J. Korean Math. Soc. **57** (2020), No. 4, pp. 915–933
https://doi.org/10.4134/JKMS.j190449
pISSN: 0304-9914 / eISSN: 2234-3008

A TWO-LEVEL FINITE ELEMENT METHOD FOR THE STEADY-STATE NAVIER–STOKES/DARCY MODEL

Jilin Fang, Pengzhan Huang, and Yi Qin

Abstract. A two-level finite element method based on the Newton iterative method is proposed for solving the Navier–Stokes/Darcy model. The algorithm solves a nonlinear system on a coarse mesh H and two linearized problems of different loads on a fine mesh $h = O(H^{4-\epsilon})$. Compared with the common two-grid finite element methods for the considered problem, the presented two-level method allows for larger scaling between the coarse and fine meshes. Moreover, we prove the stability and convergence of the considered two-level method. Finally, we provide numerical experiment to exhibit the effectiveness of the presented method.

1. Introduction

The Navier–Stokes/Darcy model, coupled by certain transmission conditions at the interface, describes the coupling of incompressible fluid flow with porous media flow. This model plays an important role in many current industrial and technological applications, including hydrogeological mechanics, soil pollution simulation, biohydrodynamics, oil drilling and production engineering, industrial filtration and so on. Therefore, much effort has been devoted to the development of efficient numerical approaches for investigating this model. At the time of writing, some efficient numerical methods have been proposed [3,9,10,15,20,22,23]. In addition, Chidyagwai and Rivière [7] have used continuous finite elements in the incompressible flow region and discontinuous finite elements in the porous medium for solving the Navier–Stokes/Darcy model. A non-conforming finite volume element method has designed by Wu and Mei [29]. In [13], Girault and Rivière have proposed a numerical scheme based on discontinuous finite element methods and given the optimal error estimates. A discontinuous Galerkin finite element method for the discretization of this problem is applied by Hadji et al. [14]. Then the authors have developed a posteriori

Received June 30, 2019; Revised December 8, 2019; Accepted February 13, 2020.
2010 *Mathematics Subject Classification.* Primary 65N30, 65N12.
Key words and phrases. Navier–Stokes/Darcy model, interface conditions, two-level method, Newton iteration, scaling.
The second author was supported by the Natural Science Foundation of China (grant number 11861067).

error analysis for the resulting discrete problem. Cao et al. [5] have proposed a domain decomposition method to improve the efficiency of the finite element method and applied Newton iteration to deal with the nonlinear systems. In particular, based on two-grid discretization, Du et al. [11, 12] have constructed some local and parallel finite element algorithms for the considered problem. Besides, Chidyagwai [6] have designed a multilevel decoupling method for the Navier–Stokes/Darcy model and obtained the optimal error estimates.

When it comes to the two-grid method which is firstly introduced by Xu [30, 31] and can save a large amount of CPU time, it is a significant method to deal with the nonlinear problem. Its basic idea is solving one nonlinear system on a coarse mesh as an iterative initial value approximation of a fine mesh and then solving one linear system on the fine mesh. By employing the two-grid strategy, a two-grid finite element method for the Navier–Stokes/Darcy model is given and the efficiency through numerical analysis and experiments is verified [4]. However, it has not theoretically achieved the optimal error estimates. The scaling between the coarse mesh size and fine mesh size is $h = O(H^{\frac{3}{2}})$. Further, Qin and Hou [25] have proved the optimal error estimates for the velocity and the pressure in the fluid flow region, and improved the scaling between the coarse and fine mesh size from $h = O(H^{\frac{3}{2}})$ to $h = O(H^2)$. In addition, according to the work in [4], Jia et al. [19] have proposed and analyzed a modified two-grid decoupling method for the mixed Navier–Stokes/Darcy model, where the scaling between the coarse and fine mesh size is $h = O(H^2)$. This scaling is also obtained in [26, 33]. Moreover, based on the two-grid method and a recent local and parallel finite element method, a parallel two-grid linearized method for the coupled Navier–Stokes-Darcy problem is proposed and analyzed [34]. Similarly, it has the same order of accuracy as the standard finite element method when one takes $h = O(H^2)$. However, it is known that the two-level method is considered to be more effective for the case $h \ll H$. Hence, it is important to find an efficient algorithm to increase the ratio between the coarse and fine meshes of the two-level method.

Recently, Dai and Cheng [8] have shown a two-grid method for solving the Navier–Stokes equations based on Newton iteration. This method involves solving one small nonlinear system on a coarse mesh and two large linear problems on the fine mesh, which allows a much higher order scaling between the coarse grid size and fine grid size. Inspired by the idea of [8, 16, 17, 28], a two-level method for the Navier–Stokes/Darcy model based on the Newton iteration is given in this article. This method consists of solving a small nonlinear problem on a coarse mesh and two large linearized problems of different loads on a fine mesh based on the Newton iteration.

The rest of the paper is arranged as follows: In the next section, we introduce some notations, function spaces and some significant results of the steady Navier–Stokes/Darcy model. In Section 3, a two-level finite element method for the Navier–Stokes/Darcy model is presented. In Section 4, numerical experiment is implemented to verify the effectiveness of this presented method.

2. Notation and preliminaries

In this article, we consider the coupled fluid and the porous media flows on the domain $\Omega \subset \mathbb{R}^2$, which consists of two subdomains Ω_f and Ω_p separated by an interface Γ, i.e., $\overline{\Omega}_f \cup \overline{\Omega}_p = \overline{\Omega}$, $\Omega_f \cap \Omega_p = \varnothing$ and $\partial\Omega_f \cap \partial\Omega_p = \Gamma$. Here, we suppose Γ is sufficiently smooth as in [4]. Besides, $\mathbf{n}_f$ and $\mathbf{n}_p$ represent the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively.

In Ω_f, the fluid flow is governed by the stationary incompressible Navier–Stokes equations [27, 33]:

$$\begin{cases} -\nabla \cdot \mathbf{T}(\mathbf{u}_f, p_f) + \rho_f(\mathbf{u}_f \cdot \nabla)\mathbf{u}_f = \mathbf{f_1} & \text{in } \Omega_f, \\ \nabla \cdot \mathbf{u}_f = 0 & \text{in } \Omega_f, \end{cases} \tag{1}$$

where $\mathbf{u}_f$ and p_f denote the velocity and the kinetic pressure in Ω_f, respectively. ρ_f is the density of the fluid, $\mathbf{f_1}$ is the external force and $\mathbf{T}(\mathbf{u}_f, p_f) = -p_f\mathbf{I} + 2\nu\mathbf{D}(\mathbf{u}_f)$ is the stress tensor, where $\nu > 0$ represents the viscosity coefficient and $\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla\mathbf{u}_f + \nabla^\top\mathbf{u}_f)$ is the deformation tensor.

In Ω_p, the porous media flow is governed by the Darcy equation [24]:

$$\begin{cases} \mathbf{q} = -\mathbf{K} \cdot \nabla\phi & \text{in } \Omega_p, \\ \nabla \cdot \mathbf{q} = f_2 & \text{in } \Omega_p, \end{cases} \tag{2}$$

where $\phi = z + \frac{p_p}{\rho_f g}$ means the piezometric head, z is the elevation from a reference level, p_p is the pressure in Ω_p and g is the gravity acceleration. The discharge vector $\mathbf{q} = \varepsilon\mathbf{u}_p$, ε is the volumetric porosity [21] and $\mathbf{u}_p$ is the velocity in Ω_p. In addition, f_2 is the source term with a solvability condition $\int_{\Omega_p} f_2 = 0$. $\mathbf{K}$ is the hydraulic conductivity tensor of the porous medium. Here, we assume that $\mathbf{K}$ is a symmetric positive define matrix uniformly bounded above and below, i.e., $\exists\ \lambda_{\min} > 0$, $\lambda_{\max} > 0$ such that

$$\text{a.e. } \mathbf{x} \in \Omega_p,\ \ \lambda_{\min}\mathbf{x}\cdot\mathbf{x} \le \mathbf{K}\mathbf{x}\cdot\mathbf{x} \le \lambda_{\max}\mathbf{x}\cdot\mathbf{x}. \tag{3}$$

Using Darcy's law, (2) can be rewritten in the elliptic form:

$$-\nabla \cdot (\mathbf{K} \cdot \nabla\phi) = f_2 \text{ in } \Omega_p. \tag{4}$$

For boundaries $\partial\Omega_f\backslash\Gamma$ and $\partial\Omega_p\backslash\Gamma$, we impose homogeneous Dirichlet boundary conditions, i.e.,

$$\begin{cases} \mathbf{u}_f = 0 & \text{on } \partial\Omega_f\backslash\Gamma, \\ \phi = 0 & \text{on } \partial\Omega_p\backslash\Gamma. \end{cases} \tag{5}$$

About the interface Γ, we consider the following interface conditions as studied in [2, 18]:

$$\begin{aligned} &\mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p = 0, \\ &\boldsymbol{\tau} \cdot [-\mathbf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f] = \frac{\alpha}{\sqrt{\boldsymbol{\tau} \cdot \nu\mathbf{K} \cdot \boldsymbol{\tau}}}\mathbf{u}_f \cdot \boldsymbol{\tau}, \\ &\mathbf{n}_f \cdot [-\mathbf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f] = \rho_f g\phi, \end{aligned} \tag{6}$$

where α is a positive parameter depending on the properties of the porous medium that is experimentally determined and $\boldsymbol{\tau}$ is an unit tangential vector on Γ. For brevity, we assume that ε and ρ_f are constants.

