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PROPERTIES OF OPERATOR MATRICES

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Abstract. Let $\mathcal{S}$ be the collection of the operator matrices $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ where the range of C is closed. In this paper, we study the properties of operator matrices in the class $\mathcal{S}$. We first explore various local spectral relations, that is, the property (β), decomposable, and the property (C) between the operator matrices in the class $\mathcal{S}$ and their component operators. Moreover, we investigate Weyl and Browder type spectra of operator matrices in the class $\mathcal{S}$, and as some applications, we provide the conditions for such operator matrices to satisfy a-Weyl's theorem and a-Browder's theorem, respectively.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $\sigma_s(T)$ denote the spectrum, the point spectrum, the approximate point spectrum, and the surjective spectrum of T, respectively. For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

Many authors have studied invertibility, perturbations of spectra, etc. for upper triangular operator matrices. In particular, C. Benhida, E. H. Zerouali,

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and H. Zguitti ([3], (2005)) studied spectra of upper triangular operator matrices. In 2013, the authors ([17]) studied the local spectral properties of complex symmetric (upper triangular) operator matrices. The Weyl's theorem for upper triangular operator matrices has been studied by many authors (see [2], [9], [10], [14], [21], [20]).

The study of operator matrices has been developed from the following fact; if $\mathcal{H}$ is a complex Hilbert space and we decompose $\mathcal{H}$ as a direct sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, each bounded linear operator T can be expressed as the operator matrix form

$$T = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$$

with respect to the space of decomposition, where A, B, C, Z are operators from $\mathcal{H}_i$ into $\mathcal{H}_j$ for $i, j = 1, 2$. Recently, D. S. Cvetkvic-Ilic has studied the existence of some component Z of the operator matrix T and the problem of completion of T ([8]). Our goal is to find various connections between T and its components. As some applications of these results, we next consider the structure of T. First of all, we begin with the following notation.

Notation 1.1. *Throughout this paper, we denote the collection* $\mathcal{S}$ *as follows*:

$$\mathcal{S} = \left\{ \begin{pmatrix} A & C \\ Z & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K} \mid R(C) \text{ is closed} \right\}. \tag{1}$$

For example, if C is a semi-Fredholm operator or semi-regular, i.e., $N(C) \subset \cap_{n\in\mathbb{N}} C^n(\mathcal{H})$ and $R(C)$ is closed, then the operator matrices $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ are in the class $\mathcal{S}$. For another example, if for given $x \in \mathcal{H}$ there exist $c > 0$ and a $y \in \mathcal{H}$ such that (i) $Cx = Cy$ and (ii) $\|y\| \leq c\|Cx\|$, then $R(C)$ is closed from [12, Corollary 2]. Hence the operator matrices $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ are in the class $\mathcal{S}$.

Lemma 1.2 ([2]). *If* $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$, *then* M *has the following matrix representation*;

$$M = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & 0 & C_1 \\ Z & B_1 & B_2 \end{pmatrix} \tag{2}$$

which maps from $\mathcal{H} \oplus N(C) \oplus N(C)^\perp$ *to* $R(C)^\perp \oplus R(C) \oplus \mathcal{K}$ *where* $C_1 = C|_{N(C)^\perp}$, $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$, $A_2 = P_{R(C)} A|_{\mathcal{H}}$, B_1 *denotes a mapping* B *from* $N(C)$ *into* $\mathcal{K}$, B_2 *denotes a mapping* B *from* $N(C)^\perp$ *into* $\mathcal{K}$, $P_{R(C)^\perp}$ *denotes the projection of* $\mathcal{H}$ *onto* $R(C)^\perp$, *and* $P_{R(C)}$ *denotes the projection of* $\mathcal{H}$ *onto* $R(C)$.

In this paper, we study the class $\mathcal{S}$ the collection of the operator matrices $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ where $R(C)$ is closed. In Section 3, we explore several local spectral relations, i.e., the property (β), decomposable, and the property (C) between the 2×2, not necessarily upper triangular, operator matrices in the class $\mathcal{S}$ and their component operators. In particular, in Section 4, we study the Weyl spectrum and the Browder essential approximate point spectrum for operator matrices $M \in \mathcal{S}$. In Section 5, we give the conditions for such operator matrices to satisfy a-Weyl's theorem and a-Browder's theorem, respectively.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well known that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

Any of the converse implications does not hold, in general (see [19] for more details). Since decomposability or the property (β) provides a partial solution to the invariant subspace (see [11]), it is worth to research decomposability (or the property (β)). For example, M. Putinar [24] showed that every hyponormal operator (i.e., $T^*T \geq TT^*$) has the property (β) and such an operator with thick spectrum has a nontrivial invariant subspace, a result due to S. Brown (see [4]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$ind(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent, respectively. The left essential spectrum $\sigma_{SF+}(T)$, the right essential spectrum $\sigma_{SF-}(T)$, the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\begin{aligned}\sigma_{SF+}(T) &:= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not upper semi-Fredholm}\};\\ \sigma_{SF-}(T) &:= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not lower semi-Fredholm}\};\\ \sigma_e(T) &:= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\};\\ \sigma_w(T) &:= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\};\\ \sigma_b(T) &:= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}.\end{aligned}$$

Evidently, we get the next inclusions

$$\sigma_{SF+}(T)\cup\sigma_{SF-}(T)=\sigma_e(T)\subseteq\sigma_w(T)\subseteq\sigma_b(T)=\sigma_e(T)\cup \operatorname{acc}\sigma(T),$$

where we write acc $\sigma(T)$ for the set of all accumulation points of $\sigma(T)$.

Let iso $\sigma(T)$ be the set of all isolated points of $\sigma(T)$. We write $\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T-\lambda) < \infty \}$, and $p_{00}(T) := \sigma(T)\setminus\sigma_b(T)$. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T)\setminus\sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T)\setminus\sigma_w(T) = p_{00}(T)$. We recall the definitions of *Weyl essential approximate point spectrum* $\sigma_{ea}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T)$ given by

$$\begin{aligned}\sigma_{ea}(T) &:= \bigcap \{\sigma_a(T+K) : K \in \mathcal{K}(\mathcal{H})\},\\ \sigma_{ab}(T) &:= \bigcap \{\sigma_a(T+K) : TK=KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}.\end{aligned}$$

We say that *a-Weyl's theorem holds* for T if $\sigma_a(T)\setminus\sigma_{ea}(T)=\pi_{00}^a(T)$ and that *a-Browder's theorem holds* for T if $\sigma_a(T)\setminus\sigma_{ea}(T)=p_{00}^a(T)$, where $\pi_{00}^a(T) := \{\lambda \in \text{iso } \sigma_a(T) : 0<\alpha(T-\lambda)<\infty\}$ and $p_{00}^a(T) := \sigma_a(T)\setminus\sigma_{ab}(T)$. It is known that

$$\begin{aligned}&a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem},\\ &a\text{-Weyl's theorem} \Longrightarrow \text{Weyl's theorem} \Longrightarrow \text{Browder's theorem}.\end{aligned}$$

3. Local spectral properties

Let $M = \left(\begin{smallmatrix} A & C\\ Z & B\end{smallmatrix}\right)$ be an operator matrix in the class $\mathcal{S}$. Since $R(C)$ is closed, $C_1 = C|_{N(C)^\perp} : N(C)^\perp \to R(C)$ is invertible. Given $\lambda \in \mathbb{C}$, using the representation of Lemma 1.2, we write $M-\lambda$ as follows;

$$\begin{aligned} M-\lambda &= \begin{pmatrix} A_1-\lambda & 0 & 0\\ A_2-\lambda & 0 & C_1\\ Z & B_1-\lambda & B_2-\lambda\end{pmatrix}\\ &= \begin{pmatrix} 0 & I & 0\\ 0 & 0 & I\\ I & 0 & (B_2-\lambda)C_1^{-1}\end{pmatrix}\begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0\\ 0 & A_1-\lambda & 0\\ 0 & 0 & C_1\end{pmatrix}\begin{pmatrix} 0 & I & 0\\ I & 0 & 0\\ C_1^{-1}(A_2-\lambda) & 0 & I\end{pmatrix}, \end{aligned} \tag{3}$$

where $A_1-\lambda = P_{R(C)^\perp}(A-\lambda)|_{\mathcal{H}}$, $A_2-\lambda = P_{R(C)}(A-\lambda)|_{\mathcal{H}}$, $B_1-\lambda = (B-\lambda)|_{N(C)}$, $B_2-\lambda = (B-\lambda)|_{N(C)^\perp}$ and $\triangle_\lambda = Z-(B_2-\lambda)C_1^{-1}(A_2-\lambda)$

(see [2, Page 714] for more details). Note that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & B_2C_1^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}A_2 & 0 & I \end{pmatrix} \text{ are invertible.} \tag{4}$$

In this section, we study the local spectral properties of the operator matrices in the class $\mathcal{S}$.

