J. Korean Math. Soc. **57** (2020), No. 4, pp. 845–871
https://doi.org/10.4134/JKMS.j190406
pISSN: 0304-9914 / eISSN: 2234-3008

EQUIDISTRIBUTION OF HIGHER DIMENSIONAL GENERALIZED DEDEKIND SUMS AND EXPONENTIAL SUMS

Hi-joon Chae, Byungheup Jun, and Jungyun Lee

Abstract. We consider generalized Dedekind sums in dimension n, defined as sum of products of values of periodic Bernoulli functions. For the generalized Dedekind sums, we associate a Laurent polynomial. Using this, we associate an exponential sum of a Laurent polynomial to the generalized Dedekind sums and show that this exponential sum has a nontrivial bound that is sufficient to fulfill the equidistribution criterion of Weyl and thus the fractional part of the generalized Dedekind sums are equidistributed in $\mathbb{R}/\mathbb{Z}$.

1. Introduction

Dedekind sums are rational numbers $s(a,c)$ defined for a pair of relatively prime integers (a,c) by the following formula

$$s(a,c)=\sum_{k=1}^{|c|}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ak}{c}\right)\right),$$

where $((x))=x-[x]-1/2$ for $x\notin\mathbb{Z}$ and $((x))=0$ for $x\in\mathbb{Z}$. These were introduced by R. Dedekind to describe modular transformation of the eta function $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ where $q=e^{2\pi i\tau}$ for $\tau\in\mathfrak{h}=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}$ [5,19]. While the modular discriminant $\Delta(\tau)=\eta(\tau)^{24}$ is a cusp form of weight 12, $\eta(\tau)$ fails to be a modular form of weight 1/2 with respect to $\mathrm{SL}_2(\mathbb{Z})$. The failure is measured in the following: for $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$,

$$\log\eta(A\tau)=\log\eta(\tau)+\frac{1}{4}\log\{-(c\tau+d)^2\}+\pi i\phi(A)$$

Received June 11, 2019; Accepted September 11, 2019.
2010 *Mathematics Subject Classification.* 11F20, 11L03, 14M25.
Key words and phrases. generalized Dedekind sums, Todd series, exponential sums, equidistribution.
The first author was supported by 2019 Hongik University Research Fund.
The second author was supported by NRF-2018R1D1A1A02085748.
The third author was supported by 2019 Research Grant from Kangwon National University and NRF-2017R1A6A3A11030486.

with ϕ being the Rademacher ϕ-function:

$$\phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \begin{cases} \operatorname{sign}(c) \cdot s(a,c) - \frac{1}{12}\frac{a+d}{c}, & \text{if } c \neq 0, \\ \frac{b}{d}, & \text{if } c = 0. \end{cases}$$

Considering the weight of $\eta(\tau)$, one sees that ϕ is valued in $\frac{1}{12}\mathbb{Z}$.

The formulation of the Dedekind sum and the Rademacher ϕ-function has interesting application to the partial zeta value at $s=0$ of an ideal $\mathfrak{b}$ of a real quadratic field. If the fundamental unit ϵ acts on $\mathfrak{b}$ by multiplication of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with respect to the basis $[1,\omega]$ for a reduced element in the sense of Gauss (i.e., $\omega > 1$, $0 < \omega' < 1$), we have the following two equations due to C. Meyer ([15]) and Siegel ([20]), respectively:

$$\begin{aligned} \zeta(0,\mathfrak{b}) &= \frac{1}{12}\sum_{i=0}^{\ell-1}(b_i - 3) \\ &= s(a,c) - \frac{a+d}{12c}, \end{aligned}$$

where b_i are terms of the minus continued fraction of a/c (cf. [9]). While the modularity of $\eta(\tau)$ reveals the integrality of the partial zeta values, $s(a,c)$ is far from being an integer (cf. [10, 13, 14, 16, 21]). The nonintegrality of the Dedekind sum is measured by the rational function $\frac{a+d}{12c}$ defined over $\mathrm{SL}_2(\mathbb{Z})$. Actually the mod-1 equidistribution of $\frac{a+d}{12c}$ is a consequence of the Weil bound of Kloosterman sums.

For the special values at negative integers, a higher degree generalization of Dedekind sums is involved [8, 14, 24]. For $i, j \geq 1$ and a, c relatively prime, we define

$$s_{ij}(a,c) := \sum_{k=0}^{c-1} \tilde{B}_i\left(\frac{k}{c}\right)\tilde{B}_j\left(\frac{ak}{c}\right).$$

Here $\tilde{B}_k(x)$ denotes the k-th periodic Bernoulli function for $k \geq 1$. It is the Fourier series expansion of the Bernoulli polynomial $B_k(x)$ restricted to the unit interval $[0,1]$: $\tilde{B}_k(x) = -k!\sum_{m\neq 0}\frac{e^{2\pi i m x}}{(2\pi i m)^k}$. Recall that $B_k(x)$ is given by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty}\frac{B_k(x)}{k!}z^k.$$

We have, in particular, $\tilde{B}_k(x) = B_k(x-[x])$ for $k \geq 2$ and $\tilde{B}_1(x) = ((x))$. These sums are introduced by Apostol and Carlitz in study of modular transformation of certain Lambert series ([1], [3]). They vanish for $i+j$ odd. The classical Dedekind sum $s(a,c)$ appears as the case $i=j=1$ (cf. [13]).

In [13], it is shown that the fractional part of $R_{i+j}c^{i+j-2}s_{ij}(a,c)$ are equidistributed where R_{i+j} is a constant determined by the weight $N=i+j$ only. It is obtained by the following formula:

$$c^{i+j-2}s_{ij}(a,c)-\frac{\alpha_N r_N}{R_N c}\left(\binom{N-1}{i}a^i+\binom{N-1}{j}a'^j\right)\in\frac{1}{R_N}\mathbb{Z}. \tag{1}$$

Here a' denotes the inverse of a modulo c and α_N, r_N are integers given by N. Using this formula, one can associate an exponential sum which has a Weil bound due to Denef-Loeser ([7]). Consequently, the equidistribution of the fractional part of $R_{i+j}c^{i+j-2}s_{ij}(a,c)$ is obtained.

The goal of this article is to generalize the result to higher dimension. The generalization we take in this paper is as follows:

Definition 1.1. Let $(r_1,\dots,r_n)\in\mathbb{N}^n$ and $(p_1,\dots,p_{n-1},q)\in\mathbb{Z}^n$ such that $(p_i,q)=1$ for every i. We define the generalized Dedekind sum as

$$\begin{aligned}(2)\qquad & s_{r_1,\dots,r_n}(q;p_1,\dots,p_{n-1})\\ &:=\sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}/q\mathbb{Z}}\tilde{B}_{r_1}\left(\frac{k_1}{q}\right)\cdots\tilde{B}_{r_{n-1}}\left(\frac{k_{n-1}}{q}\right)\tilde{B}_{r_n}\left(\frac{\sum_{i=1}^{n-1}p_ik_i}{q}\right).\end{aligned}$$

The version considered by Zagier ([23]) is the case $r_1=\cdots=r_n=1$. We identify these sums as the coefficients of the Todd series of lattice cones in higher dimension as in [13]. The Todd series are introduced by Brion-Vergne ([2]) in the formulation of the Euler-Maclaurin formula for simple polytopes. The Todd series of lattice cones have certain additivity under barycentric decomposition (cf. [18]).

