to \infty\}.$$

It is well known that if a homeomorphism f is expansive on a compact metric space, then there is $e > 0$ such that

$$W^s_e(p) \subset W^s(p) \text{ and } W^u_e(p) \subset W^u(p)$$

for all $p \in Per(f)$. It is a key property to prove the spectral decomposition theorem, and it also holds for strongly measure expansive homeomorphisms (see Theorem 5.1 in [5]). However, we note that the property does not hold for invariantly measure expanding homeomorphisms. Indeed, let f be the homeomorphism given in Example 2.1. We can check that f is invariantly measure expanding. For any $e > 0$, there is $n \in \mathbb{N}$ such that $(0, a_{n0}) \in W^s_e((0, a_{n1})) \setminus W^s((0, a_{n1}))$.

We need the following lemma for the proof of the spectral decomposition.

Lemma 4.3. *Suppose that a homeomorphism f is invariantly measure expanding on its chain recurrent set $CR(f)$. Then there is a constant $e > 0$ such that if*

$$d(f^i(x), f^i(p)) \le e,\ \forall i > 0 \ (\textit{resp. } \forall i < 0)$$

for some $x \in X$ and $p \in Per(f)$, then

$$d(f^i(x), \mathcal{O}_f(p)) \to 0 \textit{ as } i \to \infty \ (\textit{resp. } i \to -\infty).$$

Proof. By Theorem 3.1, we suppose that f is an invariantly measure expanding homeomorphism on a compact metric space X. Let e be a constant such that $\mu(\Gamma^f_e(x) \setminus \mathcal{O}_f(x)) = 0$ for all $x \in X$ and $\mu \in \mathcal{M}(X, f)$. We claim that if $d(f^i(x), f^i(p)) \le e$ for all $i > 0$ and for some $x \in X, p \in Per(f)$, then $d(f^i(x), \mathcal{O}_f(p)) \to 0$ as $i \to \infty$. By contradiction, we assume that there are $x \in X$, $p \in Per(f)$ and $r > 0$ such that $d(f^i(x), f^i(p)) \le e$ for all $i > 0$, and there is a sequence $\{i_k \in \mathbb{N}\}_{k\in\mathbb{N}}$ converging to infinity as $k \to \infty$ and $d(f^{i_k}(x), \mathcal{O}_f(p)) > r$ for all $k \in \mathbb{N}$. Since X is compact, taking a subsequence if necessary, we suppose that $f^{i_k}(x) \to x_0$ and $f^{i_k}(p) \to p_0$ as $k \to \infty$. Since

$$d(x_0, \mathcal{O}_f(p)) = \lim_{k\to\infty} d(f^{i_k}(x), \mathcal{O}_f(p)) \ge r,$$

we observe that $x_0 \notin \mathcal{O}_f(p)$. Moreover, for each $i \in \mathbb{Z}$, we have

$$d(f^i(x_0), f^i(p_0)) = \lim_{k\to\infty} d(f^{i+i_k}(x), f^{i+i_k}(p)) \le e,$$

and so $p_0 \in \Gamma^f_e(x_0)$. We define an invariant Borel measure μ of f by

$$\mu(A) = \frac{1}{n_p} \sum_{i=0}^{n_p-1} \delta_{f^i(p_0)}(A),\ \forall A \in \beta(X),$$

where n_p is the period of p_0. Since f is invariantly measure expanding, we have $\mu(\Gamma^f_e(x_0) \setminus \mathcal{O}_f(x_0)) \ge \mu(p_0) > 0$. The contradiction implies $d(f^i(x), \mathcal{O}_f(p)) \to 0$ as $i \to \infty$.