In general, even though A has the property (β), A_1, its projection of A, may not have the property (β). For example, if the multiplication operator M_φ is normal on L^2 and so it has property (β). But, the Toeplitz operator $T_\varphi = P(M_\varphi)$ on H^2 may not have property (β). So we study the following theorem with respect to A_1 and B_1 which have the property (β).

Theorem 3.1. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$ and let $A_1 = P_{R(C)^\perp}A|_{\mathcal{H}}$ and $B_1 = B|_{N(C)}$. Then the following statements hold.*

(i) *If A_1 and B_1 have the property (β), then M has the property (β).*

(ii) *If* 0 *is not an eigenvalue of C^*, then M has the property (β) if and only if B_1 has the property (β).*

Proof. (i) Suppose that A_1 and B_1 have the property (β). Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H} \oplus N(C) \oplus N(C)^\perp$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(M-\lambda)\begin{pmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \\ f_{n,3}(\lambda) \end{pmatrix}\|_K = 0 \tag{5}$$

for every compact set K in D, where $\|f\|_K = \sup_{\lambda\in K}\|f(\lambda)\|$ for an $\mathcal{H} \oplus N(C) \oplus N(C)^\perp$-valued function $f(\lambda)$. Since $\left(\begin{smallmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2-\lambda)C_1^{-1} \end{smallmatrix}\right)$ is invertible, it follows from (5) that

$$\lim_{n\to\infty} \left\| \begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0 \\ 0 & A_1-\lambda & 0 \\ 0 & 0 & C_1 \end{pmatrix} \begin{pmatrix} g_{n,1}(\lambda) \\ g_{n,2}(\lambda) \\ g_{n,3}(\lambda) \end{pmatrix} \right\|_K = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\left(\begin{smallmatrix} g_{n,1}(\lambda) \\ g_{n,2}(\lambda) \\ g_{n,3}(\lambda) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \\ f_{n,3}(\lambda) \end{smallmatrix}\right)$. Therefore, we get that

$$\begin{cases} \lim_{n\to\infty} \|(B_1-\lambda)g_{n,1}(\lambda) + \triangle_\lambda g_{n,2}(\lambda)\|_K = 0, \\ \lim_{n\to\infty} \|(A_1-\lambda)g_{n,2}(\lambda)\|_K = 0, \\ \lim_{n\to\infty} \|C_1 g_{n,3}(\lambda)\|_K = 0. \end{cases} \tag{6}$$

Since C_1 is invertible, it follows from (6) that $\lim_{n\to\infty}\|g_{n,3}(\lambda)\|_K = 0$. Moreover, A_1 and B_1 have the property (β), hence $\lim_{n\to\infty}\|g_{n,2}(\lambda)\|_K = 0$ and so

$\lim_{n\to\infty}\|g_{n,1}(\lambda)\|_K = 0$. Therefore

$$0 = \lim_{n\to\infty}\left\|\begin{pmatrix} g_{n,1}(\lambda)\\ g_{n,2}(\lambda)\\ g_{n,3}(\lambda)\end{pmatrix}\right\|_K = \lim_{n\to\infty}\left\|\begin{pmatrix} 0 & I & 0\\ I & 0 & 0\\ C_1^{-1}(A_2-\lambda) & 0 & I\end{pmatrix}\begin{pmatrix} f_{n,1}(\lambda)\\ f_{n,2}(\lambda)\\ f_{n,3}(\lambda)\end{pmatrix}\right\|_K.$$

Since $\begin{pmatrix} 0 & I & 0\\ I & 0 & 0\\ C_1^{-1}(A_2-\lambda) & 0 & I\end{pmatrix}$ is invertible, it follows that

$$\lim_{n\to\infty}\left\|\begin{pmatrix} f_{n,1}(\lambda)\\ f_{n,2}(\lambda)\\ f_{n,3}(\lambda)\end{pmatrix}\right\|_K = \begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}.$$

Hence M has the property (β).

(ii) Assume that M has the property (β). Let D be an open set in $\mathbb{C}$ and let $h_n : D \to N(C)$ be a sequence of analytic functions such that

$$\lim_{n\to\infty}\|(B_1-\lambda)h_n(\lambda)\|_K = 0$$

for every compact set K in D, where $\|h\|_K$ denotes $\sup_{\lambda\in K}\|h(\lambda)\|$ for an $N(C)$-valued function $h(\lambda)$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty}\|(M-\lambda)(0\oplus h_n(\lambda)\oplus 0)\|_K\\
&= \lim_{n\to\infty}\left\|\begin{pmatrix} A_1-\lambda & 0 & 0\\ A_2-\lambda & 0 & C_1\\ Z & B_1-\lambda & B_2-\lambda\end{pmatrix}\begin{pmatrix} 0\\ h_n(\lambda)\\ 0\end{pmatrix}\right\|_K\\
&= \lim_{n\to\infty}\left\|\begin{pmatrix} 0\\ 0\\ (B_1-\lambda)h_n(\lambda)\end{pmatrix}\right\|_K = 0.
\end{aligned}$$

Since M has the property (β), it follows that $\lim_{n\to\infty}\|h_n(\lambda)\|_K = 0$. Hence B_1 has the property (β).

Conversely, assume that 0 is not an eigenvalue of C^* and B_1 has the property (β). Then $R(C) = \mathcal{H}$ and $A_1 = 0$. Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H}\oplus N(C)\oplus N(C)^\perp$ be a sequence of analytic functions such that

$$\lim_{n\to\infty}\left\|(M-\lambda)\begin{pmatrix} f_{n,1}(\lambda)\\ f_{n,2}(\lambda)\\ f_{n,3}(\lambda)\end{pmatrix}\right\|_K = 0 \tag{7}$$

for every compact set K in D, where $\|f\|_K = \sup_{\lambda\in K}\|f(\lambda)\|$ for an $\mathcal{H}\oplus N(C)\oplus N(C)^\perp$-valued function $f(\lambda)$. Since $\begin{pmatrix} 0 & I & 0\\ 0 & 0 & I\\ I & 0 & (B_2-\lambda)C_1^{-1}\end{pmatrix}$ is invertible, it follows from (7) that

$$\lim_{n\to\infty}\left\|\begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0\\ 0 & -\lambda & 0\\ 0 & 0 & C_1\end{pmatrix}\begin{pmatrix} g_{n,1}(\lambda)\\ g_{n,2}(\lambda)\\ g_{n,3}(\lambda)\end{pmatrix}\right\|_K = \begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}, \tag{8}$$

where $\begin{pmatrix} g_{n,1}(\lambda) \\ g_{n,2}(\lambda) \\ g_{n,3}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix} \begin{pmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \\ f_{n,3}(\lambda) \end{pmatrix}$. Then from (8) we have

$$(9) \qquad \begin{cases} \lim_{n\to\infty} \|(B_1-\lambda)g_{n,1}(\lambda) + \triangle_\lambda g_{n,2}(\lambda)\|_K = 0, \\ \lim_{n\to\infty} \|-\lambda g_{n,2}(\lambda)\|_K = 0, \text{and} \\ \lim_{n\to\infty} \|C_1 g_{n,3}(\lambda)\|_K = 0. \end{cases}$$

Moreover, since C_1 is invertible, it follows that

$$\lim_{n\to\infty} \|g_{n,3}(\lambda)\|_K = \lim_{n\to\infty} \|g_{n,2}(\lambda)\|_K = 0.$$

Hence from (9), $\lim_{n\to\infty} \|(B_1-\lambda)g_{n,1}(\lambda)\|_K = 0$. Since B_1 has the property (β), it follows that $\lim_{n\to\infty} \|g_{n,1}(\lambda)\|_K = 0$. Since $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix}$ is invertible, we have

$$\lim_{n\to\infty} \|f_{n,1}(\lambda)\|_K = \lim_{n\to\infty} \|f_{n,2}(\lambda)\|_K = \lim_{n\to\infty} \|f_{n,3}(\lambda)\|_K = 0.$$

Hence M has the property (β). □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, *paranormal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$, and *totally paranormal* if $T - \lambda I$ is paranormal for every $\lambda \in \mathbb{C}$.