Higher dimensional Dedekind sums are introduced and investigated by many authors in diverse contexts (e.g. Zagier [23], Chapman [4], Hu-Solomon [11]). In particular, the generalization we take is an 'inhomogeneous' version of Chapman's for which he obtained a nice reciprocity formula at the level of generating function. If one puts $p_n=-1$ and $\lambda_1=\lambda_2=\cdots=\lambda_n=0$ in Chapman's definition of higher dimensional generalized Dedekind sums, one obtains (2). On the other hand, Chapman's homogeneous version can be easily recovered from the inhomogeneous version.

The method of this paper is continuation of [13]. But in higher dimension we have some technical difficulties. We are lack of notion of continued fractions and it is inevitable to replace the technique with nonsingular decomposition of cones. As a result, we don't have full control of the individual cones appearing in the decomposition but the components supported by the facets of the original cone. We will show that the inner cones do not contribute after reduction modulo q. Then we read off the coefficients of the mod-q generating function to obtain the fractional part as a Laurent polynomial in $p_1,\dots,p_{n-1}$, where we need the notion of higher dimensional residue of rational functions with non-field coefficient (cf. App. B.). The following is the first of our main results:

Theorem 1.2. *Let $(r_1,r_2,\dots,r_n)\in\mathbb{N}^n$ and $(p_1,\dots,p_{n-1},q)\in\mathbb{N}^n$ with $(p_i,q)=1$ for $i=1,\dots,n-1$. Suppose $N=r_1+r_2+\cdots+r_n$ is even. Let $p_n=-1$.*

Then for an integer $d_{N,n}$ determined by N and n,

$$\frac{d_{N,n}q^{N-n+1}}{r_1!\cdots r_n!}s_{r_1,r_2,\ldots,r_n}(q;p_1,p_2,\ldots,p_{n-1})$$

is an integer and we have

$$\begin{aligned}\frac{d_{N,n}q^{N-n+1}}{r_1!\cdots r_n!}&s_{r_1,\ldots,r_n}(q;p_1,\ldots,p_{n-1})\\ \equiv \sum_{(m_1,\ldots,m_n)}&(-d_{N,n})\prod_{i=1}^{n}\frac{B_{m_i}}{m_i!}\binom{m_i-1}{r_i-1}p_i^{m_i-r_i} \mod q.\end{aligned} \tag{3}$$

Here the summation is taken over the set of n-tuples $(m_1,\ldots,m_n)$ of non-negative even integers such that $\sum_{i=1}^{n}m_i=N$ and at least one of m_i is zero.

In the above, B_k denotes the k-th Bernoulli number. When N is odd, the sum vanishes (Proposition 3.3). The $d_{N,n}$ can be given explicitly as

$$d_{N,n} := \underset{\substack{m_1+\cdots+m_n=N\\ m_1,\ldots,m_n\geq 0}}{\mathrm{lcm}}\left\{\text{denominator of }\prod_{i=1}^{n}\frac{B_{m_i}}{m_i!}\right\}. \tag{4}$$

As an application of the above theorem, we associate an exponential sum of a certain Laurent polynomial to higher dimensional Dedekind sums. In higher dimension, it is not as simple to check the nondegeneracy of a Newton polytope and the result of Denef-Loeser ([7]) is not directly applicable as in [13]. Instead we put a condition (B) in §6. Roughly speaking it is nondegeneracy of Newton polytope in codimension one. This will give us a loose bound but enough to fulfill the Weyl's criterion for equidistribution. Consequently, we show that the fractional part of

$$\frac{dq^{N-n}}{r_1!\cdots r_n!}s_{r_1,\ldots,r_n}(q;p_1,\ldots,p_{n-1})\in\frac{1}{q}\mathbb{Z}$$

are equidistributed in $[0,1)$ (or in $\mathbb{R}/\mathbb{Z}$) (Theorem 6.2). In particular, the higher dimensional Dedekind sums introduced by Zagier ([23]) are equidistributed in the unit interval when we take the fractional part.

This paper is composed as follows: The definition of Todd series of lattice cones and the additivity under cone decomposition are reviewed in §2. A precise relation between coefficients of Todd series and generalized Dedekind sums is given in §3, which will be used in the subsequent sections. The integrality of Todd coefficients and generalized Dedekind sums are shown in §4. A formula for reduction mod q of generalized Dedekind sums is given in §5. In §6, we prove the equidistribution of fractional parts of generalized Dedekind sums by estimating the exponential sum of the associated Laurent polynomial. Finally, in §7 we explicitly compute the Laurent polynomials for two cases: generalized Dedekind sums in 3-dimension and Zagier-Dedekind sums. Additionally, two appendices are attached: one on number of congruence solutions modulo prime power and the other on the notion of iterated constant term for rational functions with coefficient from a general commutative ring.

Comment on notation

Throughout the paper, we often abbreviate a multi-index $(r_1, \dots, r_n)$ to $\mathbf{r}$. By $|\mathbf{r}|$ and $\mathbf{r}!$ we denote $r_1+r_2+\cdots+r_n$ and $(r_1!)\cdots(r_n!)$ respectively. A lattice vector $(p_1, \dots, p_{n-1}, q)$ satisfying $(p_i, q) = 1$ for $i = 1, \dots, n-1$ is denoted by $(q; p_1, \dots, p_{n-1})$. Again it is abbreviated to $(q; \mathbf{p})$. The set of such vectors $(q; \mathbf{p})$ will be denoted by I_n.

2. Lattice cones and Todd series

2.1. Lattice cones

Consider the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. We recall the notion of lattice cones with simplicial structure. A *m-simplicial lattice cone* is an ordered m-tuple $(v_1, v_2, \dots, v_m)$ of primitive lattice vectors v_i in $\mathbb{Z}^n$ such that the convex hull of $\{v_1, \dots, v_m\}$ does not contain the origin. We denote the simplicial cone of $(v_1, v_2, \dots, v_m)$ by $\mathrm{Cone}(v_1, v_2, \dots, v_m)$. Since we will deal only with simplicial lattice cones in this paper, we often abbreviate simplicial lattice cones to cones if there is no danger of confusion. The simplicial lattice cone corresponding to a subtuple of $(v_1, v_2, \dots, v_m)$ is called a *face* of $\mathrm{Cone}(v_1, \dots, v_m)$. A *facet* is a face of codimension one. The underlying topological space of $C = \mathrm{Cone}(v_1, v_2, \dots, v_m)$ is a closed subset of $\mathbb{R}^n$

$$|C| = |\mathrm{Cone}(v_1, v_2, \dots, v_m)| := \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_m.$$

A m-dimensional simplicial cone C is said to be *degenerate* (resp. *nondegenernate*) if the vectors defining C are linearly dependent (resp. linearly independent). If $m > n$, then a m-simplicial cone is necessarily degenerate. There is an obvious action of $g \in \mathrm{GL}_n(\mathbb{Z})$ on the set of lattice cones, by $(v_1, \dots, v_m) \mapsto (gv_1, \dots, gv_m)$. The nondegeneracy is preserved under $\mathrm{GL}_n(\mathbb{Z})$-action.

Let $C = \mathrm{Cone}(v_1, v_2, \dots, v_n)$ be a nondegenerate n-simplicial lattice cone. We define following objects associated to C.