Similarly we can show that if $d(f^i(x), f^i(p)) \ge e$ for all $i < 0$ for some $x \in X$ and $p \in Per(f)$, then $d(f^i(x), \mathcal{O}_f(p)) \to 0$ as $i \to -\infty$. $\square$

Remark 4.4. Suppose that f is invariantly measure expanding on its chain recurrent set $CR(f)$. For given $p \in Per(f)$ with period n_p, there is $\varepsilon = \varepsilon(p)$ such that $W^s_\varepsilon(p) \subset W^s(p)$ and $W^u_\varepsilon(p) \subset W^u(p)$. Indeed, let $e > 0$ be given by Lemma 4.3, and $\varepsilon < e$ be such that

$$B(f^i(p), \varepsilon) = \{f^i(p)\}, \ \forall\ 0 \le i \le n_p - 1.$$

Let $x \in W^s_\varepsilon(p)$. Since $\{f^{kn_p+i}(x)\}_{k\in\mathbb{N}} \subset B(f^i(p), \varepsilon)$ for all $0 \le i \le n_p - 1$ and $\omega(x) \subset \mathcal{O}_f(p)$, we have $f^{kn_p+i}(x) \to f^i(p)$ as $k \to \infty$. Then $x \in W^s(p)$, and so $W^s_\varepsilon(p) \subset W^s(p)$. Similarly, we also have $W^u_\varepsilon(p) \subset W^u(p)$.

Lemma 4.5. *Suppose f is invariant measure expanding on its chain recurrent $CR(f)$ and has the eventually shadowing property on $CR(f)$. Then $\overline{Per(f)} = CR(f)$.*

Proof. By Theorems 2.6 and 3.1, we can suppose that f is invariantly measure expanding on compact metric space X and $f|_{\Omega(f)}$ has the shadowing property. Let $x \in CR(f)$. It is sufficient to show that for any $\varepsilon > 0$, there is a periodic point p_ε of f such that $d(p_\varepsilon, x) < \varepsilon$. We fix $0 < \varepsilon < e/2$, where e is an invariantly measure expanding constant of f. Take $\delta > 0$ corresponding to ε by the shadowing property of $f|_{\Omega(f)}$. Since $x \in CR(f)$, there is a δ-chain $\{x_i\}_{i=0}^n$ of f from x to itself. We extend it to a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f by defining $x_{k(n+1)+i} = x_i$ for all $k \in \mathbb{Z}$ and $0 \le i \le n$. By the shadowing property of $f|_{\Omega(f)}$, there is $p_\varepsilon \in \Omega(f)$ such that

$$d(f^i(p_\varepsilon), x_i) < \varepsilon \ \text{ for all } i \in \mathbb{Z}.$$

For each $0 \le i \le n$ we have

$$\begin{aligned} &d(f^j(f^i(p_\varepsilon)), f^j(f^{k(n+1)+i}(p_\varepsilon))) \\ &\le d(f^j(f^i(p_\varepsilon)), x_{i+j}) + d(x_{j+i+k(n+1)}, f^j(f^{k(n+1)+i}(p_\varepsilon))) \le e. \end{aligned}$$

Then $f^{k(n+1)}(f^i(p_\varepsilon)) \in \Gamma^f_e(f^i(p_\varepsilon))$ for all $0 \le i \le n$, and so

$$\overline{\mathcal{O}_f(p_\varepsilon)} \subset \bigcup_{i=0}^n \Gamma^f_e(f^i(p_\varepsilon)).$$

Let μ be an invariant Borel measure on X such that $\mu(\overline{\mathcal{O}(p_\varepsilon)}) = 1$. Then there is $0 \le i \le n$ such that $\mu(\Gamma^f_e(f^i(p_\varepsilon))) > 0$. Since f is invariantly measure expanding on $CR(f)$, we get

$$\mu(\Gamma^f_e(f^i(p_\varepsilon)) \setminus \mathcal{O}_f(f^i(p_\varepsilon))) = 0.$$

It implies that $\mu(\mathcal{O}_f(f^i(p_\varepsilon))) > 0$, and so p_ε is a periodic point of f with $d(p_\varepsilon, x) < \varepsilon$. Consequently we have $\overline{Per(f)} = CR(f)$. $\square$