Denote $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$, where

$$H_f = \{\mathbf{v} \in H^1(\Omega_f)^2 : \mathbf{v} = 0 \text{ on } \partial\Omega_f \backslash \Gamma\},$$
$$H_p = \{\phi \in H^1(\Omega_p) : \phi = 0 \text{ on } \partial\Omega_p \backslash \Gamma\}.$$

We equip the space $L^2(\Lambda)$ ($\Lambda = \Omega_f$ or Ω_p) with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_{L^2(\Lambda)}$. The space W is equipped with the following norm: $\forall u = (\mathbf{u}_f, \phi) \in W$,

$$\|u\|_W = \sqrt{2\nu(\mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{u}_f))_{\Omega_f} + \mathbf{K}(\nabla\phi, \nabla\phi)_{\Omega_p}}.$$

Set $f = (\mathbf{f_1}, f_2)$, then the weak formulation of the steady Navier–Stokes/Darcy model as follows: Find $u = (\mathbf{u}_f, \phi) \in W$, $p_f \in Q$ such that

$$\left\{ \begin{array}{ll} a(u,v) + d(v,p_f) + b(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) = (f,v) & \forall v = (\mathbf{v}, \psi) \in W, \\ d(u,q) = 0 & \forall q \in Q, \end{array} \right. \tag{7}$$

where

$$\begin{aligned}
a(u,v) &= a_{\Omega_f}(\mathbf{u}_f, \mathbf{v}) + a_{\Omega_p}(\phi, \psi) + a_\Gamma(u,v), \\
a_{\Omega_f}(\mathbf{u}_f, \mathbf{v}) &= \varepsilon \int_{\Omega_f} 2\nu \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}) + \varepsilon \int_\Gamma \frac{\alpha}{\sqrt{\boldsymbol{\tau} \cdot \nu \mathbf{K} \cdot \boldsymbol{\tau}}} (\mathbf{u}_f \cdot \boldsymbol{\tau}) \cdot (\mathbf{v} \cdot \boldsymbol{\tau}), \\
a_{\Omega_p}(\phi, \psi) &= \rho_f g \int_{\Omega_p} \mathbf{K} \cdot \nabla\phi \cdot \nabla\psi, \quad a_\Gamma(u,v) = \varepsilon \rho_f g \int_\Gamma (\phi \mathbf{v} \cdot \mathbf{n}_f - \psi \mathbf{u}_f \cdot \mathbf{n}_f), \\
b(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) &= \varepsilon \rho_f \left(\int_{\Omega_f} (\mathbf{u}_f \cdot \nabla) \mathbf{u}_f \cdot \mathbf{v} + \frac{1}{2} \int_{\Omega_f} (\nabla \cdot \mathbf{u}_f) \mathbf{u}_f \cdot \mathbf{v} \right), \\
d(v, p_f) &= -\varepsilon \int_{\Omega_f} p_f \nabla \cdot \mathbf{v}, \quad (f,v) = \varepsilon \int_{\Omega_f} \mathbf{f_1} \cdot \mathbf{v} + \rho_f g \int_{\Omega_p} f_2 \psi.
\end{aligned}$$

In addition, for the trilinear form $b(\cdot;\cdot,\cdot)$, we list the following estimates [8]: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H_f$,

$$\begin{aligned}
|b(\mathbf{u}; \mathbf{v}, \mathbf{w})| &\leq C_0 \|\nabla \mathbf{u}\|_{L^2(\Omega_f)^2} \|\nabla \mathbf{v}\|_{L^2(\Omega_f)^2} \|\nabla \mathbf{w}\|_{L^2(\Omega_f)^2}, \\
|b(\mathbf{u}; \mathbf{v}, \mathbf{w})| &\leq C_1 \|\mathbf{u}\|_{L^2(\Omega_f)^2}^{1-\epsilon} \|\nabla \mathbf{u}\|_{L^2(\Omega_f)^2}^{\epsilon} \|\nabla \mathbf{v}\|_{L^2(\Omega_f)^2} \|\nabla \mathbf{w}\|_{L^2(\Omega_f)^2},
\end{aligned} \tag{8}$$

where C_0 and C_1 denote the positive constants and $\epsilon > 0$ is arbitrarily small.

We also recall the Poincaré and Korn's inequalities, trace inequalities [32] and Sobolev inequalities that are useful in the analysis. There exist constants C_2, C_3, C_4, C_5, C_6, C_7 and C_8 that depend on Ω_f or Ω_p such that $\forall \mathbf{v} \in H_f$,

$\forall \phi \in H_p$, we have the following bounds [33]

$$
\begin{aligned}
&\|\mathbf{v}\|_{L^2(\Omega_f)^2} \le C_2 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)^2}, \qquad \|\phi\|_{L^2(\Omega_p)} \le C_3 \|\nabla \phi\|_{L^2(\Omega_p)},\\
&\|\nabla \mathbf{v}\|_{L^2(\Omega_f)^2} \le C_4 \|D(\mathbf{v})\|_{L^2(\Omega_f)^2}, \quad C_5 \|\mathbf{K}^{1/2}\nabla\phi\|^2_{L^2(\Omega_p)} \le a_{\Omega_p}(\phi,\phi),\\
&\|\phi\|_{L^2(\Omega_p)} \le C_6 \|\mathbf{K}^{1/2}\nabla\phi\|_{L^2(\Omega_p)}, \quad \|\mathbf{v}\|_{L^2(\Gamma)^2} \le C_7 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)^2},\\
&\|\phi\|_{L^2(\Gamma)} \le C_8 \|\nabla\phi\|_{L^2(\Omega_p)}.
\end{aligned} \tag{9}
$$

Given (3), we obtain

$$
\frac{1}{\sqrt{\lambda_{\max}}}\|\mathbf{K}^{1/2}\nabla\phi\|_{L^2(\Omega_p)} \le \|\nabla\phi\|_{L^2(\Omega_p)} \le \frac{1}{\sqrt{\lambda_{\min}}}\|\mathbf{K}^{1/2}\nabla\phi\|_{L^2(\Omega_p)}. \tag{10}
$$

The following well-posedness for the coupled Navier–Stokes/Darcy model (7) is classical.

Theorem 2.1 ([1,33])**.** *Assume that the data satisfies*:

$$
\frac{C_2^2 C_4^2}{4\nu}\|\mathbf{f}_1\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f g C_3^2}{2\varepsilon\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_p)} < \frac{4\varepsilon^2\nu^3}{C_0^2 C_4^6}.
$$

Then the problem (7) *has at most one weak solution satisfying*

$$
\|\mathbf{D}(\mathbf{u}_f)\|^2_{L^2(\Omega_f)^2} \le \frac{C_2^2 C_4^2}{4\nu^2}\|\mathbf{f}_1\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f g C_3^2}{2\varepsilon\nu\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_p)}.
$$

We partition Ω_f and Ω_p by quasi-uniform triangulations $K_{f,\mu}$ and $K_{p,\mu}$ with a real positive parameter μ ($\mu = h$ or H with $h \ll H$). For given $K_{p,\mu}$ and $K_{f,\mu}$, we consider the following finite element spaces $W_\mu = H_{f,\mu} \times H_{p,\mu} \subset W$ and $Q_\mu \subset Q$:

$$
H_{f,\mu} = (P^b_{1,\mu})^d \cap H_f, \quad H_{p,\mu} = \{\psi_\mu \in C^0(\Omega_p) : \psi_\mu|_K \in P_1(K), \forall K \in K_{p,\mu}\},
$$
$$
Q_\mu = \{q_\mu \in C^0(\Omega_f) : q_\mu|_K \in P_1(K), \forall K \in K_{f,\mu}\},
$$

where

$$
P^b_{1,\mu} = \{\mathbf{v}_\mu \in C^0(\Omega_f) : \mathbf{v}_\mu|_K \in P_1(K) \oplus \operatorname{span}\{\widehat{b}\}, \forall K \in K_{f,\mu}\},
$$