- A $(n \times n)$ integral matrix $M_C = (v_1|v_2|\cdots|v_n)$ where we take v_i as column vectors in $\mathbb{Z}^n$.
- A sublattice $\Lambda_C = \sum_{i=1}^n \mathbb{Z}v_i$ of $\mathbb{Z}^n$ and the quotient group $\Gamma_C = \mathbb{Z}^n/\Lambda_C$.
- A n-tuple of characters $(\chi_1^C, \dots, \chi_n^C)$ on $\mathbb{Z}^n$ (or on Γ_C):

$$\chi_j^C(v) := \exp(2\pi i a_j) \quad \text{ if } v = \sum_{j=1}^n a_j v_j.$$

- The fundamental parallelepiped P_C of the torus $\mathbb{R}^n/\Lambda_C$:

$$P_C := \left\{ \sum_{i=1}^{n} a_i v_i \mid a_i \in [0,1) \quad \text{for } i = 1, \ldots, n \right\}.$$

We denote by $P_C(\mathbb{Z})$ the set of lattice points in P_C.

For a nondegenerate cone C, $|\Gamma_C| = |\det(M_C)|$. A cone C is said to be *nonsingular* if $|\det(M_C)| = 1$ or equivalently $\Lambda_C = \mathbb{Z}^n$. Under $\mathrm{GL}_n(\mathbb{Z})$-action, the nonsingularity and the characters χ_i^C of C are preserved. The *orientation* of C is the sign of $\det(M_C)$. If there appears only a single cone C, we will often abbreviate M_C, Λ_C, χ_i^C and Γ_C to M, Λ, χ_i and Γ, respectively.

Remark 2.1. The notion of simplicial cones defined above is different from that appearing in usual texts of convex geometry. For a simplicial cone C, $|C|$ is generally referred as rational polyhedral cone. In this article, we stress more on the simplicial structure of C. Let $\tilde{C}$ be a degenerate $(n+1)$-simplicial cone such that one of the facet is a n-simplicial cone C and the opposite ray belongs to $|C|$. Then one can relate a barycentric decomposition of n-simplicial cone C as in the following subsection.

2.2. Barycentric and nonsingular decomposition

By a *decomposition* of a nondegenerate n-dimensional simplicial cone C, we mean a finite set of n-dimensional nondegenerate simplicial cones

$$\mathcal{C} = \{C_1, \ldots, C_m\}$$

satisfying

- $|C| = \bigcup_{i=1}^m |C_m|$ and $|C_i| \cap |C_j| = |C_{ij}|$ for C_{ij} a proper face of both C_i and C_j for $i \neq j$. (Thus $|C_i|$ for $i = 1, \ldots, m$ have pairwise disjoint interiors).
- each C_i has the same orientation with C.

If every nondegenerate cone of $\mathcal{C}$ is nonsingular, it is said to be *nonsingular.*

Barycentric subdivision is an example of decomposition of a nondegenerate cone. Let C be a nondegenerate lattice cone $\mathrm{Cone}(v_1, \ldots, v_n)$. For a nonzero primitive lattice vector v in $|C|$ and $1 \leq k \leq n$, let $C(v,k) := \mathrm{Cone}(v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_n)$. The *barycentric subdivision* $\mathcal{B}(C,v)$ of C by v is the set of cones

$$\mathcal{B}(C,v) := \{C(v,k) \mid 1 \leq k \leq n, C(v,k) \text{ is nondegenerate}\}.$$

It is easy to check that $C(v,k)$ and C share the same orientation if $C(v,k)$ is nondegenerate. A barycentric subdivision is a typical example of a decomposition of a cone. Notice that the number of cones in $\mathcal{B}(C,v)$ is

$$|\mathcal{B}(C,v)| = \min_{\substack{F:\text{face of } C \\ v \in |F|}} \dim F.$$

For example, if v is in the interior of C, $\mathcal{B}(C,v)$ consists of n cones. Or if $v = v_i$, then $\mathcal{B}(C,v) = \{C\}$.

Let v be a primitive lattice vector of $|C|$ and $\mathcal{C} = \{C_\alpha \mid \alpha \in \mathcal{A}\}$ be a decomposition of C. Let $\mathcal{C}(v)$ be a subset of $\mathcal{C}$ composed of C_α such that $v \in |C_\alpha|$. We denote by $\mathcal{BC}(v)$ another decomposition of C obtained by replacing those $D \in \mathcal{C}$ which contains v with $\mathcal{B}(D,v)$:

$$\mathcal{BC}(v) := \{D \in \mathcal{C} | v \notin |D|\} \cup \bigcup_{D \in \mathcal{C}(v)} \mathcal{B}(D,v).$$

The following technical lemma is a crucial step in this article.

Lemma 2.2. *A nondegenerate n-simplicial lattice cone C admits a nonsingular decomposition.*

Proof. Let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be a decomposition of C. If $|\det(M_{C_k})| = 1$ for every $k \in \{1, \dots, m\}$, then $\mathcal{C}$ is a nonsingular decomposition.

We use the induction on $\max_{D \in \mathcal{C}} |\det(M_D)|$. Then it is enough to construct, starting from $\mathcal{C}$, another (finer) decomposition $\mathcal{C}'$ of C with property

$$\max_{D \in \mathcal{C}'} |\Gamma_D| < \max_{D \in \mathcal{C}} |\Gamma_D|.$$

Let k be such that $|\Gamma_{C_k}| = \max_{D \in \mathcal{C}} |\Gamma_D|$. Since $\Gamma_{C_k} \neq \{0\}$, there exists a non-zero lattice vector v in $P_{C_k}(\mathbb{Z})$. Then it follows that for any lattice cone $D' \in \cup_{D \in \mathcal{C}(v)} \mathcal{B}(D,v)$, we have $|\Gamma_{D'}| < |\Gamma_{C_k}|$. In other words, in the decomposition $\mathcal{BC}(v)$ of C, the cones (including C_k) in $\mathcal{C}$ which contain v are decomposed into cones of smaller determinant. Thus $\mathcal{BC}(v)$ satisfies either

$$\max_{D \in \mathcal{BC}(v)} |\Gamma_D| < \max_{D \in \mathcal{C}} |\Gamma_D|$$

or the equality holds but the maximum is attained at fewer cones. In the latter case, we can repeat the process with $\mathcal{BC}(v)$ in place of $\mathcal{C}$. ☐

Remark 2.3. In the proof of the lemma, it is crucial to choose a vector v in $P_C(\mathbb{Z})$. One may choose any primitive lattice vector in $|C|$ to obtain a subdivision. But this choice does not ensure smaller $|\Gamma_{C_k}|$.

For example, let us consider $C = \mathrm{Cone}(v_1, v_2)$ with $v_1 = (1,0)$ and $v_2 = (1,2)$. Then $|\Gamma_C| = 2$. The subdivision of C by $v = (1,1)$ is a nonsingular decomposition. However for $v' = (2,1) = \frac{3}{2}v_1 + \frac{1}{2}v_2$, in the corresponding barycentric subdivision, we have $C_1 = \mathrm{Cone}(v', v_2)$ and $C_2 = \mathrm{Cone}(v_1, v')$ with $|\Gamma_{C_1}| = \frac{3}{2}|\Gamma_C| = 3$ and $|\Gamma_{C_2}| = \frac{1}{2}|\Gamma_C| = 1$.

Remark 2.4. Suppose that a facet is generated by vectors which is a part of a lattice basis. If we use the barycentric subdivision as in the lemma, the facet is not decomposed and belongs to a unique nonsingular cone in the nonsingular decomposition.