Proof of Theorem 4.1. By Theorems 2.6 and 3.1, we can suppose that f is invariantly measure expanding on X and its restriction $f|_{\Omega(f)}$ has the shadowing property. By Lemma 4.5, we have

$$\Omega(f) = CR(f) = \bigcup_{\lambda\in\Lambda} B_\lambda,$$

where B_λ's are chain components of f. We claim that B_λ is open in $\Omega(f)$. Let $e > 0$ be given in Lemma 4.3. Take $\delta > 0$ corresponding to e by the shadowing property of $f|_{\Omega(f)}$. Fix $\lambda \in \Lambda$, we denote by

$$U_\lambda = \{y \in \Omega(f) \,|\, d(y, \Omega(f)) > \delta\}$$

the δ-neighborhood of B_λ in $\Omega(f)$. Since $\overline{Per(f)} = \Omega(f)$, we take $p \in U \cap Per(f)$, and $y \in B_\lambda$ such that $d(y,p) < \delta$.

We show that $p \sim y$. For given $\tau > 0$, let $p_\tau \in Per(f)$ be such that $d(p_\tau, y) < \tau$. It is easy to see that $p_\tau \sim^\tau y$. Moreover, let $\{x_i\}_{i\in\mathbb{Z}}$ be a δ-pseudo orbit defined by $x_i = f^i(p_\tau)$ if $i < 0$ and $x_i = f^i(p)$ if $i \geq 0$. By the shadowing property of $f|_{\Omega(f)}$, there is $z \in \Omega(f)$ such that

$$d(f^i(z), f^i(p_\tau)) \leq e \text{ for all } i > 0,$$

and

$$d(f^i(z), f^i(p_\tau)) \leq e \text{ for all } i < 0.$$

By Lemma 4.3, there are $i_1 < 0 < i_2$ such that $d(f^{i_1}(z), f(p_\tau)) < \tau$ and $d(f^{i_2}(z), p) < \tau$. Then $\{p, f^{i_1}(z), \ldots, f^{i_2-1}(z)\}$ is a δ-chain of f from p_τ to p. Similarly, we can construct a δ-chain of f from p to p_τ. Then $p \sim^\tau p_\tau \sim^\tau y$. Since τ is arbitrary, we have $p \sim y$. This means $p \in B_\lambda$. Hence

$$B_\lambda \supset \overline{U \cap Per(f)} \supset U \cap \overline{Per(f)} = U.$$

Therefore, B_λ is open in $\Omega(f)$, and so

$$\Omega(f) = \bigcup_{i=1}^{n} B_i.$$

Next, we prove that f is topologically transitive on each chain component B_i. Let $\gamma > 0$ be such that γ-neighborhoods of B_i with $1 \leq i \leq n$ in $\Omega(f)$ are disjoint. For fixed $1 \leq i \leq n$, let U, V be two open subsets in B_i. There is $0 < r < \gamma$ such that $\{z \in B_i \,|\, d(z,x) < r\} \subset U$ and $\{z \in B_i \,|\, d(z,y) < r\} \subset V$ for some $x \in U$ and $y \in V$. Take $\delta_1 > 0$ corresponding to r by the shadowing property of $f|_{\Omega(f)}$. Since x, y belong to the same chain component B_i, there is a δ_1-chain $\{z_i\}_{i=0}^{m-1}$ from x to y. By the shadowing property of $f|_{\Omega(f)}$, there is $z \in \Omega(f)$ such that

$$d(z,x) \leq r \text{ and } d(f^m(z), y) \leq r.$$

By the choice of γ, we note that $z \in B_i$, and so $f^m(z) \in f^m(U) \cap V \neq \emptyset$. It implies that f is topologically transitive on each chain component B_i.

To prove part (ii), we need some notations. For convenience, we fix $B = B_i$ for some $1 \leq i \leq n$. For each $q \in Per(f) \cap B$, let $C_q := \overline{W^s(q) \cap B}$. Fix

$p \in Per(f) \cap B$ with period n_p. By Remark 4.4, there is $0 < \varepsilon_0 < \gamma$ with respect to p such that $W^s_{\varepsilon_0}(f^i(p)) \subset W^s(f^i(p))$ for all $0 \le i \le n_p - 1$.