Corollary 3.2. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$ and let $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$ and $B_1 = B|_{N(C)}$. Then the following statements hold.*

(i) *Suppose that A_1 and B_1 have the property (β). If $\sigma(M)$ has nonempty interior in $\mathbb{C}$, then M has a nontrivial invariant subspace.*

(ii) *Suppose A_1 and B_1 have the single-valued extension property. Then M has the single-valued extension property. Moreover, if 0 is not an eigenvalue of C^*, then M has the single-valued extension property if and only if B_1 has the single-valued extension property.*

Proof. (i) Since A_1 and B_1 have the property (β), it follows from Theorem 3.1 that M has the property (β). Hence M has a nontrivial invariant subspace from [11, Theorem 2.1].

(ii) The proof follows from a similar way of the proof of Theorem 3.1. □

Example 3.3. Let $A, B,$ and C be defined on $\ell^2(\mathbb{N})$ by

$$Ax := (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots),$$
$$Bx := (\beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \ldots),$$
$$Cx := (x_1, 0, x_2, 0, x_3, 0, \cdots),$$

where $x = (x_n) \in \ell^2(\mathbb{N})$ and $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, 3, \ldots$. Since C is bounded below, it follows that $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$ for arbitrary $Z \in \mathcal{L}(\ell^2(\mathbb{N}))$. Also, since $N(C) = \{0\}$ and $R(C)^\perp = N(C^*) = \bigvee_{k\geq 1}\{e_{2k}\}$, we have that $A_1(x) = P_{R(C)^\perp} A|_{\ell^2(\mathbb{N})}(x) = (0, \alpha_1 x_2, 0, \alpha_4 x_4, \ldots)$ and $B_1 = 0$. Then A_1 and B_1 are normal. Therefore M has the property (β) from Theorem 3.1(i).

Example 3.4. Let U be the unilateral shift given by $Ue_n = e_{n+1}$ on $\ell^2(\mathbb{N})$ for $n \in \mathbb{N}$. If B is hyponormal and $C = U^*$, then 0 is not an eigenvalue of C^* and $B_1 = B|_{N(C)}$ is hyponormal. Since U^* is surjective, it follows that $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$ for arbitrary A and $Z \in \mathcal{L}(\ell^2(\mathbb{N}))$. Moreover, since B_1 has the property (β), it follows that M has the property (β) from Theorem 3.1(ii).

Example 3.5. Let C be defined on $\ell^2(\mathbb{N})$ by

$$Cx := (x_2, x_3, x_4, \ldots)$$

for all $x = (x_n) \in \ell^2(\mathbb{N})$, and let W be the weighted shift given by $We_n = \frac{1}{n+1} e_{n+1}$ on $\ell^2(\mathbb{N})$ for $n \in \mathbb{N}$ with $W_1 = W|_{N(C)}$. Then W_1 has the property (β) from [1] and 0 is not an eigenvalue of C^*. Thus $\left(\begin{smallmatrix} A & C \\ Z & W \end{smallmatrix}\right) \in \mathcal{S}$ and has the property (β) from Theorem 3.1(ii).

In the following theorem, we investigate the decomposablity of the operator matrix $M \in \mathcal{S}$.

Theorem 3.6. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ where $R(C)$ and $R(Z)$ are closed and let $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$ and $B_1 = B|_{N(C)}$. If $P_{R(Z^*)^\perp} A^*|_{\mathcal{H}}$ and A_1 have the property (β), and B_1 is decomposable, then M is decomposable. Moreover, if 0 is not an eigenvalue of both C^* and Z^*, then M is decomposable if and only if B_1 is decomposable.*

Proof. Let $R(C)$ and $R(Z)$ be closed. Then $M, M^* \in \mathcal{S}$. Since B_1 is decomposable, it follows that B_1 and ${B_1}^*$ have the property (β). Moreover, since A_1 and $P_{R(Z^*)^\perp} A^*|_{\mathcal{H}}$ have the property (β), it follows from Theorem 3.1 that M and M^* have the property (β). Hence M is decomposable.

On the other hand, if M is decomposable, then M and M^* have the property (β). Thus, by Theorem 3.1, B_1 and ${B_1}^*$ have the property (β). Hence B_1 is decomposable. The converse implication holds by a similar way. □

Corollary 3.7. *Let $M = \left(\begin{smallmatrix} A & C \\ C^* & B \end{smallmatrix}\right)$ where $R(C)$ is closed and A is self-adjoint, let $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$ have the property (β) and let $B_1 = B|_{N(C)}$ be decomposable. Then M is decomposable.*

Proof. Since $R(C)$ is closed, M and M^* are in the class $\mathcal{S}$. Thus M and M^* have the property (β), so this implies that M is decomposable. □

Corollary 3.8. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$. If $B_1 = B|_{N(C)}$ is normal or compact, and C and Z are surjective, then $M \in \mathcal{S}$ and is decomposable.*

Proof. Let B_1 be normal or compact. Then B_1 is decomposable from [19]. Since C and Z are surjective, these have closed range and so $M \in \mathcal{S}$. The result follows from Theorem 3.6. □

Example 3.9. Let $M = \left(\begin{smallmatrix} A & U \\ U^* & B \end{smallmatrix}\right)$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ where U is the unilateral shift given by $Ue_n = e_{n+1}$ for $n \in \mathbb{N}$ and $B_1 = B|_{N(U)}$ is a zero operator. Then B_1 is normal and so B_1 is decomposable. Since $R(U)$ and $R(U^*)$ are closed, M is decomposable from Theorem 3.8.

Next, we focus on the Dunford property (C) of the operator matrix $M \in \mathcal{S}$. We need the following lemma.

Lemma 3.10. *If $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$, $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$, and $B_1 = B|_{N(C)}$, then the following properties hold.*

(i) *If 0 is not an eigenvalue of C^*, then $\sigma_M(0 \oplus 0 \oplus x) = \sigma_{B_1}(x)$ for $x \in \mathcal{K}$.*

(ii) *$\sigma_{A_1}(x) \subset \sigma_M(x \oplus y \oplus z)$ for $x \oplus y \oplus z \in R(C)^\perp \oplus R(C) \oplus \mathcal{K}$.*

(iii) *$\{0\} \oplus \{0\} \oplus H_{B_1}(F) = H_M(F)$ and $H_M(F) \subset H_{A_1}(F) \oplus N(C) \oplus N(C)^\perp$ hold where $H_M(F) := \{x \oplus y \oplus z : \sigma_M(x \oplus y \oplus z) \subset F\}$.*

Proof. (i) Suppose that $\lambda_0 \in \rho_M(0 \oplus 0 \oplus x)$. Then there is an $\mathcal{H} \oplus N(C) \oplus N(C)^\perp$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that

$$(M - \lambda) \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

for every $\lambda \in D$. Hence we obtain from (3) that

$$\begin{cases} (A_1 - \lambda) f_1(\lambda) = 0, \\ (A_2 - \lambda) f_1(\lambda) + C_1 f_3(\lambda) = 0, \\ Z f_1(\lambda) + (B_1 - \lambda) f_2(\lambda) + (B_2 - \lambda) f_3(\lambda) = x. \end{cases} \tag{10}$$

Let 0 be not an eigenvalue of C^*. Then $A_1 = 0$ and A_1 has the single-valued extension property. By (10), we have $f_1(\lambda) = 0$. Moreover, since C_1 is invertible, $C_1 f_3(\lambda) = 0$ implies $f_3(\lambda) = 0$. Therefore, (10) becomes $(B_1 - \lambda) f_2(\lambda) = x$. Hence $\lambda_0 \in \rho_{B_1}(x)$ and so $\rho_M(0 \oplus 0 \oplus x) \subset \rho_{B_1}(x)$ for $x \in \mathcal{K}$.