$\widehat{b}$ is a bubble function, and $P_1(K)$ is a space of linear polynomials on element K. Furthermore, we need the subspace $H_{f,\mu 0}$ of $H_{f,\mu}$ which is defined as

$$
H_{f,\mu 0} = \{\mathbf{v}_\mu \in H_{f,\mu}; (\nabla\cdot\mathbf{v}_\mu, q_\mu) = 0, \forall q_\mu \in Q_\mu\}.
$$

Note that the inf-sup condition holds, i.e., there is a positive constant β independent of μ such that

$$
d(v_\mu, q_\mu) \ge \beta\|v_\mu\|_W \|q_\mu\|_Q, \quad \forall v_\mu \in W_\mu, \ q_\mu \in Q_\mu. \tag{11}
$$

Further, the finite element scheme of (7) is defined as the following coupled system: Find $u_\mu = (\mathbf{u}_{f,\mu}, \phi_\mu) \in W_\mu$, and $p_{f,\mu} \in Q_\mu$, such that

$$
\begin{cases}
a(u_\mu, v_\mu) + d(v_\mu, p_{f,\mu}) + b(\mathbf{u}_{f,\mu}; \mathbf{u}_{f,\mu}, \mathbf{v}_\mu) = (f, v_\mu) & \forall v_\mu = (\mathbf{v}_\mu, \psi_\mu) \in W_\mu,\\
d(u_\mu, q_\mu) = 0 & \forall q_\mu \in Q_\mu.
\end{cases} \tag{12}
$$

The following theorems establish the stability and error estimate results for the finite element discretization (12) of the considered problem.

Theorem 2.2 ([4,7,33])**.** *Let*

$$R = \left(\max\{\frac{1}{\varepsilon}, \frac{1}{C_5}\}\right)^{1/2} \left(\frac{\varepsilon C_2^2 C_4^2}{\nu}\|\mathbf{f}_1\|^2_{L^2(\Omega_f)^2} + \frac{2\rho_f^2 g^2 C_6^2}{C_5}\|f_2\|^2_{L^2(\Omega_p)}\right)^{1/2}.$$

Then, under the assumption $R^2 \le \frac{8\varepsilon^2\nu^3}{C_0^2 C_4^6}$, *the problem* (12) *admits a unique solution satisfying*

$$2\nu\|\mathbf{D}(\mathbf{u}_{f,\mu})\|^2_{L^2(\Omega_f)^2} + \|\mathbf{K}^{1/2}\nabla\phi_\mu\|^2_{L^2(\Omega_p)} \le R^2. \tag{13}$$

Besides, let $(\mathbf{u}_f, \phi, p_f) \in H^2(\Omega_f)^2 \times H^2(\Omega_p) \times H^1(\Omega_f)$ *be the solution of* (7), *we have the following error estimate*

$$\begin{aligned} &\|\mathbf{D}(\mathbf{u}_f - \mathbf{u}_{f,\mu})\|_{L^2(\Omega_f)^2} + \|\nabla(\phi - \phi_\mu)\|_{L^2(\Omega_p)} + \|p_f - p_{f,\mu}\|_{L^2(\Omega_f)} \\ &\le c\mu(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}). \end{aligned} \tag{14}$$

Theorem 2.3 ([4])**.** *Under the assumption of Theorem 2.2, let* $(\mathbf{u}_{f,h}, \phi_h, p_{f,h})$ *be the finite element solution of* (12) $(\mu = h)$, *we have the* L^2*-error estimate*

$$\begin{aligned} &\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_f)^2} + \|\phi - \phi_h\|_{L^2(\Omega_p)} \\ &\le ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}) \end{aligned} \tag{15}$$

and for $h \ll H$, *we then have*

$$\|\mathbf{u}_{f,h} - \mathbf{u}_{f,H}\|_{H^1(\Omega_f)^2} \le cH(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}). \tag{16}$$

3. A two-level finite element method

In this section, we show an effective two-level finite element method for the Navier-Stokes/Darcy model. The algorithm is shown as follows:

Algorithm 3.1.
Step I. Solve the nonlinear problem on a coarse grid: Find $u_H = (\mathbf{u}_{f,H}, \phi_H) \in W_H$, $p_{f,H} \in Q_H$ such that for all $v = (\mathbf{v}, \psi) \in W_H$, $q \in Q_H$,

$$\begin{cases} a(u_H, v) + d(v, p_{f,H}) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,H}, \mathbf{v}) = (f, v), \\ d(u_H, q) = 0. \end{cases} \tag{17}$$

Step II. Solve a linear problem on a fine grid based on Newton iteration: Find $u_h^* = (\mathbf{u}_{f,h}^*, \phi_h^*) \in W_h$, $p_{f,h}^* \in Q_h$ such that for all $v = (\mathbf{v}, \psi) \in W_h$, $q \in Q_h$,

$$\begin{cases} a(u_h^*, v) + d(v, p_{f,h}^*) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{v}) + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H}, \mathbf{v}) \\ \quad = (f, v) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,H}, \mathbf{v}), \\ d(u_h^*, q) = 0. \end{cases} \tag{18}$$

Step III. Update on the same fine mesh: Find $u^h = (\mathbf{u}_f^h, \phi^h) \in W_h$, $p_f^h \in Q_h$ such that for all $v = (\mathbf{v}, \psi)$, $q \in Q_h$,

$$\begin{cases} a(u^h, v) + d(v, p_f^h) + b(\mathbf{u}_{f,H}; \mathbf{u}_f^h, \mathbf{v}) + b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{v}) \\ \quad = (f, v) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{v}) + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{v}), \\ d(u^h, q) = 0. \end{cases} \tag{19}$$

Next, we derive stability and error estimates of the presented method for the Navier–Stokes/Darcy equations.

Theorem 3.1. *Suppose* $0 < \delta < 1$ *with* $\delta = 1 - \frac{C_0 R C_4^3}{\sqrt{2}\nu^{3/2}\varepsilon}$. *Under the assumption of Theorem 2.2,* $(\mathbf{u}_{f,h}^*, \phi_h^*)$ *defined by Step II of Algorithm 3.1 satisfies*

$$2\nu \|\mathbf{D}(\mathbf{u}_{f,h}^*)\|_{L^2(\Omega_f)^2}^2 + \sigma^{-1}\|\mathbf{K}^{1/2}\nabla \phi_h^*\|_{L^2(\Omega_p)}^2 \le \sigma^{-1} R_1^2. \tag{20}$$

Moreover, $(\mathbf{u}_f^h, \phi^h)$ *defined by Step III of Algorithm 3.1 satisfies*

$$2\nu \|\mathbf{D}\left(\mathbf{u}_f^h\right)\|_{L^2(\Omega_f)^2}^2 + \sigma^{-1}\|\mathbf{K}^{1/2}\nabla \phi^h\|_{L^2(\Omega_p)}^2 \le \sigma^{-1} R_2^2, \tag{21}$$

where

$$\sigma = \frac{\varepsilon}{C_5}\delta, \quad R_1^2 = \frac{\varepsilon C_2^2 C_4^2}{\nu C_5}\|\mathbf{f_1}\|_{L^2(\Omega_f)^2}^2 + \frac{\rho_f^2 g^2 C_3^2}{\lambda_{\min} C_5^2}\|f_2\|_{L^2(\Omega_p)}^2 + \frac{C_0^2 R^4 C_4^6}{4\varepsilon\nu^3 C_5},$$

and

$$R_2^2 = \frac{3\varepsilon C_2^2 C_4^2}{2\nu C_5}\|\mathbf{f}_1\|_{L^2(\Omega_f)^2}^2 + \frac{\rho_f^2 g^2 C_3^2}{\lambda_{\min} C_5^2}\|f_2\|_{L^2(\Omega_p)}^2 + \frac{3C_0^2 R^2 R_1^2 C_4^6}{2\varepsilon\sigma\nu^3 C_5} + \frac{3C_0^2 R_1^4 C_4^6}{8\varepsilon\sigma^2\nu^3 C_5}.$$