Claim 1: $B = \bigcup_{i=0}^{n_p-1} C_{f^i(p)}$.

Let $x \in B$ and $\varepsilon < \varepsilon_0$. Take $\delta > 0$ corresponding to ε by the shadowing property of $f|_{\Omega(f)}$. Since $x, p \in B$, there is a δ-chain $\{x_i\}_{i=0}^k$ of f from x to p. Consider the δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ of f given by

$$y_i = \begin{cases} f^{-i}(x) & \text{if } i < 0, \\ x_i & \text{if } 0 \le i \le k, \\ f^{i-k-1}(p) & \text{if } i \ge k+1. \end{cases}$$

By the shadowing property of $f|_{\Omega(f)}$, there is $y \in \Omega(f)$ such that

$$d(f^j(f^{k+1}(y)), f^j(p)) \le \varepsilon, \quad \forall j \ge 0.$$

Since $\varepsilon < \varepsilon_0$, we can assume that $y \in B$. By the choice of ε_0, we get $f^{k+1}(y) \in W^s(p)$ and so $y \in \bigcup_{i=0}^{n_p-1} W^s(f^i(p)) \cap B$. Since ε is arbitrary small, we derive that $x \in \bigcup_{i=0}^{n_p-1} C_{f^i(p)}$.

Claim 2: If $q \in C_p \cap Per(f)$, then $C_q = C_p$.

Let n_q be the period of q. First, we prove that $C_q \subset C_p$. Indeed, since $q \in C_p$, there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset W^s(p) \cap B$ which converges to q as n tends to infinity. Fix $x \in W^s(q) \cap B$. For each $i \in \mathbb{N}$, we let $\gamma_i = \frac{\varepsilon_0}{2(i+1)}$, and take $\delta_i > 0$ corresponding to γ_i by the shadowing property of $f|_{\Omega(f)}$. Let $L_i \ge 2$ be such that

$$d(y_{L_i}, q) < \frac{\delta_i}{2} \text{ and } d(f^{n_q(n_pL_i)}(x), q) < \frac{\delta_i}{2}.$$

Since

$$d(f^{n_pn_qL_i}(x), y_{L_i}) \le d(f^{n_pn_qL_i}(x), q) + d(q, y_{L_i}) \le \delta_i,$$

we can define a δ_i-pseudo orbit by defining $z_j = f^j(x)$ if $j \le n_pn_qL_i - 1$ and $z_j = f^{j-n_pn_qL_i}(y_{L_i})$ if $j \ge n_pn_qL_i$. By the shadowing property of $f|_{\Omega(f)}$, there is $x_i \in B$ such that $d(x_i, x) < \gamma_i$. Then we obtain a sequence $\{x_i\}_{i\in\mathbb{N}}$ in B which converges to x as i tends to infinity.

Moreover, we check that $x_i \in W^s(p)$ for all $i \in \mathbb{N}$. Indeed, for each $i \in \mathbb{N}$, we take $N_i > 0$ such that

$$f^{N_i}(p) = p \text{ and } d(f^{j+N_i}(y_{L_i}), f^{j+N_i}(p)) < \frac{\varepsilon_0}{2}$$

for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, we have

$$\begin{aligned} d(f^j(f^{n_pn_qL_i+N_i}(x_i)), f^j(f^{N_i}(p))) &\le d(f^j(f^{n_pn_qL_i+N_i}(x_i)), f^{j+N_i}(y_{L_i})) \\ &\quad + d(f^{j+N_i}(y_{L_i}), f^j(f^{N_i}(p))) \\ &\le \gamma_i + \frac{\varepsilon_0}{2} < \varepsilon_0. \end{aligned}$$

Then we get

$$f^{n_pn_qL_i+N_i}(x_i) \in W^s_{\varepsilon_0}(f^{N_i}(p)) \subset W^s(f^{N_i}(p)) = W^s(p),$$

and so $x_i \in W^s(p)$. Hence $x \in \overline{W^s(p) \cap B} = C_p$.