Conversely, assume that $\lambda_0 \in \rho_{B_1}(x)$. Then there exists an $N(C)$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that $(B_1 - \lambda) f(\lambda) = x$ for every $\lambda \in D$. So we get that

$$\begin{aligned} (M - \lambda)(0 \oplus f(\lambda) \oplus 0) &= \begin{pmatrix} A_1 - \lambda & 0 & 0 \\ A_2 - \lambda & 0 & C_1 \\ Z & B_1 - \lambda & B_2 - \lambda \end{pmatrix} \begin{pmatrix} 0 \\ f(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ (B_1 - \lambda) f(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \end{aligned} \tag{11}$$

on D. Hence $\lambda_0 \in \rho_M(0 \oplus 0 \oplus x)$, and so $\rho_{B_1}(x) \subset \rho_M(0 \oplus 0 \oplus x)$. Therefore $\sigma_M(0 \oplus 0 \oplus x) = \sigma_{B_1}(x)$ for $x \in \mathcal{K}$.

(ii) Let $\lambda_0 \in \rho_M(x \oplus y \oplus z)$. Then there is an $\mathcal{H} \oplus N(C) \oplus N(C)^\perp$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that

$$(M - \lambda) \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for every $\lambda \in D$. So we obtain from (3) that $(A_1 - \lambda) f_1(\lambda) = x$. Hence $\lambda_0 \in \rho_{A_1}(x)$ and so $\rho_M(x \oplus y \oplus z) \subset \rho_{A_1}(x)$ for $x \oplus y \oplus z \in R(C)^\perp \oplus R(C) \oplus \mathcal{K}$.

(iii) If $x \in H_{B_1}(F)$, then $\sigma_{B_1}(x) \subset F$. Since $\sigma_M(0 \oplus 0 \oplus x) = \sigma_{B_1}(x)$, it follows that $\sigma_M(0 \oplus 0 \oplus x) \subset F$. Therefore we have $0 \oplus 0 \oplus x \in H_M(F)$. Hence $\{0\} \oplus \{0\} \oplus H_{B_1}(F) \subset H_M(F)$ holds. Conversely, if $0 \oplus 0 \oplus x \in H_M(F)$, then $\sigma_M(0 \oplus 0 \oplus x) \subset F$. Since $\sigma_M(0 \oplus 0 \oplus x) = \sigma_{B_1}(x)$, it follows that $\sigma_{B_1}(x) \subset F$. Hence $x \in H_{B_1}(F)$ and so $H_M(F) \subset \{0\} \oplus \{0\} \oplus H_{B_1}(F)$. For the second inclusion, if $x \oplus y \oplus z \in H_M(F)$, then $\sigma_M(x \oplus y \oplus z) \subset F$. Since $\sigma_{A_1}(x) \subset \sigma_M(x \oplus y \oplus z)$, it follows that $\sigma_{A_1}(x) \subset F$. Therefore we have $x \in H_{A_1}(F)$. Hence $H_M(F) \subset H_{A_1}(F) \oplus N(C) \oplus N(C)^{\perp}$ holds. □

Theorem 3.11. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$, $A_1 = P_{R(C)^{\perp}} A|_{\mathcal{H}}$, and $B_1 = B|_{N(C)}$. Then the following statements hold.*

(i) *If 0 is not an eigenvalue of C^*, then M has the property (C) if and only if B_1 has the property (C).*

(ii) *Assume that $H_{A_1}(F) \oplus N(C) \oplus N(C)^{\perp} \subset H_M(F)$ holds. If A_1 has the property (C), then M has the property (C).*

Proof. (i) Suppose that M has the property (C). Then $H_M(F)$ is closed. By Lemma 3.10, $\{0\} \oplus \{0\} \oplus H_{B_1}(F)$ is closed and so $H_{B_1}(F)$ is closed. Hence B_1 has the property (C). The converse implication holds by a similar way.

(ii) Assume that $H_{A_1}(F) \oplus N(C) \oplus N(C)^{\perp} \subset H_M(F)$ holds. If A_1 has the property (C), then $H_{A_1}(F)$ is closed. By Lemma 3.10, $H_M(F)$ is closed. Hence M has the property (C). □

Corollary 3.12. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ where $C : \mathcal{K} \to \mathcal{H}$ is Fredholm, 0 is not an eigenvalue of C^*, and B is totally praranormal on $\mathcal{K}$. Then $M \in \mathcal{S}$ and M has the property (C).*

Proof. Let C be Fredholm. Then $M \in \mathcal{S}$. Since B is totally praranormal, it follows from [19] that B has the property (C) and so B_1 has the property (C). Hence M has the property (C) by Theorem 3.11. □

Corollary 3.13. *Let $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ where $C : \mathcal{K} \to \mathcal{H}$ satisfies that the family $\{C_\alpha : \alpha > 0\}$ is uniformly bounded where $C_\alpha := (C^*C + \alpha I)^{-1}C^*$ and let 0 be not an eigenvalue of C^*. If B is totally praranormal on $\mathcal{K}$, then $M \in \mathcal{S}$ and M has the property (C).*

Proof. By hypotheses, $ran(C)$ is closed from [25, Proposition 3.2]. Then $M \in \mathcal{S}$. Since B is totally praranormal, it follows from [19] that B has the property (C) and so B_1 has the property (C). Hence M has the property (C) by Theorem 3.11. □

Example 3.14. Let $B : L^2[0,1] \to L^2[0,1]$ be the Volterra operators given by

$$Bf(x) = \int_0^x f(t)dt.$$

It is known that every Volterra operator is quasinilpotent, that is, $\sigma(B) = \{0\}$. So B is totally paranormal, and then it has the property (C). And let $C : L^2[0,1] \to L^2[0,1]$ be defined by $Cf(x) = f(1-x)$. Then C is invertible, so it is Fredholm and $M = \left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$. Moreover, 0 is not an eigenvalue of C^* on $L^2[0,1]$. Therefore it follows from Corollary 3.12 that M has the property (C).

4. Weyl and Browder type spectra

In this section, we consider the various spectra for the operator matrices in the class $\mathcal{S}$. So M is an operator matrix in $\mathcal{S}$ with the representation (2).

Lemma 4.1. *For $M \in \mathcal{S}$, the following properties hold*:
(i) $\sigma(M) \subseteq \sigma(B_1 \oplus A_1) = \sigma(B_1) \cup \sigma(A_1)$.
(ii) $\sigma_e(M) \subseteq \sigma_e(B_1 \oplus A_1) = \sigma_e(B_1) \cup \sigma_e(A_1)$.
(iii) $\sigma_w(M) \subseteq \sigma_w(B_1 \oplus A_1) \subseteq \sigma_w(B_1) \cup \sigma_w(A_1)$.

Proof. (i) For given $\lambda \in \mathbb{C}$, we have the factorization (3) for $M - \lambda$. Thus if $B_1 \oplus A_1$ are invertible, then it is obvious from (3) that $M - \lambda$ is invertible.

(ii) Suppose that $(B_1 \oplus A_1) - \lambda$ is Fredholm. Note that

$$\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A_1 - \lambda \end{pmatrix} \begin{pmatrix} I & \triangle_\lambda \\ 0 & I \end{pmatrix} \begin{pmatrix} B_1 - \lambda & 0 \\ 0 & I \end{pmatrix}. \tag{12}$$

From (12), $\left(\begin{smallmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{smallmatrix}\right)$ is also Fredholm. On the other hand, since C_1 is invertible, it follows from (3) and (4) that $M - \lambda$ is Fredholm.