Proof. Firstly, we consider the stability of $(\mathbf{u}_{f,h}^*, \phi_h^*)$. Taking $(v, q) = (u_h^*, p_{f,h}^*)$, i.e., $v = (\mathbf{v}, \psi) = (\mathbf{u}_{f,h}^*, \phi_h^*) = u_h^*$ and $q = p_{f,h}^*$ in (18), we have

$$\begin{cases} a(u_h^*, u_h^*) + d(u_h^*, p_{f,h}^*) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{u}_{f,h}^*) + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*) \\ \quad = (f, u_h^*) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*), \\ d(u_h^*, p_{f,h}^*) = 0. \end{cases} \tag{22}$$

Noting that $a_\Gamma(u_h^*, u_h^*) = 0$ and $b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{u}_{f,h}^*) = 0$ yields

$$\begin{aligned} &a_{\Omega_f}(\mathbf{u}_{f,h}^*, \mathbf{u}_{f,h}^*) + a_{\Omega_p}(\phi_h^*, \phi_h^*) \\ &\quad = (f, u_h^*) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*) - b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*). \end{aligned} \tag{23}$$

By using (9) and the triangle inequality, we get

$$\begin{aligned} &2\nu\varepsilon \|\mathbf{D}(\mathbf{u}_{f,h}^*)\|_{L^2(\Omega_f)^2}^2 + C_5\|\mathbf{K}^{1/2}\nabla\phi_h^*\|_{L^2(\Omega_p)}^2 \\ &\le \varepsilon|(\mathbf{f_1}, \mathbf{u}_{f,h}^*)_{\Omega_f}| + \rho_f g|(f_2, \phi_h^*)_{\Omega_p}| + |b(\mathbf{u}_{f,H}; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*)| + |b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H}, \mathbf{u}_{f,h}^*)|. \end{aligned} \tag{24}$$

Then, applying the Hölder, Poincaré, Young inequalities and Theorem 2.2, it follows that

$$\frac{2\nu\varepsilon}{C_5}\|\mathbf{D}(\mathbf{u}_{f,h}^*)\|_{L^2(\Omega_f)^2}^2 + \|\mathbf{K}^{1/2}\nabla\phi_h^*\|_{L^2(\Omega_p)}^2$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{C_5}|(\mathbf{f_1},\mathbf{u}^*_{f,h})_{\Omega_f}| + \frac{\rho_f g}{C_5}|(f_2,\phi^*_h)_{\Omega_p}| + \frac{1}{C_5}|b(\mathbf{u}_{f,H};\mathbf{u}_{f,H},\mathbf{u}^*_{f,h})| \\
&\quad + \frac{1}{C_5}|b(\mathbf{u}^*_{f,h};\mathbf{u}_{f,H},\mathbf{u}^*_{f,h})| \\
&\leq \frac{\varepsilon C_2 C_4}{C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|_{L^2(\Omega_f)^2}\|\mathbf{f_1}\|_{L^2(\Omega_f)^2} + \frac{\rho_f g C_3}{C_5}\|\nabla\phi^*_h\|^2_{L^2(\Omega_p)}\|f_2\|_{L^2(\Omega_p)} \\
&\quad + \frac{C_0 C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,H})\|^2_{L^2(\Omega_f)^2} \\
&\quad + \frac{C_0 C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2} \\
&\leq \frac{\nu\varepsilon}{2C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2} + \frac{\varepsilon C_2^2 C_4^2}{2\nu C_5}\|\mathbf{f_1}\|^2_{L^2(\Omega_f)^2} + \frac{\lambda_{\min}}{2}\|\nabla\phi^*_h\|^2_{L^2(\Omega_p)} \\
&\quad + \frac{\rho_f^2 g^2 C_3^2}{2\lambda_{\min} C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{C_0^2 C_4^6}{2\varepsilon\nu C_5}\|\mathbf{D}(\mathbf{u}_{f,H})\|^4_{L^2(\Omega_f)^2} + \frac{\nu\varepsilon}{2C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2} \\
&\quad + \frac{C_0 C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2} \\
&\leq \frac{\nu\varepsilon}{C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2} + \frac{\varepsilon C_2^2 C_4^2}{2\nu C_5}\|\mathbf{f_1}\|^2_{L^2(\Omega_f)^2} + \frac{1}{2}\|\mathbf{K}^{1/2}\nabla\phi^*_h\|^2_{L^2(\Omega_p)} \\
&\quad + \frac{\rho_f^2 g^2 C_3^2}{2\lambda_{\min} C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{C_0^2 R^4 C_4^6}{8\varepsilon C_5 \nu^3} + \frac{C_0 R C_4^3}{\sqrt{2\nu} C_5}\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2}.
\end{aligned}
$$

Let $\delta = 1 - \frac{C_0 R C_4^3}{\sqrt{2}\nu^{3/2}\varepsilon}$, and assume $0 < \delta < 1$. Then we have

$$
\begin{aligned}
&\frac{\varepsilon}{C_5}\delta 2\nu\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2} + \|\mathbf{K}^{1/2}\nabla\phi^*_h\|^2_{L^2(\Omega_p)} \\
&\leq \frac{\varepsilon C_2^2 C_4^2}{\nu C_5}\|\mathbf{f_1}\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f^2 g^2 C_3^2}{\lambda_{\min} C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{C_0^2 R^4 C_4^6}{4\varepsilon\nu^3 C_5}.
\end{aligned} \tag{25}
$$

Further, set $\sigma = \frac{\varepsilon}{C_5}\delta$, we arrive at

$$
\begin{aligned}
&2\nu\|\mathbf{D}(\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2} + \sigma^{-1}\|\mathbf{K}^{1/2}\nabla\phi^*_h\|^2_{L^2(\Omega_p)} \\
&\leq \sigma^{-1}\left(\frac{\varepsilon C_2^2 C_4^2}{\nu C_5}\|\mathbf{f_1}\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f^2 g^2 C_3^2}{\lambda_{\min} C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{C_0^2 R^4 C_4^6}{4\varepsilon\nu^3 C_5}\right).
\end{aligned} \tag{26}
$$

Next, taking $(v,q) = \left(u^h, p_f^h\right)$, i.e., $(\mathbf{v},\psi) = (\mathbf{u}_f^h, \phi^h)$ and $q = p_f^h$ in (19), we know that

$$
\begin{cases}
a(u^h,u^h) + d(u^h,p_f^h) + b(\mathbf{u}_{f,H};\mathbf{u}_f^h,\mathbf{u}_f^h) + b(\mathbf{u}_f^h;\mathbf{u}_{f,H},\mathbf{u}_f^h) \\
\ = (f,u^h) + b(\mathbf{u}_{f,H};\mathbf{u}^*_{f,h},\mathbf{u}_f^h) + b(\mathbf{u}^*_{f,h};\mathbf{u}_{f,H} - \mathbf{u}^*_{f,h},\mathbf{u}_f^h), \\
d(u^h,p_f^h) = 0.
\end{cases} \tag{27}
$$

By $a_\Gamma(u^h, u^h) = 0$ and $b(\mathbf{u}_{f,H}; \mathbf{u}_f^h, \mathbf{u}_f^h) = 0$, we obtain

$$\begin{aligned} &a_{\Omega_f}(\mathbf{u}_f^h, \mathbf{u}_f^h) + a_{\Omega_p}(\phi^h, \phi^h) \\ &= (f, u^h) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{u}_f^h) + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{u}_f^h) - b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{u}_f^h). \end{aligned} \tag{28}$$