To prove $C_p \subset C_q$, it is sufficient to show that $p \in C_q$. By contradiction, we assume that $p \notin C_q$. Let $d = \inf\{d(q,y) \,|\, y \in C_p \setminus C_q\} > 0$. Since $q \in C_p$, there is $x \in W^s(p) \cap B$ such that $d(x,q) < d/2$. We observe that $x \in C_q$. Otherwise, $x \in C_p \setminus C_q$ implies that $d(q,x) > d$ which is a contradiction. Then $f^{n_p n_q k}(x) \in C_q$ for all $k \in \mathbb{N}$. Since

$$d(f^{n_p n_q k}(x), p) = d(f^{n_p n_q k}(x), f^{n_p n_q k}(p)) \to 0 \text{ as } k \to \infty,$$

we have $p \in C_q$. The contradiction shows that $C_p = C_q$.

Denote by $C^o_{f^i(p)}$ the interior of $C_{f^i(p)}$ in B for all $0 \le i \le n_p - 1$. By Claim 1 and Baire category theorem, there is $0 \le i_0 \le n_p - 1$ such that $C^o_{f^{i_0}(p)}$ is nonempty. Since $f(C^o_{f^i(p)}) = C^o_{f^{i+1}(p)}$, we have $C^o_{f^i(p)}$ is nonempty for all $0 \le i \le n_p - 1$. Suppose that $C^o_{f^i(p)} \cap C^o_{f^j(p)} \neq \emptyset$. Since $Per(f)$ is dense in B, there is $q \in C^o_{f^i(p)} \cap C^o_{f^j(p)}$. By Claim 2, we get $C_{f^i(p)} = C_{f^j(p)}$. Let $0 \le a \le n_p$ be the smallest nonnegative number such that $C_{f^a(f^i(p))} = C_{f^i(p)}$ for all $i \in \mathbb{N}$. Then we have $B = \bigcup_{i=0}^{a-1} C_{f^i(p)}$.

Claim 3: f^a is topologically mixing on C^o_p.

Let U and V be two nonempty open subsets of C^o_p and q be a periodic point in V with period n_q. Then $n_q = a.r$ for some $r \in \mathbb{N}$, and take $\eta > 0$ such that $B(q,\eta) \subset V$. Since for any $0 \le j < n_q$, U is open in $C^o_p = C^o_{f^{aj}(p)} = C^o_{f^{aj}(q)}$, we can take $z_j \in U \cap W^s(f^{aj}(q))$. Then there is $N_j \in \mathbb{N}$ such that $d(f^i(z_j), f^i(f^{aj}(q))) < \eta$ for $i \ge N_j$. For $i > 0$ with $an_q i - aj \ge N_j$, we have

$$d(f^{an_q i - aj}(z_j), q) = d(f^{an_q i - aj}(z_j), f^{an_q i - aj}(f^{aj}(q))) < \eta.$$

Let $N \in \mathbb{N}$ be such that $N \ge \frac{1}{a}\max\{N_j \mid 0 \le j < n_q\}$. For $k \ge N$, we have $ak = a(n_q i - j) \ge N_j$ for some $i, j \in \mathbb{N}, 0 \le j < n_q$, and so

$$d(f^{ka}(z_j), q) = d(f^{an_q i - aj}(z_j), q) < \eta.$$

It means that $f^{ka}(U) \cap V \neq \emptyset$ for all $k \ge N$. Hence f^a is topologically mixing on C^o_p. $\square$

5. Ω-stability of diffeomorphisms with expanding measures

In this section, we characterize the invariantly measure expanding diffeomorphisms on compact smooth manifolds by using the notion of Ω-stability. Precisely, we show that a diffeomorphism f on a compact smooth manifold is C^1 stably invariantly measure expanding if and only if it is Ω-stable. Moreover, we claim that C^1-generically, a diffeomorphism f is invariantly measure expanding if and only if it is Ω-stable.