(iii) Suppose that $(B_1 \oplus A_1) - \lambda$ is Weyl. Then $B_1 - \lambda$ and $A_1 - \lambda$ are Fredholm, and $ind(B_1 - \lambda) + ind(A_1 - \lambda) = 0$. Thus we have that

$$\begin{aligned} ind(M - \lambda) &= ind \begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} \\ &= ind(B_1 - \lambda) + ind(A_1 - \lambda) = 0. \end{aligned}$$

Since $M - \lambda$ is Fredholm by the relation (ii), we prove that $M - \lambda$ is Weyl. □

Proposition 4.2. *Let $M \in \mathcal{S}$. If one of the following statements holds*;
(i) $\sigma_{SF_+}(A_1) \subset \sigma_e(M)$ *or* $\sigma_{SF_-}(B_1) \subset \sigma_e(M)$.
(ii) B_1^* *has the single-valued extension property at* $\lambda \notin \sigma_{SF_+}(B_1)$ *or* A_1 *has the single-valued extension property at* $\lambda \notin \sigma_{SF_-}(A_1)$, *then*

$$\sigma_e(M) = \sigma_e(B_1) \cup \sigma_e(A_1).$$

Proof. From Lemma 4.1, it suffices to show that $\sigma_e(B_1) \cup \sigma_e(A_1) \subset \sigma_e(M)$.

(i) Suppose that $\sigma_{SF_+}(A_1) \subset \sigma_e(M)$. For the contrary, we assume that $\sigma_e(M) \neq \sigma_e(B_1) \cup \sigma_e(A_1)$. Then there exists $\lambda \in \mathbb{C}$ such that

$$\lambda \in [\sigma_e(B_1) \cup \sigma_e(A_1)] \setminus \sigma_e(M).$$

Then $M-\lambda$ is Fredholm and hence $A_1-\lambda$ is upper semi-Fredholm by hypothesis. On the other hand, (3) and (4) yield that $\left(\begin{smallmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{smallmatrix}\right) \oplus C_1$ is Fredholm. Since C_1 is invertible, it means that $\left(\begin{smallmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{smallmatrix}\right)$ is also Fredholm. It follows from [20, Lemma 4] that $A_1 - \lambda$ is lower semi-Fredholm. Thus $A_1 - \lambda$ is Fredholm, so that $B_1 - \lambda$ is also Fredholm from (12). So, this is a contradiction. Therefore,

$\sigma_e(M) = \sigma_e(B_1) \cup \sigma_e(A_1)$. If $\sigma_{SF_-}(B_1) \subset \sigma_e(M)$, then the proof follows from the previous arguments.

(ii) Assume that $\lambda \notin \sigma_e(M)$. Since $M - \lambda$ is Fredholm, $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is Fredholm from the proof of (i). This ensures from [20, Lemma 4] that $A_1 - \lambda$ is lower semi-Fredholm and $B_1 - \lambda$ is upper semi-Fredholm. If B_1^* has the single-valued extension property at $\lambda \notin \sigma_{SF_+}(B_1)$, then $\beta(B_1 - \lambda) \leq \alpha(B_1 - \lambda) < \infty$ by [1, Corollary 3.19]. Hence $B_1 - \lambda$ is Fredholm. Since $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is Fredholm, it follows from (12) that $A_1 - \lambda$ is also Fredholm. Thus, $\sigma_e(B_1) \cup \sigma_e(A_1) \subset \sigma_e(M)$. Similarly, if A_1 has the single-valued extension property at $\lambda \notin \sigma_{SF_-}(A_1)$, then $B_1 - \lambda$ is Fredholm, so that $\sigma_e(B_1) \cup \sigma_e(A_1) \subset \sigma_e(M)$. Hence this completes the proof. □

We next state the relation between Weyl spectrum of $M \in \mathcal{S}$ and the union of Weyl spectra of A_1 and B_1.

Theorem 4.3. *For $M \in \mathcal{S}$, the following equality satisfies*;

$$\sigma_w(A_1) \cup \sigma_w(B_1) = \sigma_w(M) \cup \mathcal{Q},$$

where $\mathcal{Q}$ is the union of certain of the holes in $\sigma_w(M)$ which happen to be subsets of $\sigma_w(A_1) \cap \sigma_w(B_1)$.

Proof. It suffices to show that Claims 1 and 2 hold.

Claim 1. For $M \in \mathcal{S}$, the following inclusions hold;

$$[\sigma_w(B_1) \cup \sigma_w(A_1)] \setminus [\sigma_w(B_1) \cap \sigma_w(A_1)] \subset \sigma_w(M) \subset \sigma_w(B_1) \cup \sigma_w(A_1). \tag{13}$$

The second inclusion in (13) holds by Lemma 4.1. To show the first inclusion, we let $\lambda \in [\sigma_w(B_1) \cup \sigma_w(A_1)] \setminus \sigma_w(M)$. Then $M - \lambda$ is Fredholm and $ind(M - \lambda) = 0$. If $A_1 - \lambda$ is Weyl, then it follows from (3) and (12) that $B_1 - \lambda$ is Fredholm. On the other hand, $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is Weyl and from [21, page 134],

$$ind \begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} = ind(A_1 - \lambda) + ind(B_1 - \lambda),$$

and hence $ind(B_1 - \lambda) = 0$. Therefore $B_1 - \lambda$ is Weyl. Then this means that $\lambda \in \sigma_w(B_1) \cap \sigma_w(A_1)$. Thus (13) can be proved.

Claim 2. For $M \in \mathcal{S}$, we have

$$\eta(\sigma_w(M)) = \eta(\sigma_w(B_1) \cup \sigma_w(A_1)), \tag{14}$$

where ηK denotes the polynomially convex hull of the compact set $K \subset \mathbb{C}$.

If $M - \lambda$ is Weyl, then $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is Fredholm as in the proof of Lemma 4.1(ii). By [20, Lemma 4], we get that $A_1 - \lambda$ is lower semi-Fredholm and $B_1 - \lambda$ is upper semi-Fredholm. This means that

$$\sigma_{SF+}(B_1) \cup \sigma_{SF-}(A_1) \subset \sigma_w(M).$$

Since $int(\sigma_w(M)) \subset int(\sigma_w(A_1) \cup \sigma_w(B_1))$ by (13), it follows from the previous fact and from punctured neighborhood theorem ([15] and [20]) that

$$\begin{aligned}\partial(\sigma_w(B_1) \cup \sigma_w(A_1)) &\subset \partial(\sigma_w(B_1)) \cup \partial(\sigma_w(A_1)) \\ &\subset \sigma_{SF_+}(B_1) \cup \sigma_{SF_-}(A_1) \subset \sigma_w(M).\end{aligned}$$

Therefore it follows from (13) that (14) can be proved, so that the passage from $\sigma_w(B_1) \cup \sigma_w(A_1)$ to $\sigma_w(M)$ is the filling in certain of the holes in $\sigma_w(B_1) \cap \sigma_w(A_1)$. Hence this completes the proof of this theorem. □

The following corollary follows from Theorem 4.3.

Corollary 4.4. *Let $M \in \mathcal{S}$. If $\sigma_w(A_1) \cap \sigma_w(B_1)$ has no interior points, then $\sigma_w(M) = \sigma_w(B_1) \cup \sigma_w(A_1)$.*

Proof. If $\sigma_w(A_1) \cap \sigma_w(B_1)$ has no interior points, then $\sigma_w \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix} = \sigma_w(B_1) \cup \sigma_w(A_1)$ from [21, Corollary 7] where $\triangle = Z - B_2C_1^{-1}A_2$. Since $\sigma_w(M) = \sigma_w \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$ by the proof of Theorem 4.3 and Lemma 4.1, we obtain $\sigma_w(M) = \sigma_w(B_1) \cup \sigma_w(A_1)$. □

Recall that for an operator $T \in \mathcal{L}(\mathcal{H})$, a *hole* in $\sigma_e(T)$ is a nonempty bounded component of $\mathbb{C} \backslash \sigma_e(T)$ and a *pseudohole* in $\sigma_e(T)$ is a nonempty component of $\sigma_e(T) \setminus \sigma_{SF_+}(T)$ or of $\sigma_e(T) \setminus \sigma_{SF_-}(T)$, where $\sigma_{SF+}(T)$ and $\sigma_{SF-}(T)$ denote the left and the right essential spectrum, respectively. The *spectral picture* of an operator $T \in \mathcal{L}(\mathcal{H})$ (notation: $SP(T)$) is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes, and the indices associated with these holes and pseudoholes (see [23] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. In the following theorem, we give the result of Weyl's theorem for 2×2 operator matrices.