2.3. Dual cones

For a nondegenerate lattice cone $C = \operatorname{Cone}(v_1, \ldots, v_n)$ in $\mathbb{R}^n$, let us define its dual lattice cone $\check{C} = \operatorname{Cone}(u_1, \ldots, u_n)$ lying in $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}) \simeq \mathbb{R}^n$. Let $\{v_1^*, \ldots, v_n^*\}$ be the basis dual to $\{v_1, \ldots, v_n\}$. Then u_i is the positive (integral) multiple of v_i^* which is primitive. Geometrically, u_i is given as the primitive inward normal vector to the i-th facet $C(i) = \operatorname{Cone}(v_1, \ldots, \hat{v}_i, \ldots, v_n)$. We will write the dual vectors u_i as row vectors in $\mathbb{Z}^n$, and similarly we define the matrix $M_{\check{C}}$ of $\check{C}$ as the $(n \times n)$-matrix whose i-th row is u_i. It can be written as a product of a diagonal matrix with positive diagonal entries and M_C^{-1}.

2.4. Todd series

Let $C = \operatorname{Cone}(v_1, v_2, \ldots, v_n)$ be a nondegenerate lattice cone in $\mathbb{R}^n$. Define the *Todd series* of C as

$$\operatorname{Todd}_C(\mathbf{x}) := \sum_{\gamma \in \Gamma_C} \prod_{i=1}^{n} \frac{x_i}{1 - \chi_i^C(\gamma) e^{-x_i}} = \sum_{\mathbf{r}} \frac{a_{\mathbf{r}}(C)}{\mathbf{r}!} \mathbf{x}^{\mathbf{r}}, \tag{5}$$

which is a holomorphic function of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ at a neighborhood of 0 in $\mathbb{C}^n$. The variables $x_1, \ldots, x_n$ in (5) should be viewed as coordinates with respect to $\{v_1, \ldots, v_n\}$ (See §2.5 below).

The Todd series is invariant under $\mathrm{GL}_n(\mathbb{Z})$-action on cones due to the invariance of the characters of the cone. In particular, the Todd series of nonsingular lattice cones in $\mathbb{R}^n$ are all equal to the *Todd power series in n variables*:

$$\operatorname{Todd}(\mathbf{x}) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} = \sum_{\mathbf{r}} (-1)^{|\mathbf{r}|} \frac{B_{\mathbf{r}}}{\mathbf{r}!} \mathbf{x}^{\mathbf{r}}, \tag{6}$$

where the summation is over $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{Z}_{\geq 0}^n$.

Since the summation of the values of χ_i^C has Galois invariance, it is easy to see that the Taylor series of Todd_C has coefficients in $\mathbb{Q}$. In the following section, we will see that the coefficients of $\operatorname{Todd}_C(\mathbf{x})$ are closely related to the higher dimensional generalized Dedekind sums.

For a nonnegative integer N, let $\operatorname{Todd}_C^N(\mathbf{x}) \in \mathbb{Q}[x_1, \ldots, x_n]$ be the homogeneous part of the total degree N of $\operatorname{Todd}_C(\mathbf{x})$. It is called the *$N$-th Todd polynomial* of C. It is the partial sum over $|\mathbf{r}| = N$ of the above sum and is given by

$$\operatorname{Todd}_C^N(\mathbf{x}) = \frac{1}{N!} \frac{\partial^N}{\partial t^N} \operatorname{Todd}_C(t\mathbf{x}) \bigg|_{t=0}.$$

The homogeneous part $\operatorname{Todd}^N(\mathbf{x})$ of total degree N of $\operatorname{Todd}(\mathbf{x})$ is defined similarly. It is called the *N-th Todd polynomial in n variables.*

$$\operatorname{Todd}^N(\mathbf{x}) = (-1)^N \sum_{|\mathbf{r}|=N} \frac{B_{\mathbf{r}}}{\mathbf{r}!} \, \mathbf{x}^{\mathbf{r}}. \tag{7}$$

2.5. Additivity of Todd series

Definition 2.5. The *normalized Todd series* of a nondegenerate n-simplicial cone C in $\mathbb{R}^n$ is a meromorphic function around 0

$$S_C(\mathbf{x}) := \frac{\mathrm{Todd}_C(\mathbf{x})}{(\det M_C)x_1x_2\cdots x_n}$$

in $\mathbb{C}^n$ with poles along the coordinate hyperplanes. The normalized Todd series of a degenerate cone is 0.

For a nonnegative integer N, denote by $S_C^N(x_1,\dots,x_n)$ the homogeneous part of total degree $N-n$. It is a homogeneous rational function:

$$S_C^N(\mathbf{x}) = \mathrm{Todd}_C^N(\mathbf{x})/(\det M_C)x_1x_2\cdots x_n.$$

To deal with Todd series for various cones in $V=\mathbb{R}^n$ simultaneously, it is necessary to view $\mathrm{Todd}_C(\mathbf{x})$ and $S_C(\mathbf{x})$ as functions on V (or on $V\otimes\mathbb{C}$) by taking variables $x_1,x_2,\dots,x_n$ as coordinates on V with respect to the ordered basis $\{v_1,v_2,\dots,v_n\}$ if $C=\mathrm{Cone}(v_1,v_2,\dots,v_n)$ is nondegenerate.

Definition 2.6. Let $C=\mathrm{Cone}(v_1,v_2,\dots,v_n)$ be a nondegenerate n-simplicial lattice cone in $V=\mathbb{R}^n$. Let $T(C)$ and $S(C)$ be the meromorphic functions on $V_{\mathbb{C}}=V\otimes\mathbb{C}$ respectively given by

$$\begin{aligned} T(C) &: x_1v_1+\cdots+x_nv_n \mapsto \mathrm{Todd}_C(\mathbf{x}),\\ S(C) &: x_1v_1+\cdots+x_nv_n \mapsto S_C(\mathbf{x}). \end{aligned}$$

For a nonnegative integer N, the homogeneous polynomial $T^N(C)$ and the homogeneous rational function $S^N(C)$ on V are defined similarly.

Let $y_1,y_2,\dots,y_n$ be the coordinates with respect to the standard basis of V. Then the function $S(C)$ is given by $S_C((y_1,\dots,y_n)(M_C^{-1})^T)\in\mathbb{Q}((y_1,\dots,y_n))$ in terms of these coordinates.

The following proposition describes the additivity of normalized Todd series under barycentric subdivision. It is a restatement of [18, Theorem 3].

Proposition 2.7 (Pommersheim)**.** *Let $\mathcal{C}(v)=\{C_1,C_2,\dots,C_n\}$ be the barycentric subdivision of C with respect to $v\in P_C$. Then we have*

$$S(C)=S(C_1)+\cdots+S(C_n) \quad \textit{and} \quad S^N(C)=S^N(C_1)+\cdots+S^N(C_n).$$

3. Dedekind sums and Todd coefficients

Let $(q;\mathbf{p})=(q;p_1,p_2,\dots,p_{n-1})\in I_n$. We will identify $s_{\mathbf{j}}(q;\mathbf{p})$ using the coefficients of the Todd series of cones of special type. For $\mathbf{j}=(j_1,j_2,\dots,j_n)\in\mathbb{N}^n$, let us introduce $t_{\mathbf{j}}(q;\mathbf{p})$ which is close to $s_{\mathbf{j}}(q;\mathbf{p})$.

Definition 3.1. For $(q;\mathbf{p})\in I_n$ and $\mathbf{j}\in\mathbb{N}^n$ as above, we define $t_{\mathbf{j}}(q;\mathbf{p})$ as

$$t_{\mathbf{j}}(q;\mathbf{p}) := \sum_{k_1,k_2,\dots,k_{n-1}=0}^{q-1} B_{j_1}\left(\frac{k_1}{q}\right)\cdots B_{j_{n-1}}\left(\frac{k_{n-1}}{q}\right) B_{j_n}\left(\left\langle\frac{\sum_{i=1}^{n-1}p_ik_i}{q}\right\rangle\right),$$

where $\langle x\rangle = x - [x]$ denotes the fractional part of $x \in \mathbb{R}$.