Analogously,

$$\begin{aligned} &2\nu\varepsilon\|\mathbf{D}(\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2} + C_5\|\mathbf{K}^{1/2}\nabla\phi^h\|^2_{L^2(\Omega_p)} \\ &\leq \varepsilon|(\mathbf{f_1}, \mathbf{u}_f^h)_{\Omega_f}| + \rho_f g|(f_2, \phi^h)_{\Omega_p}| + |b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{u}_f^h)| \\ &\quad + |b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{u}_f^h)| + |b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{u}_f^h)|. \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} &\frac{2\nu\varepsilon}{C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2} + \|\mathbf{K}^{1/2}\nabla\phi^h\|^2_{L^2(\Omega_p)} \\ &\leq \frac{\varepsilon}{C_5}|(\mathbf{f_1}, \mathbf{u}_f^h)_{\Omega_f}| + \frac{\rho_f g}{C_5}|(f_2, \phi^h)_{\Omega_p}| \\ &\quad + \frac{1}{C_5}|b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{u}_f^h)| + \frac{1}{C_5}|b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{u}_f^h)| + \frac{1}{C_5}|b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{u}_f^h)| \\ &\leq \frac{\varepsilon C_2 C_4}{C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2}\|\mathbf{f_1}\|_{L^2(\Omega_f)^2} + \frac{\rho_f g C_3}{C_5}\|\nabla\phi^h\|_{L^2(\Omega_p)}\|f_2\|_{L^2(\Omega_p)} \\ &\quad + \frac{2C_0C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,h}^*)\|_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2} \\ &\quad + \frac{C_0C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}_{f,h}^*)\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2} \\ &\quad + \frac{C_0C_4^3}{C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2} \\ &\leq \frac{\nu\varepsilon}{3C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2} + \frac{3\varepsilon C_2^2C_4^2}{4\nu C_5}\|\mathbf{f}_1\|^2_{L^2(\Omega_f)^2} + \frac{\lambda_{\min}}{2}\|\nabla\phi^h\|^2_{L^2(\Omega_p)} \\ &\quad + \frac{\rho_f^2 g^2 C_3^2}{2\lambda_{\min}C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{3C_0^2C_4^6}{\nu\varepsilon C_5}\|\mathbf{D}(\mathbf{u}_{f,H})\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_{f,h}^*)\|^2_{L^2(\Omega_f)^2} \\ &\quad + \frac{\nu\varepsilon}{3C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2} + \frac{3C_0^2C_4^6}{4\nu\varepsilon C_5}\|\mathbf{D}(\mathbf{u}_{f,h}^*)\|^4_{L^2(\Omega_f)^2} + \frac{\nu\varepsilon}{3C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2} \\ &\quad + \frac{C_0RC_4^3}{\sqrt{2\nu}C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2} \\ &\leq \frac{\nu\varepsilon}{C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2} + \frac{3\varepsilon C_2^2C_4^2}{4\nu C_5}\|\mathbf{f}_1\|^2_{L^2(\Omega_f)^2} + \frac{1}{2}\|\mathbf{K}^{1/2}\nabla\phi^h\|^2_{L^2(\Omega_p)} \\ &\quad + \frac{\rho_f^2 g^2 C_3^2}{2\lambda_{\min}C_5^2}\|f_2\|^2_{L^2(\Omega_p)} + \frac{3C_0^2R^2R_1^2C_4^6}{4\varepsilon\sigma\nu^3C_5} + \frac{3C_0^2R_1^4C_4^6}{16\varepsilon\sigma^2\nu^3C_5} \\ &\quad + \frac{C_0RC_4^3}{\sqrt{2\nu}C_5}\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2}. \end{aligned} \tag{30}$$

By simplifying, we can get

$$
\begin{aligned}
&\frac{\varepsilon}{C_5}(1-\frac{C_0RC_4^3}{\sqrt{2}\nu^{3/2}\varepsilon})2\nu\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2}+\|\mathbf{K}^{1/2}\nabla\phi^h\|^2_{L^2(\Omega_p)}\\
&\leq \frac{3\varepsilon C_2^2C_4^2}{2\nu C_5}\|\mathbf{f}_1\|_{L^2(\Omega_f)^2}+\frac{\rho_f^2g^2C_3^2}{\lambda_{\min}C_5^2}\|f_2\|^2_{L^2(\Omega_p)}+\frac{3C_0^2R^2R_1^2C_4^6}{2\varepsilon\sigma\nu^3C_5}\\
&\quad+\frac{3C_0^2R_1^4C_4^6}{8\varepsilon\sigma^2\nu^3C_5}.
\end{aligned}\tag{31}
$$

Set $\sigma=\frac{\varepsilon}{C_5}\delta=\frac{\varepsilon}{C_5}\left(1-\frac{C_0RC_4^3}{\sqrt{2}\nu^{3/2}\varepsilon}\right)$ and use $0<\delta<1$ to yield

$$
\begin{aligned}
&2\nu\|\mathbf{D}(\mathbf{u}_f^h)\|^2_{L^2(\Omega_f)^2}+\sigma^{-1}\|\mathbf{K}^{1/2}\nabla\phi^h\|^2_{L^2(\Omega_p)}\\
&\leq\sigma^{-1}\left(\frac{3\varepsilon C_2^2C_4^2}{2\nu C_5}\|\mathbf{f_1}\|_{L^2(\Omega_f)^2}+\frac{\rho_f^2g^2C_3^2}{\lambda_{\min}C_5^2}\|f_2\|^2_{L^2(\Omega_p)}+\frac{3C_0^2R^2R_1^2C_4^6}{2\varepsilon\sigma\nu^3C_5}+\frac{3C_0^2R_1^4C_4^6}{8\varepsilon\sigma^2\nu^3C_5}\right).
\end{aligned}
$$

□

Theorem 3.2. *Let* $(\mathbf{u}_f,\phi,p_f)\in H^2(\Omega_f)^2\times H^2(\Omega_p)\times H^1(\Omega_f)$ *be the solution of* (7)*. Then, under the assumption of Theorem 3.1, we have the following estimate*:

$$
\begin{aligned}
&\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2}+\|\nabla(\phi-\phi^h)\|_{L^2(\Omega_p)}+\|p_f-p_f^h\|_{L^2(\Omega_f)}\\
&\leq C(h+H^{4-\epsilon})(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}),
\end{aligned}\tag{32}
$$

where ϵ *is arbitrarily small.*

Proof. Firstly, we consider the error of Step II of Algorithm 3.1. Subtracting (18) from (7), we obtain

$$
\begin{cases}
a_{\Omega_f}(\mathbf{u}_f-\mathbf{u}^*_{f,h},\mathbf{v})+a_{\Omega_p}(\phi-\phi_h^*,\psi)+a_\Gamma(u-u_h^*,v)+d(v,p_f-p^*_{f,h})\\
\quad+b(\mathbf{u}_f;\mathbf{u}_f,\mathbf{v})+b(\mathbf{u}_{f,H};\mathbf{u}_{f,H},\mathbf{v})-b(\mathbf{u}_{f,H};\mathbf{u}^*_{f,h},\mathbf{v})-b(\mathbf{u}^*_{f,h};\mathbf{u}_{f,H},\mathbf{v})=0,\\
d(u-u_h^*,q)=0.
\end{cases}\tag{33}
$$

For the trilinear terms, it is easy to verify that

$$
\begin{aligned}
&b(\mathbf{u}_f;\mathbf{u}_f,\mathbf{v})+b(\mathbf{u}_{f,H};\mathbf{u}_{f,H},\mathbf{v})-b(\mathbf{u}_{f,H};\mathbf{u}^*_{f,h},\mathbf{v})-b(\mathbf{u}^*_{f,h};\mathbf{u}_{f,H},\mathbf{v})\\
&=b(\mathbf{u}_{f,H};\mathbf{u}_f-\mathbf{u}^*_{f,h},\mathbf{v})+b(\mathbf{u}_f-\mathbf{u}^*_{f,h};\mathbf{u}_{f,H},\mathbf{v})\\
&\quad-b(\mathbf{u}_f-\mathbf{u}_{f,H};\mathbf{u}_{f,H}-\mathbf{u}_f,\mathbf{v}).
\end{aligned}\tag{34}
$$

Let us denote

$$
\begin{aligned}
\varsigma_1-\chi_1&=(\mathbf{u}_f-\Pi_h\mathbf{u}_f)-(\mathbf{u}^*_{f,h}-\Pi_h\mathbf{u}_f),\\
\eta_1-\xi_1&=(\phi-\Pi_h\phi)-(\phi_h^*-\Pi_h\phi),\\
\eta_2-\xi_2&=(p_f-\Pi_hp_f)-(p^*_{f,h}-\Pi_hp_f),
\end{aligned}
$$

where Π_hs denotes interpolation of s in its finite element space, and $s=\mathbf{u}_f$, ϕ and p_f.