Theorem 4.5. *Let $M \in \mathcal{S}$. Assume that the following statements hold*:

(i) *either $SP(A_1)$ or $SP(B_1)$ has no pseudoholes.*

(ii) *B_1 satisfies Weyl's theorem.*

(iii) *B_1 is isoloid.*

If Weyl's theorem holds for $(B_1 \oplus A_1)$, then Weyl's theorem holds for M.

Proof. If $R(C)$ is closed, the statements (i), (ii), and (iii) are satisfied and Weyl's theorem holds for $(B_1 \oplus A_1)$, then it follows from [21, Theorem 2.4] that Weyl's theorem holds for $\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. By (3), we know that $\lambda \in \sigma(M)$ if and only if $\lambda \in \sigma \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$ and $\lambda \in \sigma_w(M)$ if and only if $\lambda \in \sigma_w \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$, respectively. Hence Weyl's theorem holds for M. □

Corollary 4.6. *Let $M \in \mathcal{S}$. Suppose that A_1 and B_1 are hyponormal. If either $SP(A_1)$ or $SP(B_1)$ has no pseudoholes, then Weyl's theorem holds for M.*

Proof. Since A_1 and B_1 are hyponormal, they are isoloid. Moreover, since A_1 and B_1 satisfy Weyl's theorem by [6] and [1], it follows that $B_1 \oplus A_1$ satisfies Weyl's theorem from [20]. Hence Weyl's theorem holds for M from Theorem 4.5 □

If u and v are nonzero vectors in $\mathcal{H}$, we write $u \otimes v$ for the operator of *rank one* defined by $(u \otimes v)x = \langle x,\ v\rangle u$, for $x \in \mathcal{H}$, where $\langle\ ,\ \rangle$ denotes the inner product of $\mathcal{H}$.

Corollary 4.7. *Let $M \in \mathcal{S}$. If an isoloid operator $A_1 = N + (u \otimes v)$ where N is normal and $(u \otimes v)$ is a rank one operator with $N(u \otimes v) = (u \otimes v)N$ and B_1 is paranormal, then Weyl's theorem holds for M.*

Proof. Since $A_1 = N + (u \otimes v)$ is essentially normal, $SP(A_1)$ has no pseudo-holes. Also, N satisfies Weyl's theorem and $(u \otimes v)$ is a rank one operator commuting with N, it implies from [22] that Weyl's theorem holds for A_1. Moreover, it holds from [7] that B_1 is isoloid and satisfies Weyl's theorem. Hence $\sigma_w(B_1 \oplus A_1) = \sigma_w(B_1) \cup \sigma_w(A_1)$, equivalently, Weyl's theorem holds for $B_1 \oplus A_1$. Consequently, this means that Weyl's theorem holds for M from Theorem 4.5. □

Next, we begin with the following proposition. Proposition 4.8 says that the passage from $\sigma_a(A_1) \cup \sigma_a(B_1)$ to $\sigma_a(M)$ is the punching of some open sets in $\sigma_s(B_1) \cap \sigma_a(A_1)$ for M in the class $\mathcal{S}$.

Proposition 4.8. *Let $M \in \mathcal{S}$. Then the following equation holds*;

$$\sigma_a(A_1) \cup \sigma_a(B_1) = \sigma_a(M) \cup \mathcal{Q},$$

where $\mathcal{Q}$ is the union of certain of the holes in $\sigma_a(B_1)$ which happen to be subsets of $\sigma_s(B_1) \cap \sigma_a(A_1)$. In particular, if $\sigma_s(B_1) \cap \sigma_a(A_1)$ has no interior points, then $\sigma_a(A_1) \cup \sigma_a(B_1) = \sigma_a(M)$.

Proof. Suppose that $\lambda \notin \sigma_a(A_1) \cup \sigma_a(B_1)$. Then $A_1 - \lambda$ and $B_1 - \lambda$ are bounded below. It ensures from [16, page 269] that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is bounded below. Hence from (3) and (4), we have that $M - \lambda$ is also bounded below. Hence

$$\sigma_a(M) = \sigma_a \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}, \tag{15}$$

where $\triangle = Z - B_2 C_1^{-1} A_2$. Hence we get this result from [16, Theorem 2]. From the above result, we have immediately the second statement. □

We next investigate the connection from $\sigma_{ab}(A_1) \cup \sigma_{ab}(B_1)$ to $\sigma_{ab}(M)$ for $M \in \mathcal{S}$. The following lemmas provide a clue.

Lemma 4.9. *Let $M \in \mathcal{S}$. Then the following inclusions hold*;

$$\sigma_{ab}(B_1) \subseteq \sigma_{ab}(M) \subseteq \sigma_{ab}(B_1) \cup \sigma_{ab}(A_1). \tag{16}$$

Proof. Let $\lambda \notin \sigma_{ab}(M)$. Then $M-\lambda$ is upper semi-Fredholm with finite ascent. Since C_1 is invertible, it follows from (3) that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is upper semi-Fredholm with finite ascent and so $B_1-\lambda$ is upper semi-Fredholm. Moreover, since $N\left((B_1-\lambda)^n\right) \oplus \{0\} \subset N\left(\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}^n\right)$ for every $n \in \mathbb{N}$, it follows that $B_1-\lambda$ has finite ascent. Hence $\lambda \notin \sigma_{ab}(B_1)$.

Let $\lambda \notin \sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)$. Then $B_1-\lambda$ and $A_1-\lambda$ are upper semi-Fredholm operators with finite ascent. Since both $B_1-\lambda$ and $A_1-\lambda$ have finite ascents, it ensures from [5, Lemma 2.2] that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ has finite ascent. Since C_1 is invertible, it follows from (3) that $M-\lambda$ has finite ascent. Since $\begin{pmatrix} I & \triangle_\lambda \\ 0 & I \end{pmatrix}$ is invertible, it gives from (12) that $\begin{pmatrix} B_1-\lambda & \triangle_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is upper semi-Fredholm. Therefore, M is upper semi-Fredholm by the previous statements. Hence, $\lambda \notin \sigma_{ab}(M)$. □

Lemma 4.10. *Let $M \in \mathcal{S}$. Then the following equality holds*;

$$\eta(\sigma_{ab}(M)) = \eta(\sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)), \tag{17}$$

where ηK denotes the polynomially convex hull of the compact set $K \subset \mathbb{C}$.

Proof. It is well known that for every operator $T \in \mathcal{L}(\mathcal{H})$,

$$\partial\sigma_b(T) \subset \sigma_{ab}(T) \subset \sigma_b(T),$$

so that $\eta(\sigma_b(T)) = \eta(\sigma_{ab}(T))$. Similarly, it satisfies that

$$\eta(\sigma_b(B_1) \cup \sigma_b(A_1)) = \eta(\sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)).$$

Therefore we have that

$$\begin{aligned}\eta(\sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)) &= \eta(\sigma_b(B_1) \cup \sigma_b(A_1)) \\ &= \eta(\sigma_b(M)) = \eta(\sigma_{ab}(M)).\end{aligned}$$ □

Using Lemmas 4.9 and 4.10, we have the following theorem.

Theorem 4.11. *Let $M \in \mathcal{S}$. Then the following relations hold*;

$$\sigma_{ab}(A_1) \cup \sigma_{ab}(B_1) = \sigma_{ab}(M) \cup \mathcal{Q},$$

where $\mathcal{Q}$ is the union of certain of the holes in $\sigma_{ab}(M)$ which happen to be subsets of $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$.

Proof. Lemmas 4.9 and 4.10 imply that

$$(\sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)) \setminus \sigma_{ab}(M) \subset \sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1). \tag{18}$$

Therefore it follows from (17) that (18) can be proved, so that the passage from $\sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)$ to $\sigma_{ab}(M)$ is the filling in certain of the holes in $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$. Hence this completes the proof of this theorem. □

Corollary 4.12. *Let $M \in \mathcal{S}$. If $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points, then $\sigma_{ab}(A_1) \cup \sigma_{ab}(B_1) = \sigma_{ab}(M)$ and $\sigma_a(A_1) \cup \sigma_a(B_1) = \sigma_a(M)$.*

Proof. Since acc $\sigma_a(T) \subseteq \sigma_{ab}(T)$ for every operator $T \in \mathcal{L}(\mathcal{H})$, it follows that $\sigma_{ab}(A_1)$ has no interior points if and only if $\sigma_a(A_1)$ has no interior points. From Theorem 4.11 and [16], we get this result. □

5. Weyl type theorems

In this section, we study a-Weyl's theorem and a-Browder's theorem for operator matrices in the class $\mathcal{S}$. So we start with the following theorem.