Let $C = \mathrm{Cone}(w_1, w_2, \ldots, w_n)$ be the cone generated by $w_i = e_i$ (for $i = 1, \ldots, n-1$) and $w_n = (p_1, \ldots, p_{n-1}, q)$ where e_i is the i-th standard unit vector in $\mathbb{R}^n$. Let us denote this cone by $C(q; \mathbf{p})$. Note that w_i are primitive and the generators of the dual cone $\check{C} = \mathrm{Cone}(u_1, \ldots, u_n)$ are $u_i = (0, \ldots, q, \ldots, 0, -p_i)$ with q in i-th coordinate for $1 \le i \le n-1$ and $u_n = e_n$.

They are coefficients of Todd_C after multiplied by some power of q.

Proposition 3.2. *Let $C = C(q; \mathbf{p})$ be as above. Then we have*

$$\mathrm{Todd}_C(\mathbf{x}) = \sum_{N=0}^{\infty} \sum_{|\mathbf{j}|=N} (-1)^N q^{N-n+1} \frac{t_{\mathbf{j}}(q; \mathbf{p})}{\mathbf{j}!} \mathbf{x}^{\mathbf{j}}.$$

Proof. Expanding the denominators in (5), we have

(8)

$$\mathrm{Todd}_C(\mathbf{x}) = x_1 x_2 \cdots x_n \sum_{\gamma \in \Gamma_C} \sum_{\ell_1, \ldots, \ell_n = 0}^{\infty} \chi_1(\gamma)^{\ell_1} \cdots \chi_n(\gamma)^{\ell_n} e^{-\ell_1 x_1} \cdots e^{-\ell_n x_n}.$$

The above expansion is absolutely convergent in the region: $(x_1, \ldots, x_n) \in \mathbb{R}^n_{>0}$ and is analytically continued to a neighborhood of 0.

The summation of $\chi_1(\gamma)^{\ell_1} \cdots \chi_n(\gamma)^{\ell_n}$ over $\gamma \in \Gamma_C$ is 0 unless $\chi_1^{\ell_1} \cdots \chi_n^{\ell_n}$ is the trivial character of Γ_C. In this case, the sum is equal to $|\Gamma_C| = q$. It happens if and only if $\sum \ell_i w_i^* = \sum \ell_i u_i / q \in \mathbb{Z}^n$. Thus by summing over $\gamma \in \Gamma_C$ first in (8), we may rewrite Todd_C as summation over the lattice points inside $|\check{C}|$:

$$(9)\qquad \begin{aligned} \mathrm{Todd}_C(\mathbf{x}) &= q x_1 x_2 \cdots x_n \sum_{w \in |\check{C}| \cap \mathbb{Z}^n} e^{-\sum_{i=1}^n \langle w, w_i\rangle x_i} \\ &= q x_1 \cdots x_n \sum_{u \in P_{\check{C}} \cap \mathbb{Z}^n} \frac{e^{-\sum_{i=1}^n \langle u, w_i\rangle x_i}}{(1 - e^{-q x_1}) \cdots (1 - e^{-q x_n})}. \end{aligned}$$

Here we identified $w \in |\check{C}| \cap \mathbb{Z}^n$ with $w = u + \ell_1 u_1 + \cdots + \ell_n u_n$ for $u \in P_{\check{C}}(\mathbb{Z})$ and $\ell_1, \ldots, \ell_n \in \mathbb{Z}_{\ge o}$. As $P_{\check{C}}(\mathbb{Z})$ is given by

$$P_{\check{C}}(\mathbb{Z}) = \left\{ \sum_{i=1}^{n-1} \frac{k_i}{q} u_i + \left\langle \frac{p_1 k_1 + \cdots + p_{n-1} k_{n-1}}{q} \right\rangle u_n \,\middle|\, \text{for } k_i = 0, 1, \ldots, q-1 \right\}$$

we obtain the wanted formula for $\mathrm{Todd}_C(\mathbf{x})$:

$$\sum_{\mathbf{j} \in \mathbb{Z}^n_{\ge 0}} \sum_{k_1, \ldots, k_{n-1}} (-1)^{|\mathbf{j}|} q^{|\mathbf{j}|-n+1} \frac{B_{j_1}\left(\frac{k_1}{q}\right) \cdots B_{j_{n-1}}\left(\frac{k_{n-1}}{q}\right) B_{j_n}\left(\left\langle \frac{\sum_{i=1}^{n-1} p_i k_i}{q} \right\rangle\right)}{\mathbf{j}!} \mathbf{x}^{\mathbf{j}},$$

where the inner sum is over $(k_1, \ldots, k_{n-1}) \in \{0, \ldots, q-1\}^n$. $\square$

From the relation of the Todd series and the generalized Dedekind sums, one can easily obtain the following vanishing property.

Proposition 3.3. *If $|\mathbf{j}|$ is odd, then $s_{\mathbf{j}}(q;\mathbf{p})$ vanishes.*

Proof. This generalizes Corollary 4.2 in [13] and the proof is similar. $\square$

If $j>1$, $\widetilde{B}_j(t)=B_j(t)$ on $[0,1)$. Thus if $j_1,\dots,j_n>1$, we have

$$s_{\mathbf{j}}(q;\mathbf{p})=t_{\mathbf{j}}(q;\mathbf{p}).$$

But if $\mathbf{j}$ contains 1, as $\tilde{B}_1(0)\neq B_1(0)$, we need some care. Let $\mathbf{i}\in\mathbb{Z}^n_{>0}$ and $(q;\mathbf{p})=(q;p_1,\dots,p_{n-1})\in I_n$. We set $p_n=-1$. (This convention will be also used in later sections.) For a subset T of $\{1,\dots,n\}$, we set $\mathbf{i}_T=(i_k)_{k\notin T}$. We set $\tilde{\mathbf{p}}_T=(p'_{j_1},\dots,p'_{j_{r-1}})\in\mathbb{Z}^{r-1}$ for $\{j_1,\dots,j_r\}=\{1,\dots,n\}\backslash T$ (as ordered sets) and $p'_{j_i}p_{j_r}\equiv -p_{j_i} \pmod q$ for each $i=1,\dots,r-1$. Moreover we may impose the condition $1\le p'_{j_i}<q$ for each i. This uniquely determines $\tilde{\mathbf{p}}_T$. Then we have the following conversion formula between $s_{\mathbf{i}}$ and $t_{\mathbf{i}}$:

Theorem 3.4. *For $\mathbf{i}=(i_1,i_2,\dots,i_n)$, let $J=\{j\mid i_j=1\}$. Suppose $|\mathbf{i}|$ is even. Then we have*

$$s_{\mathbf{i}}(q;\mathbf{p})=t_{\mathbf{i}}(q;\mathbf{p})+\sum_{\emptyset\neq T\subset J}\left(\tfrac12\right)^{|T|}t_{\mathbf{i}_T}(q;\tilde{\mathbf{p}}_T),$$

$$t_{\mathbf{i}}(q;\mathbf{p})=s_{\mathbf{i}}(q;\mathbf{p})+\sum_{\emptyset\neq T\subset J}\left(\tfrac12\right)^{|T|}s_{\mathbf{i}_T}(q;\tilde{\mathbf{p}}_T),$$

where the second summation runs over nonempty subsets $T\subset J$ with $|T|$ even.