Let M be a compact smooth manifold with a metric d induced by a Riemannian metric on tangent bundle TM. We denote by $\mathrm{Diff}^1(\mathrm{M})$ the collection of all C^1 diffeomorphisms on M endowed with C^1 topology. We say that an invariant subset Λ of M is said to be *hyperbolic* for $f \in \mathrm{Diff}^1(\mathrm{M})$ if the tangent bundle $T_\Lambda M$ admits a Df-invariant splitting $E^s \oplus E^u$ such that there are $C \geq 1$ and $\lambda > 0$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \in \mathbb{N}$.

We recall that a diffeomorphism f satisfies Axiom A if $\overline{Per(f)} = \Omega(f)$ and $\Omega(f)$ is hyperbolic. By the Smale's spectral decomposition theorem, the nonwandering set $\Omega(f)$ can be decomposed into the disjoint union of basic sets $\Omega(f) = B_1 \cup \cdots \cup B_n$. A collection $B_{i_1}, \ldots, B_{i_k}$ is called a cycle if there are a_j for $1 \leq j \leq k$ such that $\omega(a_j) \subset B_{i_j+1}$ and $\alpha(a_j) \subset B_{i_j}$ with $k+1 \equiv 1$. We say that a diffeomorphism f satisfies the no-cycle condition if there does not exist any cycles among the basic sets of $\Omega(f)$.

Definition. A diffeomorphism f on a compact smooth manifold M is said to be C^1 *stably invariantly measure expanding* if there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, g is invariantly measure expanding.

A diffeomorphism $f : M \to M$ is said to be Ω-*stable* if there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $h \circ f = g \circ h$. We say that $f \in \mathcal{F}^1(M)$ if there is a C^1 neighborhood $\mathcal{U}$ of f such that every periodic orbit of $g \in \mathcal{U}$ is hyperbolic. It is well known that the following statements are pairwise equivalent:

(i) f is Ω-stable,
(ii) $f \in \mathcal{F}^1(M)$,
(iii) f satisfies both Axiom A and no-cycle condition.

In the following theorem, we characterize the Ω-stability of diffeomorphisms via the notion of C^1 stably invariantly measure expanding.

Theorem 5.1. *A diffeomorphism f on compact smooth manifold M is C^1 stably invariantly measure expanding if and only if it is Ω-stable.*

To prove Theorem 5.1, we need the following lemma which is proved in [7].

Lemma 5.2. *Let $f \in \mathrm{Diff}^1(\mathrm{M})$ and $\mathcal{U}$ be a C^1 neighborhood of f. Then there is $\delta > 0$ such that for a finite set $\{x_1, x_2, \ldots, x_n\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_n\}$ and linear maps $L_i : T_{x_i}M \to T_{f(x_i)}M$ satisfying $\|L_i - D_{x_i}f\| \leq \delta$ for all $1 \leq i \leq n$, there are $\varepsilon > 0$ and $g \in \mathcal{U}$ such that*

(i) *$g(x) = f(x)$ if $x \in M \setminus U$,*
(ii) *$g(x) = \exp_{f(x_i)} \circ L_i \circ \exp_{x_i}^{-1}(x)$ if $x \in B(x_i, \varepsilon)$ for all $1 \leq i \leq n$,*

where exp *is the exponential map of the Riemannian manifold M.*

Proof of Theorem 5.1. Suppose that a diffeomorphism f on a compact smooth manifold M is C^1 stably invariantly measure expanding. Then there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, g is invariantly measure expanding. We claim that $f \in \mathcal{F}^1(M)$. By contradiction, we assume that there are $g \in \mathcal{U}$ and a non-hyperbolic periodic point p of g. Let n be the period of p. By

Lemma 5.2, we suppose that D_pg^n has an unique real eigenvalue or a pair of complex eigenvalues λ of modulus 1. Let E_p^c be the eigenspace corresponding to λ.