Theorem 5.1. *Let $M \in \mathcal{S}$. Assume that either $\sigma_{SF+}(B_1) = \sigma_{SF+}(A_1)$ or $\sigma_{SF+}(A_1) \cap \sigma_{SF-}(B_1) = \emptyset$ holds. If $B_1 \oplus A_1$ satisfies a-Browder's theorem, then M satisfies a-Browder's theorem.*

Proof. For the proof, it suffices to show that $\sigma_{ab}(M) \subseteq \sigma_{ea}(M)$. Suppose that $\lambda \notin \sigma_{ea}(M)$. First, we prove that $\sigma_{ea}(M) = \sigma_{ea}(B_1 \oplus A_1)$.

Assume that $\sigma_{SF+}(B_1) = \sigma_{SF+}(A_1)$. If $\lambda \notin \sigma_{ea}(M)$, then, since C_1 is invertible, it follows from (3) that $\lambda \notin \sigma_{ea}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. By [10, Lemma 3.2], we have (i) $\lambda \notin \sigma_{SF+}(B_1)$ and $\alpha(A_1-\lambda) < \infty$ and $ind(B_1-\lambda)+ind(A_1-\lambda) \leq 0$ or (ii) $\lambda \notin \sigma_{SF+}(B_1)$ and $\alpha(A_1 - \lambda) = \beta(B_1 - \lambda) = \infty$. On the other hand, since $\sigma_{SF+}(B_1) = \sigma_{SF+}(A_1)$, it follows that $(B_1 \oplus A_1) - \lambda$ is upper semi-Fredholm and $ind[(B_1 \oplus A_1) - \lambda] = ind(B_1 - \lambda) + ind(A_1 - \lambda) \leq 0$. Thus, $\lambda \notin \sigma_{ea}(B_1 \oplus A_1)$ and so $\sigma_{ea}(M) = \sigma_{ea}(B_1 \oplus A_1)$. Suppose that $\sigma_{SF-}(B_1) \cap \sigma_{SF+}(A_1) = \emptyset$. If $\lambda \notin \sigma_{ea}(M)$, then we consider two cases:
(Case 1) If $\lambda \in \sigma_{SF-}(B_1)$, then $\beta(B_1 - \lambda) = \infty$. The relation $\sigma_{SF-}(B_1) \cap \sigma_{SF+}(A_1) = \emptyset$ implies $\lambda \notin \sigma_{SF+}(A_1)$. Since $M - \lambda$ is upper semi-Fredholm, it follows that $\begin{pmatrix} B_1-\lambda & \Delta_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is upper semi-Fredholm and so $B_1 - \lambda$ is upper semi-Fredholm. Since $A_1 - \lambda$ and $B_1 - \lambda$ are upper semi-Fredholm, it follows that $(B_1 \oplus A_1) - \lambda$ is also upper semi-Fredholm and $ind[(B_1 \oplus A_1) - \lambda] \leq 0$. Hence $\lambda \notin \sigma_{ea}(B_1 \oplus A_1)$ and so $\sigma_{ea}(M) = \sigma_{ea}(B_1 \oplus A_1)$.
(Case 2) If $\lambda \notin \sigma_{SF-}(B_1)$, then, since $B_1 - \lambda$ is upper semi-Fredholm, $B_1 - \lambda$ is Fredholm. Now, we will show that $A_1 - \lambda$ is upper semi-Fredholm. For the contrary, let $\lambda \in \sigma_{SF+}(A_1)$. Since $A_1 - \lambda$ has closed range, it follows that $\alpha(A_1 - \lambda) = \infty$. Therefore, we have $\beta(B_1 - \lambda) = \infty$ by [10, Lemma 3.2]. This is a contradiction. Thus, $\lambda \notin \sigma_{SF-}(B_1)$ implies $\lambda \notin \sigma_{SF+}(A_1)$. Therefore, $(B_1 \oplus A_1) - \lambda$ is upper semi-Fredholm and $ind[(B_1 \oplus A_1) - \lambda] \leq 0$. Thus, $\lambda \notin \sigma_{ea}(B_1 \oplus A_1)$. Hence $\sigma_{ea}(M) = \sigma_{ea}(B_1 \oplus A_1)$. Since $\lambda \notin \sigma_{ea}(M)$, it follows that $\lambda \notin \sigma_{ea}(B_1 \oplus A_1)$ from the previous facts. Moreover, since $(B_1 \oplus A_1)$ satisfies a-Browder's theorem, it means that $\lambda \notin \sigma_{ab}(B_1 \oplus A_1)$. Since both $B_1 - \lambda$ and $A_1 - \lambda$ have finite ascents, it holds that $\begin{pmatrix} B_1-\lambda & \Delta_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ has finite ascent by [5]. Since C_1 is invertible, it follows from (3) that $M - \lambda$ has finite ascent. On the other hand, $M - \lambda$ is bounded below and so $\lambda \in iso\sigma_a(M)$. Hence $\lambda \notin \sigma_{ea}(M)$. Hence this completes the proof. □

Let us recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation J on $\mathcal{H}$ such that $T = JT^*J$. In this case, we say that T is complex symmetric with conjugation J.

Corollary 5.2. *Let $M \in \mathcal{S}$. Assume that either $\sigma_{SF+}(B_1) = \sigma_{SF+}(A_1)$ or $\sigma_{SF+}(A_1) \cap \sigma_{SF-}(B_1) = \emptyset$ holds. Suppose that A_1 and B_1 are complex symmetric. If A_1 and B_1 have the single-valued extension property, then M satisfies a-Browder's theorem.*

Proof. Let A_1 and B_1 be complex symmetric. Then it is clear that $B_1 \oplus A_1$ is also complex symmetric. Since A_1 and B_1 have the single-valued extension property, it follows that $B_1 \oplus A_1$ has also the single-valued extension property. So, $B_1 \oplus A_1$ satisfies Browder's theorem from [1]. On the other hand, since $B_1 \oplus A_1$ is complex symmetric and $B_1 \oplus A_1$ satisfies Browder's theorem, it satisfies a- Browder's theorem from [18, Theorem 4.6]. Hence, from Theorem 5.1, M satisfies a-Browder's theorem. $\square$

Example 5.3. Let $M \in \mathcal{S}$. Assume that either $\sigma_{SF+}(B_1) = \sigma_{SF+}(A_1)$ or $\sigma_{SF+}(A_1) \cap \sigma_{SF-}(B_1) = \emptyset$. If A_1 and B_1 are normal operators, then M satisfies a-Browder's theorem. Indeed, if A_1 and B_1 are normal operators, then A_1 and B_1 are complex symmetric from [13]. So, it is obvious that $B_1 \oplus A_1$ is also complex symmetric. Moreover, in this case, since A_1 and B_1 have the single-valued extension property, it follows that $B_1 \oplus A_1$ has also the single-valued extension property. Thus, $B_1 \oplus A_1$ satisfies Browder's theorem from [1]. Hence M satisfies a-Browder's theorem from Corollary 5.2.

In general, we know that $\alpha \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix} \right) < \infty$ ensures that $\alpha(A) < \infty$. If $\alpha(TS) < \infty$ and S is invertible, it is easy to show that $\alpha(TS) = \alpha(T)$ for every $T, S \in \mathcal{L}(\mathcal{H})$. In the following lemma, we consider finite multiplicity between the operator matrices M and $\left(\begin{smallmatrix} B_1 & \triangle \\ 0 & A_1 \end{smallmatrix} \right)$.

Lemma 5.4. *Let $M \in \mathcal{S}$. If $0 < \alpha(M - \lambda) < \infty$, then*

$$0 < \alpha \begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} < \infty,$$

where $A_1 - \lambda = P_{R(C)^\perp}(A - \lambda)|_{\mathcal{H}}$, $A_2 - \lambda = P_{R(C)}(A - \lambda)|_{\mathcal{H}}$, $B_1 - \lambda$ denotes a mapping $B - \lambda$ from $N(C)$ into $\mathcal{K}$, $B_2 - \lambda$ denotes a mapping $B - \lambda$ from $N(C)^\perp$ into $\mathcal{K}$, $\triangle_\lambda = Z - (B_2 - \lambda)C_1^{-1}(A_2 - \lambda)$, and $P_{R(C)}$ denotes the projection of $\mathcal{H}$ onto $R(C)$.