Proof. Setting $k_n=p_1k_1+p_2k_2+\cdots+p_{n-1}k_{n-1}$, one can write

$$\begin{aligned}s_{\mathbf{i}}(q;\mathbf{p})&=\sum_{k_j\neq 0,\forall j\in J}B_{i_1}\left(\frac{k_1}{q}\right)\cdots B_{i_n}\left(\frac{k_n}{q}\right)\\&=t_{\mathbf{i}}(q;\mathbf{p})-\sum_{k_j=0,\exists j\in J}B_{i_1}\left(\frac{k_1}{q}\right)\cdots B_{i_n}\left(\frac{k_n}{q}\right).\end{aligned}$$

Applying the inclusion-exclusion principle, we have

$$\sum_{k_j=0,\exists j\in J}B_{i_1}\left(\frac{k_1}{q}\right)\cdots B_{i_n}\left(\frac{k_n}{q}\right)=\sum_{\emptyset\neq T\subset J}(-1)^{|T|+1}\left(-\frac12\right)^{|T|}t_{\mathbf{i}_T}(q;\tilde{\mathbf{p}}_T).$$

This follows from

$$\begin{aligned}&\sum_{k_j=0,\forall j\in T}B_{i_1}(\frac{k_1}{q})\cdots B_{i_n}(\frac{k_n}{q})\\&=B_1^{|T|}\sum_{\substack{(k_{j_1},\dots,k_{j_r})\\ p_{j_1}k_{j_1}+\cdots+p_{j_r}k_{j_r}=0}}B_{i_{j_1}}\left(\frac{k_{j_1}}{q}\right)\cdots B_{i_{j_r}}\left(\frac{k_{j_r}}{q}\right)\\&=\left(-\frac12\right)^{|T|}\sum_{(k_1,\dots,k_{j_{r-1}})}B_{i_{j_1}}\left(\frac{k_{j_1}}{q}\right)\cdots B_{i_{j_r}}\left(\frac{p'_{j_1}k_{j_1}+\cdots+p'_{j_{r-1}}k_{j_{r-1}}}{q}\right),\end{aligned}$$

where $T \subset J$ and $\{j_1, \ldots, j_r\} = \{1, \ldots, n\} \setminus T$.

The proof of the 2nd formula is identical and we omit the proof. □

4. Integrality of generalized Dedekind sums

The goal of this section is to prove the integrality of generalized Dedekind sums.

Theorem 4.1. *Let $N \in \mathbb{Z}_{\geq 0}$ and $C = C(q; \mathbf{p})$ for $(q; \mathbf{p}) = (q; p_1, \ldots, p_{n-1}) \in I_n$. Then we have*

$$\mathrm{Todd}_C^N(\mathbf{x}) = q x_1 \cdots x_n S_C^N(\mathbf{x}) \in \frac{1}{d}\mathbb{Z}[x_1, x_2, \ldots, x_n]$$

for some positive integer d depending only on N and n (e.g. $d = d_{N,n}$ of (4)).

Proof. Let $w_i = e_i$ for $1 \leq i \leq n-1$ and $w_n = (p_1, \ldots, p_{n-1}, q)$. After Lemma 2.2, $C = \mathrm{Cone}(w_1, \ldots, w_n)$ admits a nonsingular decomposition $\mathcal{C}$ obtained by successive barycentric subdivision. Then we have by Proposition 2.7.

$$S^N(C) = \sum_{D \in \mathcal{C}} S^N(D). \tag{10}$$

Note that any proper subset of $\{w_1, \ldots, w_n\}$ can be extended to a basis of $\mathbb{Z}^n$. As a result, the j-th facet $C(j) = \mathrm{Cone}(w_1, \ldots, \widehat{w_j}, \ldots, w_n)$ is not affected by the subdivisions as above and is contained in a unique cone D_j in $\mathcal{C}$. D_j is given as $\mathrm{Cone}(v_1^{(j)}, \ldots, v_n^{(j)})$ with $v_i^{(j)} = w_i$ for $i \neq j$ and $v_j^{(j)}$ sitting inside C. The cones D_j for $j = 1, \ldots, n$ are the outermost cones in $\mathcal{C}$. Fig. 1 describes the outer cones in dimension 3.

Since $\det M_C = q$, we have

$$w_j = \sum_{i \neq j} a_i^j v_i^{(j)} + q v_j^{(j)} \text{ for some } a_i^j \in \mathbb{Z}. \tag{11}$$

Let $y_1, \ldots, y_n$ be the coordinate functions with respect to the standard basis. Let $D = \mathrm{Cone}(v_1^{(D)}, \ldots, v_n^{(D)}) \in \mathcal{C}$. Then $M_D \in \mathrm{SL}_n(\mathbb{Z})$ and thus $v_i^{(D)*} \in \mathbb{Z}[y_1, \ldots, y_n]$ is a primitive linear form. We have

$$S^N(D) = (-1)^N \sum_{m_1 + \cdots + m_n = N} \prod_{i=1}^{n} \frac{B_{m_i}}{m_i!} (v_i^{(D)*})^{m_i - 1}. \tag{12}$$

Since $d = d_{N,n}$ is explicitly given as (4),

$$dS^N(C) = d \sum_{D \in \mathcal{C}} S^N(D) \in \frac{1}{\prod_{D \in \mathcal{C}} \prod_{i=1}^{n} (v_i^{(D)*})} \mathbb{Z}[y_1, \ldots, y_n].$$

Moreover, since $S^N(C)$ has poles along the facets of C and $\mathbb{Z}[y_1, \ldots, y_n]$ is a UFD,

$$d v_1^{(1)*} \cdots v_n^{(n)*} S^N(C) \in \mathbb{Z}[y_1, \ldots, y_n]. \tag{13}$$

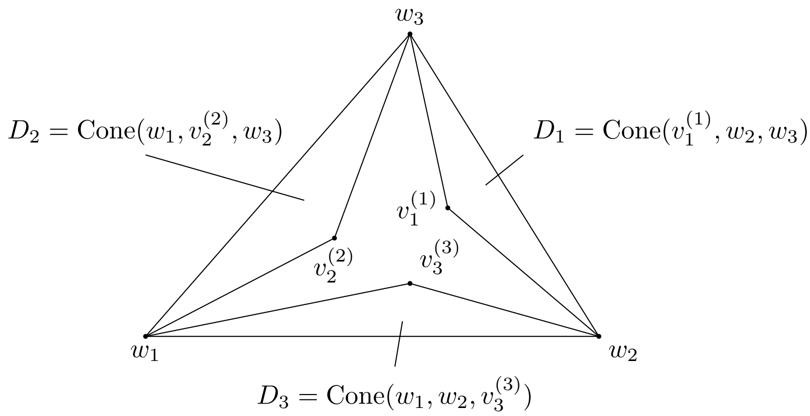

FIGURE 1. Projective picture of the outer cones in dim=3

Recall $x_1, \ldots, x_n$ are the coordinates with respect to the basis $\{w_1, \ldots, w_n\}$. For $i \neq j$, after (11),

$$v_i^{(j)*} = \sum_{k=1}^{n} \langle v_i^{(j)*}, w_k \rangle w_k^* = w_i^* + a_i^j w_j^* = x_i + a_i^j x_j. \tag{14}$$

Similarly, $v_j^{(j)*} = qx_j$. Therefore (13) turns out to be

$$dqx_1 \cdots x_n S_C^N(\mathbf{x}) \in \frac{1}{q^{n-1}} \mathbb{Z}[x_1, \ldots, x_n]. \tag{15}$$