We first consider the case that λ is a real number. We suppose that $\lambda = 1$. For the case $\lambda = -1$, the proof is similar. By Lemma 5.2, there are $\varepsilon > 0$ and $h \in \mathcal{U}$ such that $h^n(p) = g^n(p)$ and

$$h(x) = \exp_{g^{i+1}(p)} \circ D_{g^i(p)} \circ \exp^{-1}_{g^i(p)}(x)$$

for any $x \in B(g^i(p), \varepsilon)$ with $0 \leq i \leq n-1$. Let $\mathcal{I}_p \subset B(p, \varepsilon) \cap \exp_p(E_p^c(\varepsilon))$ be a small arc centered at p such that

- $h^i(\mathcal{I}_p) \cap h^j(\mathcal{I}_p)$ for $0 \leq i \neq j \leq n-1$,
- $h^n(\mathcal{I}_p) = \mathcal{I}_p$,
- $h^n|_{\mathcal{I}_p}$ is the identity map.

Let $\mathcal{L}$ be the normalized Lebesgue measure on $\mathcal{I}_p$. We define an invariant measure $\mu \in \mathcal{M}(M, h)$ by

$$\mu(A) = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}(h^{-1}(A \cap h^i(\mathcal{I}_p)))$$

for any Borel set A of M. For any $\delta > 0$, there is $\gamma > 0$ such that $B(p, \gamma) \cap \mathcal{I}_p \subset \Gamma_\delta^f(p)$ and $B(p, \gamma) \cap h^i(\mathcal{I}_p) = \emptyset$ for $1 \leq i \leq n-1$. Then

$$\mu(\Gamma_\delta^h(p) \setminus \mathcal{O}_h(p)) \geq \mu(B(p, \gamma) \cap \mathcal{I}_p \setminus \mathcal{O}_f(p)) > 0.$$

It is a contradiction.

Suppose that λ is a complex number. For simplicity, we assume that $g(p) = p$. Similarly to the first case, by Lemma 5.2, there are $\varepsilon > 0$ and $h \in \mathcal{U}$ such that $h(p) = g(p)$ and $h(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ for any $x \in B(p, \varepsilon)$. Suppose that m is the smallest integer such that $D_p g^m(v) = v$ for any $v \in E_p^c(\varepsilon)$. Let $\mathcal{I}_p$ be a small arc in $E_p^c(\varepsilon)$ such that

- $h^i(\mathcal{I}_p) \cap h^j(\mathcal{I}_p)$ for $0 \leq i \neq j \leq m-1$,
- $h^m(\mathcal{I}_p) = \mathcal{I}_p$,
- $h^m|_{\mathcal{I}_p}$ is the identity map.

We define an invariant measure $\mu \in \mathcal{M}(M, h)$ by

$$\mu(A) = \frac{1}{m} \sum_{i=0}^{m-1} \mathcal{L}(h^{-1}(A \cap h^i(\mathcal{I}_p)))$$

for any Borel set A in M. Similarly to the first case, we see that for any $\delta > 0$,

$$\mu(\Gamma_\delta^h(p) \setminus \mathcal{O}_h(p)) > 0.$$

The contradiction shows that $f \in \mathcal{F}(M)$ and so it is Ω-stable.

Conversely, suppose that f is Ω-stable. Let $\mathcal{U}$ be a C^1 neighborhood of f such that for any $g \in \mathcal{U}$, g satisfies Axiom A and the no-cycle condition. Then g is expansive on $CR(g)$ for any $g \in \mathcal{U}$. By Theorem 3.1, we obtain that g is invariantly measure expanding for any $g \in \mathcal{U}$. Hence, f is C^1 stably invariantly measure expanding. □

Next we study the invariantly measure expanding diffeomorphisms in the C^1 generic sense. We know that C^1 generically, any expansive diffeomorphism is Ω-stable (see Theorem 1.7 in [2]). By adapting the similar technique, we can characterize the notion of invariantly measure expanding for diffeomorphisms by using Ω-stability.