Then we choose $\mathbf{v} = \chi_1 \in H_{f,h0}$, $\psi = \xi_1 \in H_{p,h}$ and $q = \xi_2 \in Q_h$ in (33) and combine (34) to yield

$$\begin{aligned} & a_{\Omega_f}(\chi_1, \chi_1) + a_{\Omega_p}(\xi_1, \xi_1) + b(\chi_1; \mathbf{u}_{f,H}, \chi_1) \\ &= a_{\Omega_f}(\varsigma_1, \chi_1) + a_{\Omega_p}(\eta_1, \xi_1) + a_\Gamma(u - u_h^*, u_h^* - \Pi_h u) \\ &\quad + d(\chi_1, p_f - \Pi_h p_f) + b(\mathbf{u}_{f,H}; \varsigma_1, \chi_1) \\ &\quad + b(\varsigma_1; \mathbf{u}_{f,H}, \chi_1) - b(\mathbf{u}_f - \mathbf{u}_{f,H}; \mathbf{u}_{f,H} - \mathbf{u}_f, \chi_1) =: \sum_{i=1}^{7} R_i. \end{aligned} \tag{35}$$

Using (9), the left-hand side of (35) is bounded from below by

$$l.h.s \geq \nu\varepsilon \left(2 - \frac{C_0 R C_4^3}{\sqrt{2}\nu^{3/2}\varepsilon}\right) \|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} + \rho_f g \|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)}. \tag{36}$$

Next, applying Poincaré, Young inequalities, Theorem 2.2 and Theorem 2.3, we estimate the terms of the right-hand of (35) as follows:

$$\begin{aligned} |R_1| &\leq 2\nu\varepsilon \|\mathbf{D}(\varsigma_1)\|_{L^2(\Omega_f)^2} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_f)^2} + C\|\varsigma_1\|_{L^2(\Gamma)^2} \|\chi_1\|_{L^2(\Gamma)^2} \\ &\leq 6\nu\varepsilon \|\mathbf{D}(\varsigma_1)\|^2_{L^2(\Omega_f)^2} + \frac{1}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\quad + CC_4^2C_7^2 \|\mathbf{D}(\varsigma_1)\|_{L^2(\Omega_f)^2} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_f)^2} \\ &\leq 6\nu\varepsilon \|\mathbf{D}(\varsigma_1)\|^2_{L^2(\Omega_f)^2} + \frac{3C^2C_4^4C_7^4}{2\nu\varepsilon}\|\mathbf{D}(\varsigma_1)\|^2_{L^2(\Omega_f)^2} + \frac{\nu\varepsilon}{3}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{3}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2}, \\ |R_2| &\leq \rho_f g \|\mathbf{K}^{1/2}\nabla\eta_1\|_{L^2(\Omega_p)} \|\mathbf{K}^{1/2}\nabla\xi_1\|_{L^2(\Omega_p)} \\ &\leq \frac{\rho_f g}{2}\|\mathbf{K}^{1/2}\nabla\eta_1\|^2_{L^2(\Omega_p)} + \frac{\rho_f g}{2}\|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)} \\ &\leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\rho_f g}{2}\|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)}, \end{aligned}$$

$$\begin{aligned} |R_5 + R_6| &\leq 2C_0C_4^3 \|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2} \|\mathbf{D}(\varsigma_1)\|_{L^2(\Omega_f)^2} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_f)^2} \\ &\leq \frac{6C_0^2C_4^6}{\nu\varepsilon}\|\mathbf{D}(\mathbf{u}_{f,H})\|^2_{L^2(\Omega_f)^2} \|\mathbf{D}(\varsigma_1)\|^2_{L^2(\Omega_f)^2} + \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2}, \\ |R_7| &\leq C_1 \|\mathbf{u}_f - \mathbf{u}_{f,H}\|^{1-\epsilon}_{L^2(\Omega_f)^2} \|\nabla(\mathbf{u}_{f,H} - \mathbf{u}_f)\|^{1+\epsilon}_{L^2(\Omega_f)^2} \|\nabla\chi_1\|_{L^2(\Omega_f)^2} \\ &\leq C_1C_4^2 \|\mathbf{u}_f - \mathbf{u}_{f,H}\|^{1-\epsilon}_{L^2(\Omega_f)^2} \|\mathbf{D}(\mathbf{u}_{f,H} - \mathbf{u}_f)\|^{1+\epsilon}_{L^2(\Omega_f)^2} \|\mathbf{D}\chi_1\|_{L^2(\Omega_f)^2} \\ &\leq \frac{3C_1^2C_4^4}{2\nu\varepsilon}(\|\mathbf{u}_f - \mathbf{u}_{f,H}\|^{1-\epsilon}_{L^2(\Omega_f)^2})^2 (\|\mathbf{D}(\mathbf{u}_{f,H} - \mathbf{u}_f)\|^{1+\epsilon}_{L^2(\Omega_f)^2})^2 \\ &\quad + \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\leq C[H^{3-\epsilon}(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^{3-\epsilon}]^2 \end{aligned}$$

$$+ \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2}.$$

Similarly, we have

$$\begin{aligned} |R_4| \le |d(\chi_1,\eta_2)| &\le \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} + \frac{3C_4^2}{2\nu\varepsilon}\|\eta_2\|^2_{L^2(\Omega_f)} \\ &\le Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ &\quad + \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2}. \end{aligned} \tag{37}$$

Besides, for R_3, we obtain

$$\begin{aligned} |R_3| &= \left|\varepsilon\rho_f g\int_\Gamma [\eta_1\chi_1\cdot\mathbf{n}_f - \xi_1\varsigma_1\cdot\mathbf{n}_f]\right| \\ &\le \varepsilon\rho_f g\|\eta_1\|_{L^2(\Gamma)}\|\chi_1\cdot\mathbf{n}_f\|_{L^2(\Gamma)^2} + \varepsilon\rho_f g\|\xi_1\|_{L^2(\Gamma)}\|\varsigma_1\cdot\mathbf{n}_f\|_{L^2(\Gamma)^2} \\ &\le \varepsilon\rho_f g C_4C_7C_8(\|\nabla\eta_1\|_{L^2(\Omega_p)}\|\mathbf{D}(\chi_1)\|_{L^2(\Omega_f)^2} + \|\mathbf{D}\varsigma_1\|_{L^2(\Omega_f)^2}\|\nabla\xi_1\|_{L^2(\Omega_p)}) \\ &\le \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} + \frac{3\varepsilon\rho_f^2g^2C_4^2C_7^2C_8^2}{2\nu}\|\nabla\eta_1\|^2_{L^2(\Omega_p)} + \frac{\lambda_{\min}\rho_f g}{4}\|\nabla\xi_1\|^2_{L^2(\Omega_p)} \\ &\quad + \frac{\varepsilon^2\rho_f g C_4^2C_7^2C_8^2}{\lambda_{\min}}\|\mathbf{D}\varsigma_1\|^2_{L^2(\Omega_f)^2} \\ &\le Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{6}\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\quad + \frac{\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)}. \end{aligned}$$

Further, by using the results of the above estimates, the right-hand side of (35) is bounded by

$$\begin{aligned} r.h.s &\le Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \nu\varepsilon\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} \\ &\quad + C[H^{3-\epsilon}(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^{3-\epsilon}]^2 \\ &\quad + \frac{3\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)}. \end{aligned} \tag{38}$$

Then, combining the above two inequalities (36) and (38), we get

$$\begin{aligned} &\nu\varepsilon\delta\|\mathbf{D}(\chi_1)\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\xi_1\|^2_{L^2(\Omega_p)} \\ &\le Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ &\quad + C[H^{3-\epsilon}(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^{3-\epsilon}]^2, \end{aligned} \tag{39}$$

where $\delta = 1 - \frac{C_0RC_4^3}{\sqrt{2}\nu^{3/2}\varepsilon}$. In addition, we have

$$\begin{aligned} &\|\mathbf{D}(\mathbf{u}_f - \mathbf{u}^*_{f,h})\|_{L^2(\Omega_f)^2} + \|\nabla(\phi - \phi^*_h)\|_{L^2(\Omega_p)} \\ &\le C(h + H^{3-\epsilon})(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}). \end{aligned} \tag{40}$$

Secondly, we give the error of Step III of Algorithm 3.1. Subtracting (19) from (7), we have

$$\begin{cases} a_{\Omega_f}(\mathbf{u}_f - \mathbf{u}_f^h, \mathbf{v}) + a_{\Omega_p}(\phi - \phi^h, \psi) + a_\Gamma(u - u^h, v) + d(v, p_f - p_f^h) \\ + b(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) - b(\mathbf{u}_{f,H}; \mathbf{u}_f^h, \mathbf{v}) - b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{v}) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{v}) \\ + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{v}) = 0, \\ d(u - u^h, q) = 0. \end{cases} \tag{41}$$