Let $\mathcal{C}_{\rm in} = \mathcal{C} \setminus \{D_1, \ldots, D_j\}$ be the set of inner cones in the nonsingular decomposition. We have

$$S^N(C) = \sum_{D \in \mathcal{C}_{\rm in}} S^N(D) + \sum_{j=1}^{n} S^N(D_j).$$

Note that both

$$\begin{aligned} d \sum_{D \in \mathcal{C}_{\rm in}} S^N(D) &= dS^N(C) - d \sum_{j=1}^{n} S^N(D_j), \quad \text{and} \\ dqx_1 \cdots x_n \sum_{j=1}^{n} S^N(D_j) &= d \sum_{j=1}^{n} x_1 \cdots \widehat{x_j} \cdots x_n (v_j^{(j)*}) S^N(D_j) \end{aligned} \tag{16}$$

belong to $\frac{1}{\prod_{j=1}^n \prod_{i \neq j} (v_i^{(j)*})} \mathbb{Z}[y_1, \ldots, y_n]$. Therefore in $x_1, \ldots, x_n$, we have

$$dqx_1 \cdots x_n S_C^N(\mathbf{x}) \in \frac{1}{\prod_{j=1}^n \prod_{i \neq j} (x_i + a_i^j x_j)} \mathbb{Z}[x_1, \ldots, x_n].$$

Together with (15), this completes the proof because

$$\frac{1}{\prod_{j=1}^{n}\prod_{i\neq j}(x_i+a_i^j x_j)}\mathbb{Z}[x_1,\dots,x_n]\cap\frac{1}{q^{n-1}}\mathbb{Z}[x_1,\dots,x_n]=\mathbb{Z}[x_1,\dots,x_n].$$

□

Corollary 4.2. *For* $\mathbf{r}=(r_1,r_2,\dots,r_n)\in\mathbb{Z}^n_{>0}$, *let* $N=|\mathbf{r}|$. *Let* $d'=\frac{d}{\mathbf{r}!}$ *for* d *as in Theorem 4.1. Then for any* $(q;\mathbf{p})=(q;p_1,\dots,p_{n-1})\in I_n$, *we have*

$$d'q^{N-n+1}s_{\mathbf{r}}(q;\mathbf{p})\in\mathbb{Z}.$$

Proof. We use induction on n. By the last theorem, $d'q^{N-n+1}t_{\mathbf{r}}(q;\mathbf{p})$ is an integer. Since $2^k d_{N-k,n-k}$ divides $d_{N,n}$ for any integer $0\le k\le n$, we apply the induction hypothesis to the second equation (multiplied by $d'q^{N-n+1}$) in Theorem 3.4. This completes the proof. □

5. Reduction mod q of generalized Dedekind sums

In this section, we keep the notations of last section: C is the lattice cone in $\mathbb{R}^n$ generated by $w_1=e_1,\dots,w_{n-1}=e_{n-1},w_n=(p_1,\dots,p_{n-1},q)$. The variables $x_1,\dots,x_n$ are the coordinates with respect to $B=\{w_1,\dots,w_n\}$. In the setting of the proof of Theorem 4.1, $(v_i^{(j)*})=x_i+a_i^j x_j$ (See (14)). Now we consider the mod-q reduction of the integral polynomial $d\,\mathrm{Todd}_C^N(\mathbf{x})=dqx_1\cdots x_n S_C^N(\mathbf{x})$ where $d=d_{N,n}$.

When we take mod-q reduction of $d\,\mathrm{Todd}_C^N(\mathbf{x})$, we will see the contribution of the cones in $\mathcal{C}_{\mathrm{in}}$ vanish. Note that the reduction mod q can be extended to $\frac{1}{\prod_{j=1}^{n}\prod_{i\neq j}(x_i+a_i^j x_j)}\mathbb{Z}[x_1,\dots,x_n]$ as none of $(x_i+a_i^j x_j)$ have common factor with q. From (16), we have

$$d\sum_{D\in\mathcal{C}_{\mathrm{in}}}S^N(D)\in\frac{1}{\prod_{j=1}^{n}\prod_{i\neq j}(x_i+a_i^j x_j)}\mathbb{Z}[x_1,\dots,x_n].$$

Thus we have $dqx_1\cdots x_n\sum_{D\in\mathcal{C}_{\mathrm{in}}}S^N(D)\equiv 0\pmod q$. This implies

$$dqx_1\cdots x_n S^N(C)\equiv\sum_{j=1}^{n}dqx_1\cdots x_n S^N(D_j)\pmod q. \tag{17}$$

In $x_1,\dots,x_n$,

$$(v_j^{(j)*})=qx_j\quad\text{for }1\le j\le n, \tag{18}$$

and for $i\neq j$,

$$(v_i^{(j)*})\equiv\begin{cases}x_i-p_ip_j^{-1}x_j & \bmod q\quad\text{if }j\neq n\text{ and }i\neq n,\\ x_i+p_j^{-1}x_j & \bmod q\quad\text{if }j\neq n\text{ and }i=n,\\ x_i+p_ix_j & \bmod q\quad\text{if }j=n.\end{cases}$$

Putting $p_n=-1$, one can write $(v_i^{(j)*})$ in simpler form:

$$(v_i^{(j)*})\equiv x_i-p_ip_j^{-1}x_j\quad\bmod q\quad\text{for any }i\neq j. \tag{19}$$

It is important to note the reduction mod q of $(v_i^{(j)*})$ for $i \neq j$ does not depend on the lattice vector $v_j^{(j)}$. We remark that above formula is explicit form of (14) modulo q.

The following proposition gives an explicit formula for mod-q reduction of the integer $dq^{N-n+1}t_{\mathbf{r}}(q;\mathbf{p})/\mathbf{r}!$.

Proposition 5.1. *For $(q;\mathbf{p}) \in I_n$ and $\mathbf{r} \in \mathbb{Z}^n_{\geq 1}$, let $N = |\mathbf{r}|$ and $d = d_{N,n}$. Then $dq^{N-n+1}t_{\mathbf{r}}(q;\mathbf{p})/\mathbf{r}!$ is an integer and*

$$\frac{dq^{N-n+1}}{\mathbf{r}!}t_{\mathbf{r}}(q;\mathbf{p}) \equiv \sum_{\mathbf{m}}(-d)\frac{B_{\mathbf{m}}}{\mathbf{m}!}\prod_{i=1}^{n}\binom{m_i-1}{r_i-1}p_i^{m_i-r_i} \pmod{q}, \tag{20}$$

where the summation runs over the set of n-tuples $\mathbf{m} = (m_1,\ldots,m_n) \in \mathbb{Z}^n_{\geq 0}$ with $|\mathbf{m}| = N$ such that at least one of its coordinates is zero.

Proof. Let $C = C(q;\mathbf{p})$. Then $\frac{dq^{N-n+1}}{\mathbf{r}!}t_{\mathbf{r}}(q;\mathbf{p})$ is the coefficient of $\mathbf{x}^{\mathbf{r}}$ in the Todd polynomial $d\,\mathrm{Todd}_C^N(\mathbf{x}) = dq\mathbf{x}^{\mathbf{1}}S_C^N(\mathbf{x})$ (recall $\mathbf{x}^{\mathbf{1}} = x_1x_2\cdots x_n$) and can be computed as the *iterated constant term* $\mathrm{iCT}_{x_1,x_2,\ldots,x_n} = \mathrm{CT}_{x_1}\circ\mathrm{CT}_{x_2}\circ\cdots\circ\mathrm{CT}_{x_n}$ of $dq\mathbf{x}^{\mathbf{1}}S_C^N(\mathbf{x})/\mathbf{x}^{\mathbf{r}}$ with respect to $\mathfrak{A} = \{x_1,\ldots,x_n\}$ (cf. App. B).