Theorem 5.3. *C^1-generically, a diffeomorphism on a compact smooth manifold is invariantly measure expanding if and only if it is Ω-stable.*

For $\delta \in (0,1)$, we recall that a hyperbolic periodic point p of a diffeomorphism f with period k has a *δ-weak hyperbolic eigenvalue* if there exists an eigenvalue λ of $D_p f^k$ such that

$$(1-\delta)^k < |\lambda| < (1+\delta)^k.$$

We say that f *has no δ-weak hyperbolic eigenvalue* if f *has no hyperbolic periodic point with a δ-weak hyperbolic eigenvalue.*

Lemma 5.4. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that if $f \in \mathcal{R}$ is invariantly measure expanding, then there exists $\delta_0 > 0$ such that f has no δ_0-weak hyperbolic eigenvalue. In particular, there exists a C^1 neighborhood $\mathcal{U}$ of f such that any $g \in \mathcal{U}$ has no $\frac{\delta_0}{2}$-weak hyperbolic eigenvalue.*

Proof. Let $\mathcal{R}$ be a residual subset of $\mathrm{Diff}^1(\mathrm{M})$ as in Lemma 5.1 in [2]. Suppose that $f \in \mathcal{R}$ is invariantly measure expanding. By contradiction, we assume that for each $n \in \mathbb{N}$, there is a hyperbolic periodic point q_n of f which has a $\frac{1}{n}$-weak hyperbolic eigenvalue. For simplicity, we assume that q_n is a fixed point of f. For each $n \in \mathbb{N}$, by Lemma 5.2, there are $\varepsilon_n > 0$ and $f_n \in \mathrm{Diff}^1(\mathrm{M})$ with $d_{C^1}(f, f_n) < 1/n$ such that $f_n(q_n) = f(q_n) = q_n$, $D_{q_n} f_n$ has unique eigenvalue λ_n such that $|\lambda_n| = 1$ and $D_{q_n} f_n^{l_n}(v) = v$ for some $l_n \in \mathbb{N}$ (take l_n as the smallest one) and for all $v \in E^c_{q_n}(\varepsilon_n)$. Here $E^c_{q_n}$ is the eigenspace corresponding to λ_n. Going back to the manifold M by using exponential map, there exists a small curve $\mathcal{I}_n$ passing through q_n formed by periodic points of f_n with the same period l_n. Let e be an invariantly measure expanding constant. For any $0 < \varepsilon < e$ and C^1 neighborhood $\mathcal{U}$ of f, there are $n \in \mathbb{N}$ with $f_n \in \mathcal{U}$ and two periodic points $p_n, q_n \in \mathcal{I}_n$ of f_n with the same period such that $d(p_n, q_n) < \varepsilon$. By Lemma 5.1 in [2], there are two periodic points p, q of f which belong to different orbits and $d(f^i(p), f^i(q)) < e$ for all $i \in \mathbb{Z}$. Let $\mu \in \mathcal{M}(M, f)$ be an invariant measure given by

$$\mu(A) = \frac{1}{m} \sum_{i=0}^{m-1} \delta_{f^i(p)}(A)$$

for any Borel set A of X. We see that $\mu(\Gamma^f_e(q) \setminus \mathcal{O}_f(q)) = \mu(\{p\}) > 0$, which is a contradiction.

To prove the second part, suppose that for any C^1 neighborhood $\mathcal{U}$ of f, there exist $g \in \mathcal{U}$ and a hyperbolic periodic point p of g which has a $\frac{\delta_0}{2}$-weak hyperbolic eigenvalue. By Lemma 5.1 in [2], we obtain that f has a hyperbolic periodic point q which has a δ_0-weak hyperbolic eigenvalue. It is a contradiction, and so completes the proof. □

End of Proof of Theorem 5.3. Suppose that a diffeomorphism $f \in \mathrm{Diff}^1(\mathrm{M})$ is Ω-stable. Then it satisfies Axiom A, and so f is expansive on $CR(f)$. By Theorem 3.1, we obtain that f is invariantly measure expanding on M.

Let $\mathcal{R}$ be the residual subset of $\mathrm{Diff}^1(\mathrm{M})$ as in Lemma 5.4. By contradiction, we assume that $f \in \mathcal{R}$ is invariantly measure expanding and $f \notin \mathcal{F}^1(M)$. It means that for any C^1 neighborhood $\mathcal{U}$ of f, there is a diffeomorphism $g \in \mathcal{U}$ which has a non-hyperbolic periodic point. It contradicts to Lemma 5.4. □

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