Similarly,

$$\begin{aligned} & b(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) + b(\mathbf{u}_{f,H}; \mathbf{u}_{f,h}^*, \mathbf{v}) + b(\mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_{f,h}^*, \mathbf{v}) \\ & - b(\mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{v}) - b(\mathbf{u}_{f,H}; \mathbf{u}_f^h, \mathbf{v}) \\ = & \; b(\mathbf{u}_{f,H}; \mathbf{u}_f - \mathbf{u}_f^h, \mathbf{v}) + b(\mathbf{u}_f - \mathbf{u}_f^h; \mathbf{u}_{f,H}, \mathbf{v}) \\ & - b(\mathbf{u}_f - \mathbf{u}_{f,H}; \mathbf{u}_{f,h}^* - \mathbf{u}_f, \mathbf{v}) - b(\mathbf{u}_f - \mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_f, \mathbf{v}) \\ & - b(\mathbf{u}_{f,h}^* - \mathbf{u}_f; \mathbf{u}_{f,h}^* - \mathbf{u}_f, \mathbf{v}). \end{aligned} \tag{42}$$

Moreover, we need some analogous definitions for the later derivation:

$$\begin{aligned} \overline{\varsigma}_1 - \overline{\chi}_1 &= (\mathbf{u}_f - \Pi_h \mathbf{u}_f) - (\mathbf{u}_f^h - \Pi_h \mathbf{u}_f), \\ \overline{\eta}_1 - \overline{\xi}_1 &= (\phi - \Pi_h \phi) - (\phi^h - \Pi_h \phi), \\ \overline{\eta}_2 - \overline{\xi}_2 &= (p_f - \Pi_h p_f) - (p_f^h - \Pi_h p_f). \end{aligned}$$

Taking $\mathbf{v} = \overline{\chi}_1 \in H_{f,h0}$, $\psi = \overline{\xi}_1 \in H_{p,h}$ and $q = \overline{\xi}_2 \in Q_h$ in (41) and combining (42) yield

(43)

$$\begin{aligned} & a_{\Omega_f}(\overline{\chi}_1, \overline{\chi}_1) + a_{\Omega_p}(\overline{\xi}_1, \overline{\xi}_1) + b(\overline{\chi}_1; \mathbf{u}_{f,H}, \overline{\chi}_1) \\ = & \; a_{\Omega_f}(\overline{\varsigma}_1, \overline{\chi}_1) + a_{\Omega_p}(\overline{\eta}_1, \overline{\xi}_1) + a_\Gamma(u - u^h, u^h - \Pi_h u) + d(\overline{\chi}_1, p_f - \Pi_h p_f) \\ & + b(\overline{\varsigma}_1; \mathbf{u}_{f,H}, \overline{\chi}_1) + b(\mathbf{u}_{f,H}; \overline{\varsigma}_1, \overline{\chi}_1) - b(\mathbf{u}_f - \mathbf{u}_{f,H}; \mathbf{u}_{f,h}^* - \mathbf{u}_f, \overline{\chi}_1) \\ & - b(\mathbf{u}_f - \mathbf{u}_{f,h}^*; \mathbf{u}_{f,H} - \mathbf{u}_f, \overline{\chi}_1) - b(\mathbf{u}_{f,h}^* - \mathbf{u}_f; \mathbf{u}_{f,h}^* - \mathbf{u}_f, \overline{\chi}_1) := \sum_{i=1}^{9} \overline{R}_i. \end{aligned}$$

Do the same as (36), the left-hand side of (43) can be bounded by

$$l.h.s \geq \nu\varepsilon \left(2 - \frac{C_0 R C_4^3}{\sqrt{2}\nu^{3/2}\varepsilon} \right) \|\mathbf{D}(\overline{\chi}_1)\|_{L^2(\Omega_f)^2}^2 + \rho_f g \|K^{1/2} \nabla \overline{\xi}_1\|_{L^2(\Omega_p)}^2. \tag{44}$$

Since $\overline{R}_i$ are similar as R_i, $i = 1, 2, 3, 4, 5, 6$, it is easy to get

$$\begin{aligned} |\overline{R}_1| \leq & \; Ch^2 (\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ & + \frac{\nu\varepsilon}{4} \|\mathbf{D}(\overline{\chi}_1)\|_{L^2(\Omega_f)^2}^2, \\ |\overline{R}_2| \leq & \; Ch^2 (\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ & + \frac{\rho_f g}{2} \|\mathbf{K}^{1/2} \nabla \overline{\xi}_1\|_{L^2(\Omega_p)}^2, \end{aligned}$$

$$|\overline{R}_3| \leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{8}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2} + \frac{\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\overline{\xi}_1\|^2_{L^2(\Omega_p)},$$

$$|\overline{R}_4| \leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{8}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2},$$

$$|\overline{R}_5 + \overline{R}_6| \leq Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 + \frac{\nu\varepsilon}{8}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2}.$$

Now, we bound the last three R_i, $i = 7, 8, 9$. By using (9) and Theorem 2.2, we can attain

$$\begin{aligned}|\overline{R}_7 + \overline{R}_8| &\leq 2C_0C_4^3\|\mathbf{D}(\mathbf{u}_f - \mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2}\|(\mathbf{D}(\mathbf{u}^*_{f,h} - \mathbf{u}_f)\|_{L^2(\Omega_f)^2}\|\mathbf{D}\overline{\chi}_1\|_{L^2(\Omega_f)^2} \\ &\leq \frac{4C_0^2C_4^6}{\nu\varepsilon}\|\mathbf{D}(\mathbf{u}_f - \mathbf{u}_{f,H})\|^2_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}^*_{f,h} - \mathbf{u}_f)\|^2_{L^2(\Omega_f)^2} \\ &\quad + \frac{\nu\varepsilon}{4}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2} \\ &\leq CH^2(h^2 + H^{6-\epsilon})(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^4 \\ &\quad + \frac{\nu\varepsilon}{4}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2}.\end{aligned} \tag{45}$$

For $\overline{R}_9$, we use the second inequality in (8) to get

$$\begin{aligned}|\overline{R}_9| &\leq C_1\|\mathbf{u}^*_{f,h} - \mathbf{u}_f\|^{1-\epsilon}_{L^2(\Omega_f)^2}\|\nabla(\mathbf{u}^*_{f,h} - \mathbf{u}_f)\|^{1+\epsilon}_{L^2(\Omega_f)^2}\|\nabla\overline{\chi}_1\|_{L^2(\Omega_f)^2} \\ &\leq C_1C_4^2\|\mathbf{u}^*_{f,h} - \mathbf{u}_f\|^{1-\epsilon}_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}^*_{f,h} - \mathbf{u}_f)\|^{1+\epsilon}_{L^2(\Omega_f)^2}\|\mathbf{D}\overline{\chi}_1\|_{L^2(\Omega_f)^2} \\ &\leq \frac{2C_1^2C_4^4}{\nu\varepsilon}(\|\mathbf{u}^*_{f,h} - \mathbf{u}_f\|^{1-\epsilon}_{L^2(\Omega_f)^2}\|\mathbf{D}\left(\mathbf{u}^*_{f,h} - \mathbf{u}_f\right)\|^{1+\epsilon}_{L^2(\Omega_f)^2})^2 \\ &\quad + \frac{\nu\varepsilon}{8}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2} \\ &\leq C(h + H^{3-\epsilon})^4(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^4 \\ &\quad + \frac{\nu\varepsilon}{8}\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2}.\end{aligned}$$

By using the above estimates, the right-hand side of (43) is bounded by

$$\begin{aligned}r.h.s &\leq \nu\varepsilon\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2} + \frac{3\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\overline{\xi}_1\|^2_{L^2(\Omega_p)} \\ &\quad + Ch^2(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ &\quad + CH^2[h^2 + (H^{3-\epsilon})^2](\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2 \\ &\leq \nu\varepsilon\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2} + \frac{3\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\overline{\xi}_1\|^2_{L^2(\Omega_p)} \\ &\quad + C[h^2 + (H^{4-\epsilon})^2](\|\mathbf{u}_f\|_{H^2(\Omega_f)^2} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)})^2.\end{aligned} \tag{46}$$

Combining the above two inequalities (44) and (46), we can get

$$\begin{aligned}(47)\qquad &\nu\varepsilon\delta\|\mathbf{D}(\overline{\chi}_1)\|^2_{L^2(\Omega_f)^2}+\frac{\rho_f g}{4}\|\mathbf{K}^{1/2}\nabla\overline{\xi}_1\|^2_{L^2(\Omega_p)}\\ &\le C[h^2+(H^{4-\epsilon})^2](\|\mathbf{u}_f\|_{H^2(\Omega_f)^2}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)})^2.\end{aligned}$$

Then we can obtain

$$\begin{aligned}(48)\qquad &\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2}+\|\nabla(\phi-\phi^h)\|_{L^2(\Omega_p)}\\ &\le C(h+H^{4-\epsilon})(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}).\end{aligned}$$