Proof. By (3), (4), and the invertibility of C_1, we can see that

$$\alpha(M - \lambda) = \alpha(\begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix} \oplus C_1) = \alpha \begin{pmatrix} B_1 - \lambda & \triangle_\lambda \\ 0 & A_1 - \lambda \end{pmatrix}. \qquad \square$$

Theorem 5.5. *Let $M \in \mathcal{S}$ and let $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ have no interior points. Then M satisfies a-Browder's theorem if and only if B_1 and A_1 have the single-valued extension property at $\lambda \notin \sigma_{ea} \left(\begin{smallmatrix} B_1 & \triangle \\ 0 & A_1 \end{smallmatrix} \right)$.*

Proof. Suppose that M satisfies a-Browder's theorem. Then $\sigma_{ea}(M) = \sigma_{ab}(M)$. Let $\lambda \notin \sigma_{ea}(M)$. Since C_1 is invertible, it ensures from (3) that $\lambda \notin \sigma_{ea} \left(\begin{smallmatrix} B_1 & \triangle \\ 0 & A_1 \end{smallmatrix} \right)$

and thus $\lambda \notin \sigma_{ab}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. Since $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points, it gives that $\lambda \notin \sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)$. Therefore, $A_1 - \lambda$ and $B_1 - \lambda$ has finite ascent. Hence B_1 and A_1 have the single-valued extension property at λ.

Conversely, it suffices to show that $\sigma_{ab}(M) \subseteq \sigma_{ea}(M)$. Let $\lambda \notin \sigma_{ea}(M)$. Since C_1 is invertible, it ensures from (3) that $\lambda \notin \sigma_{ea}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. Since B_1 and A_1 have the single-valued extension property at λ, $\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$ has the single-valued extension property at λ. Moreover, since $\begin{pmatrix} B_1-\lambda & \Delta_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ is upper semi-Fredholm, it follows that $\begin{pmatrix} B_1-\lambda & \Delta_\lambda \\ 0 & A_1-\lambda \end{pmatrix}$ has finite ascent from [1]. Thus $\lambda \notin \sigma_{ab}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. On the other hand, since $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points, $\lambda \notin \sigma_{ab}(B_1) \cup \sigma_{ab}(A_1)$ so that $\lambda \notin \sigma_{ab}(M)$. Hence M satisfies a-Browder's theorem. □

Corollary 5.6. *Let $M \in \mathcal{S}$. If one of the following statements holds*;
(i) *A has finite spectrum and B is paranormal,*
(ii) *$A = I$ and B is paranormal,*
then M satisfies a-Browder's theorem.

Proof. (i) Suppose that A has finite spectrum and B is paranormal. Then B_1 is also paranormal. In this case, A_1 and B_1 have the single-valued extension property. Moreover, $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points. Hence, from Theorem 5.5, M satisfies a-Browder's theorem.

(ii) Let $A = I$ and B is paranormal. Then B_1 and A_1 are also paranormal. Moreover, in this case, $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points. In this case, since B_1 and A_1 have paranormal, they have the single-valued extension property. Hence, from Theorem 5.5, M satisfies a-Browder's theorem. □

Example 5.7. Let $M \in \mathcal{S}$. Suppose that $\sigma(A) = \{0, 1\}$ and B is a weighted shift defined by

$$B(e_0, e_1, e_2, \ldots, e_n, \ldots) = (\sqrt{\frac{1}{2}}e_1, \sqrt{\frac{2}{3}}e_2, \sqrt{\frac{3}{4}}e_3, \ldots, \sqrt{\frac{n+1}{n+2}}e_{n+1}, \ldots).$$

Then we obtain that

$$\begin{aligned} &\langle[(B^*)^2B^2 - 2\lambda(B^*B) + \lambda^2]e_n, e_n\rangle \\ &= \langle[\frac{(n+1)}{n+3} - 2\lambda\frac{n+1}{n+2} + \lambda^2]e_n, e_n\rangle \\ &= \langle[(\lambda - \frac{n+1}{n+2})^2 + \frac{(n+1)}{(n+3)(n+2)^2}]e_n, e_n\rangle \geq 0 \end{aligned}$$

for all $\lambda > 0$ and all positive n. Thus B is clearly a paranormal operator. Hence M satisfies a-Browder's theorem from Corollary 5.6.

Finally, we provide some conditions for which M satisfies a-Weyl's theorem.

Theorem 5.8. *Let $M \in \mathcal{S}$ and $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ have no interior points. If B_1 and A_1 have the single-valued extension property at $\lambda \notin \sigma_{ea}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$ and $B_1 \oplus A_1$ satisfies a-Weyl's theorem, then M satisfies a-Weyl's theorem.*

Proof. Suppose that B_1 and A_1 have the single-valued extension property at $\lambda \notin \sigma_{ea}\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}$. Then, by Theorem 5.5, a-Browder's theorem for M which means that

$$\sigma_a(M) \setminus \sigma_{ea}(M) = p_{00}^a(M) \subseteq \pi_{00}^a(M).$$

If $\lambda \in \pi_{00}^a(M)$, then $\lambda \in iso\sigma_a(M)$ and $\alpha(M - \lambda) < \infty$. Since C_1 is invertible, it ensures from (3) and Lemma 5.4 that

$$\lambda \in iso\sigma_a \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix} \text{ and } \alpha(\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix} - \lambda) < \infty.$$

Now we claim that $\sigma_a\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix} = \sigma_a(B_1 \oplus A_1)$. Since $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points, it follows that

$$\sigma_a(B_1 \oplus A_1) = \sigma_a(B_1) \cup \sigma_a(A_1) = \sigma_a \begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix}. \tag{19}$$

Thus, $\lambda \in iso\sigma_a(B_1 \oplus A_1)$. From [16], we have

$$\alpha(\begin{pmatrix} B_1 & \triangle \\ 0 & A_1 \end{pmatrix} - \lambda) < \infty \text{ implies } 0 < \alpha[(B_1 \oplus A_1) - \lambda] < \infty.$$

So, $\lambda \in \pi_{00}^a(B_1 \oplus A_1)$. Since $B_1 \oplus A_1$ satisfies a-Weyl's theorem, it follows that $\lambda \in \sigma_a(B_1 \oplus A_1) \setminus \sigma_{ea}(B_1 \oplus A_1)$. So, $\lambda \notin \sigma_{ab}(B_1 \oplus A_1)$. Since $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points, it holds that $\lambda \notin \sigma_{ab}(M)$ from (19) and Theorem 4.11. Therefore, M satisfies a-Weyl's theorem. □

Corollary 5.9. *Let $M \in \mathcal{S}$. If A has finite spectrum and B is normal, then M satisfies a-Weyl's theorem.*

Proof. Suppose that A has finite spectrum and B is normal. Then B_1 is also normal. In this case, A_1 and B_1 have the single-valued extension property. Moreover, $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points. Hence, M satisfies a-Weyl's theorem from Theorem 5.8. □

Example 5.10. Let C be the bilateral shift given by $Ce_n = e_{n+1}$ on $L^2(\mu)$ with respect to $e_n(z) = z^n$ for $n \in \mathbb{Z}$. If $A = I$ and B is a multiplication operator on a Lebesgue space $L^2(\mu)$ where μ is a planar positive Borel measure with compact support. Then $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right) \in \mathcal{S}$. In this case, since A and B are normal, B_1 and A_1 are also normal. Therefore, $B_1 \oplus A_1$ satisfies a-Weyl's theorem. Moreover, in this case, $\sigma_{ab}(A_1) \setminus \sigma_{ab}(B_1)$ has no interior points. On the other hand, since B_1 and A_1 have the single-valued extension property, we conclude from Theorem 5.8 that $\left(\begin{smallmatrix} A & C \\ Z & B \end{smallmatrix}\right)$ satisfies a-Browder's theorem for every $Z \in \mathcal{L}(L^2(\mu), L^2(\mu))$.

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