Furthermore, thanks to the discrete inf-sup condition (11) one finds

$$\begin{aligned}(49)\qquad &\beta\|p_f-p_f^h\|_{L^2(\Omega_f)}\\ &\le\frac{d(v,p_f-p_f^h)}{\|v\|_W}\\ &\le 2\nu\varepsilon\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2}+\varepsilon\frac{\alpha}{\sqrt{\boldsymbol{\tau}\cdot\nu\mathbf{K}\cdot\boldsymbol{\tau}}}\|(\mathbf{u}_f-\mathbf{u}_f^h)\cdot\boldsymbol{\tau}\|_{L^2(\Gamma)^2}\\ &\quad+\varepsilon\rho_f gC_4C_7C_8(\|\nabla(\phi-\phi^h)\|_{L^2(\Omega_p)}+\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2})\\ &\quad+\rho_f g\|K^{1/2}\nabla(\phi-\phi^h)\|_{L^2(\Omega_p)}+C_0C_4^3\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}^*_{f,h})\|^2_{L^2(\Omega_f)^2}\\ &\quad+2C_0C_4^3\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}^*_{f,h})\|_{L^2(\Omega_f)^2}\\ &\quad+2C_0C_4^3\|\mathbf{D}(\mathbf{u}_{f,H})\|_{L^2(\Omega_f)^2}\|\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\|_{L^2(\Omega_f)^2}\\ &\le C(h+H^{4-\epsilon})(\|\mathbf{u}_f\|_{H^2(\Omega_f)^2}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}).\end{aligned}$$

Finally, gathering the (48) and (49) leads to (32). □

4. Numerical experiment

In this section, numerical experiment is to verify the numerical theory of the two-level method for the Navier–Stokes/Darcy model developed in the previous section.

We present some numerical results of the Navier–Stokes/Darcy problem under a known analytical solution with the computational domain $\Omega_p=[0,1]\times[0,1]$ and $\Omega_f=[0,1]\times[1,2]$ with the interface $\Gamma=(0,1)\times\{1\}$. In this paper, the exact solution is given by

$$\mathbf{u}_f=\left(x^2(y-1)^2+y,-\frac{2}{3}x(y-1)^3+2-\pi\text{sin}(\pi x)\right),$$

$$p_f=[2-\pi\text{sin}(\pi x)]\text{sin}\left(\frac{\pi}{2}y\right),\ \phi=[2-\pi\text{sin}(\pi x)][1-y-\cos(\pi y)].$$

The force terms $\mathbf{f_1}$ and f_2 are determined by (1) and (4), respectively. For simplicity, the model parameters ρ_f, g, ε, α equal to 1 and $\mathbf{K}=\mathbf{I}$.

For a given h, we consider that the H satisfy $h=O(H^4)$ for Algorithm 3.1 and $h=O(H^2)$ for the common two-level finite element method. We list the numerical results of Algorithm 3.1, the one-level method and common two-level finite element method in Tables 1-3. From these tables, we can see that three methods work well and keep the convergence rates just like the theoretical analysis. Besides, Algorithm 3.1 is competitive with the common two-level method in accuracy, especially for the pressure. However, as expected, the coarse mesh of Algorithm 3.1 can be chosen as a coarser one.

In addition, we compare the computing time of Algorithm 3.1 with the one-level method and the common two-level method in Tables 4-5. As expected, our algorithm spends less computing time than the other two methods under nearly the same accuracy. In conclusion, Algorithm 3.1 is more efficient than the other two methods.

Table 1. The one-level method for the steady Navier–Stokes/Darcy model.

h	$\frac{\Vert\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\Vert_0}{\Vert\mathbf{D}(\mathbf{u}_f)\Vert_0}$	*Rate*	$\frac{\Vert p_f-p_f^h\Vert_0}{\Vert p_f\Vert_0}$	*Rate*	$\frac{\Vert\nabla(\phi-\phi^h)\Vert_0}{\Vert\nabla\phi\Vert_0}$	*Rate*
$\frac{1}{9}$	9.503E-2	—	4.431E-1	—	1.398E-1	—
$\frac{1}{16}$	5.223E-2	1.040	2.380E-1	1.080	8.125E-2	0.944
$\frac{1}{36}$	2.291E-2	1.016	8.645E-2	1.249	3.479E-2	1.046
$\frac{1}{64}$	1.259E-2	1.040	5.009E-2	0.948	1.930E-3	1.024

Table 2. The common two-level finite element method for the steady Navier–Stokes/Darcy model.

H	h	$\frac{\Vert\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\Vert_0}{\Vert\mathbf{D}(\mathbf{u}_f)\Vert_0}$	*Rate*	$\frac{\Vert p_f-p_f^h\Vert_0}{\Vert p_f\Vert_0}$	*Rate*	$\frac{\Vert\nabla(\phi-\phi^h)\Vert_0}{\Vert\nabla\phi\Vert_0}$	*Rate*
$\frac{1}{3}$	$\frac{1}{9}$	9.505E-2	—	4.588E-1	—	1.418E-1	—
$\frac{1}{4}$	$\frac{1}{16}$	5.223E-2	1.041	2.452E-1	1.089	8.245E-2	0.943
$\frac{1}{6}$	$\frac{1}{36}$	2.288E-2	1.017	8.685E-2	1.280	3.554E-2	1.038
$\frac{1}{8}$	$\frac{1}{64}$	1.258E-2	1.040	5.729E-2	0.723	1.968E-2	1.028

Table 3. Algorithm 3.1 for the steady Navier–Stokes/Darcy model.

H	h	$\frac{\Vert\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\Vert_0}{\Vert\mathbf{D}(\mathbf{u}_f)\Vert_0}$	*Rate*	$\frac{\Vert p_f-p_f^h\Vert_0}{\Vert p_f\Vert_0}$	*Rate*	$\frac{\Vert\nabla(\phi-\phi^h)\Vert_0}{\Vert\nabla\phi\Vert_0}$	*Rate*
$\frac{1}{2}$	$\frac{1}{9}$	1.072E-1	—	5.061E-1	—	1.562E-1	—
$\frac{1}{2}$	$\frac{1}{16}$	5.784E-2	1.073	2.040E-1	1.580	8.360E-2	1.086
$\frac{1}{3}$	$\frac{1}{36}$	2.500E-2	1.035	7.941E-2	1.163	3.495E-2	1.075
$\frac{1}{3}$	$\frac{1}{64}$	1.310E-2	1.123	4.502E-2	1.090	1.792E-2	1.111

Table 4. Comparisons of the one-level method and Algorithm 3.1.

Methods	H	h	CPU-time	$\Vert\mathbf{D}(\mathbf{u}_f-\mathbf{u}_f^h)\Vert_0$	$\Vert p_f-p_f^h\Vert_0$	$\Vert\nabla(\phi-\phi^h)\Vert_0$
One-level	–	$\frac{1}{16}$	8.320	3.738E-1	1.627E-1	3.868E-1
Algorithm 3.1	$\frac{1}{2}$	$\frac{1}{16}$	6.741	3.762E-1	1.760E-1	3.868E-1
One-level	–	$\frac{1}{81}$	2903.15	7.261E-2	2.872E-2	7.377E-2
Algorithm 3.1	$\frac{1}{3}$	$\frac{1}{81}$	1295.54	8.082E-2	7.115E-2	7.379E-2

TABLE 5. Comparisons of the common two-level method and Algorithm 3.1.

Methods	H	h	CPU-time	$\|\mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h)\|_0$	$\|p_f - p_f^h\|_0$	$\|\nabla(\phi - \phi^h)\|_0$
Common two-level	$\frac{1}{5}$	$\frac{1}{25}$	9.020	2.309E-1	8.198E-2	2.413E-1
Algorithm 3.1	$\frac{1}{2}$	$\frac{1}{16}$	6.741	3.762E-1	1.760E-1	3.868E-1
Common two-level	$\frac{1}{11}$	$\frac{1}{121}$	3701.55	5.198E-2	4.927E-2	5.023E-2
Algorithm 3.1	$\frac{1}{3}$	$\frac{1}{81}$	1295.54	8.082E-2	7.115E-2	7.379E-2

Acknowledgments. J. Fang, P. Huang and Y. Qin would like to thank the editor and reviewers for their valuable comments and suggestions which helped us to improve the results of this paper.

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Jilin Fang
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, P. R. China
Email address: fjllovely@163.com

Pengzhan Huang
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, P. R. China
Email address: hpzh007@yahoo.com

Yi Qin
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an 710049, P. R. China
Email address: qinyi1991@stu.xjtu.edu.cn