Using (17), we have to calculate the iterated constant term with respect to $\mathfrak{A}$ of $(dq\mathbf{x}^{\mathbf{1}}/\mathbf{x}^{\mathbf{r}})S^N(D_j) \pmod{q}$ for each $1 \leq j \leq n$. As D_j is nonsingular, applying (12), (18) and (19), we have

$$\begin{aligned} &dq\mathbf{x}^{\mathbf{1}}S^N(D_j)\\ &\equiv (-1)^N q\mathbf{x}^{\mathbf{1}}\sum_{\mathbf{m}\in M^N} d\frac{B_{\mathbf{m}}}{\mathbf{m}!}\prod_{\substack{1\leq i\leq n\\ i\neq j}}(x_i - p_ip_j^{-1}x_j)^{m_i-1}\cdot(qx_j)^{m_j-1} \pmod{q},\end{aligned}$$

where $M^N = \{\mathbf{m} \in \mathbb{Z}^n_{\geq 0} \mid |\mathbf{m}| = N\}$.

If $m_j \geq 1$, the summand corresponding to $\mathbf{m} = (m_1,\ldots,m_n)$ belongs to $\frac{1}{\prod_{i\neq j}(x_i+a_i^jx_j)}\mathbb{Z}[x_1,\ldots,x_n]$. Multiplied by $qx_1\cdots x_n$, it vanishes as a rational function with coefficient in $\mathbb{Z}/q\mathbb{Z}$. Thus it is enough to count $\mathbf{m}$ with $m_j = 0$ in the summation and from the last equation we have

$$\frac{dq\mathbf{x}^{\mathbf{1}}}{\mathbf{x}^{\mathbf{r}}}S^N(D_j) \equiv (-1)^N\sum_{\substack{\mathbf{m}\in M^N\\ m_j=0}} d\frac{B_{\mathbf{m}}}{\mathbf{m}!}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\frac{(x_i-p_ip_j^{-1}x_j)^{m_i-1}}{x_i^{r_i-1}}\cdot\frac{1}{x_j^{r_j}} \pmod{q}.$$

We compute the iterated constant terms of individual summands. As $m_j = 0$,

$$\begin{aligned} &\mathrm{CT}_{x_{j+1}}\circ\cdots\circ\mathrm{CT}_{x_n}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\frac{(x_i-p_ip_j^{-1}x_j)^{m_i-1}}{x_i^{r_i-1}}\cdot\frac{1}{x_j^{r_j}}\\ &= \prod_{i=1}^{j-1}\frac{(x_i-p_ip_j^{-1}x_j)^{m_i-1}}{x_i^{r_i-1}}\cdot\prod_{i=j+1}^{n}\binom{m_i-1}{r_i-1}(-p_ip_j^{-1})^{m_i-r_i}\cdot x_j^{\sum_{i\geq j}m_i-r_i}.\end{aligned}$$

Putting $e := \sum_{i\geq j} m_i - r_i$, we obtain CT_{x_j} of the last equation depending on the sign of e: if $e > 0$, then it vanishes. If $e \leq 0$, then it is equal to

$$\sum_{a_1,\ldots,a_{j-1}} \prod_{i=1}^{j-1} \binom{m_i-1}{a_i}(-p_ip_j^{-1})^{a_i} x_i^{m_i-r_i-a_i} \cdot \prod_{i=j+1}^{n} \binom{m_i-1}{r_i-1}(-p_ip_j^{-1})^{m_i-r_i},$$

where the summation is over non-negative integers $a_1, \ldots, a_{j-1}$ satisfying $a_1 + \cdots + a_{j-1} = -e = \sum_{i\geq j} r_i - m_i$. Now it is direct to see that $\mathrm{CT}_{x_1} \circ \mathrm{CT}_{x_2} \circ \cdots \circ \mathrm{CT}_{x_{j-1}}$ of the above is supported at $a_i = m_i - r_i$ for $1 \leq i \leq j-1$. This can be satisfied only if $m_i \geq r_i$ for $1 \leq i \leq j-1$. *A priori* unless $m_i > 0$ for $1 \leq i \leq j-1$, the iterated constant term vanishes (We need this later). Hence for $\mathbf{m} \in M^N$ satisfying (a) $\sum_{i\geq j} m_i - r_i \leq 0$ and (b) $m_i \geq r_i$ for $1 \leq i \leq j-1$, we have

$$\begin{aligned}
&\mathrm{iCT}_{x_1,x_2,\ldots,x_n} \prod_{\substack{1\leq i\leq n\\ i\neq j}} \frac{(x_i - p_ip_j^{-1}x_j)^{m_i-1}}{x_i^{r_i-1}} \cdot \frac{1}{x_j^{r_j}} \\
&= \prod_{\substack{1\leq i\leq n\\ i\neq j}} \binom{m_i-1}{r_i-1}(-p_ip_j^{-1})^{m_i-r_i} \\
&= \prod_{\substack{1\leq i\leq n\\ i\neq j}} \binom{m_i-1}{r_i-1} p_i^{m_i-r_i} \cdot (-p_j^{-1})^{\sum_{i\neq j} m_i - r_i} = (-1)\prod_{i=1}^{n} \binom{m_i-1}{r_i-1} p_i^{m_i-r_i},
\end{aligned}$$

where the last equality comes from $\sum_{i\neq j} m_i - r_i = r_j - m_j = r_j$ and $\binom{m_j-1}{r_j-1} = \binom{-1}{r_j-1} = (-1)^{r_j-1}$. Note that (b) implies (a) since $\sum_{i=1}^{n} m_i - r_i = 0$.

Let M_j be the subset of M^N consisting of $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_j = 0$ and $m_i \geq r_i$ for $1 \leq i \leq j-1$. Then

$$\mathrm{iCT}_{x_1,x_2,\ldots,x_n} \frac{dq\mathbf{x}^{\mathbf{1}}}{\mathbf{x}^{\mathbf{r}}} S^N(D_j) \equiv \sum_{\mathbf{m}\in M_j} (-d)\frac{B_{\mathbf{m}}}{\mathbf{m}!} \prod_{i=1}^{n} \binom{m_i-1}{r_i-1} p_i^{m_i-r_i}. \tag{21}$$

Since $M_1, \ldots, M_n$ are disjoint subsets of M^N, summing the above over $1 \leq j \leq n$, finally we obtain the formula. □

Now the proof of Theorem 1.2 follows from the previous proposition but needs some care for the difference of $t_{\mathbf{r}}$ and $s_{\mathbf{r}}$.

Proof of Theorem 1.2. Since $s_{\mathbf{r}}(q; \mathbf{p})$ vanishes for odd $N = |\mathbf{r}|$, we will assume that N is even. In the summation of (20), nontrivial summand occurs only for $\mathbf{m}$ such that m_i is even or $m_i = 1$ because B_m vanishes for odd $m > 1$. Further, if $m_i \neq 0$, then $m_i \geq r_i$ due to the binomial coefficients. In particular, for $\mathbf{m} = (m_1, \ldots, m_n)$, if $m_i = 1$, r_i is necessarily 1.

Let us denote by M the set of such $\mathbf{m} = (m_1, \dots, m_n)$ (depending on $\mathbf{r}!$):

$$M := \{ \mathbf{m} \in \mathbb{Z}^n_{\geq 0} \mid |\mathbf{m}| = N,\ m_i = 0 \text{ for some } i,\\ \text{if } m_i \neq 0, \text{ then } m_i \text{ is either even or 1 and } m_i \geq r_i \}.$$

For simplicity let $\phi(\mathbf{m}, \mathbf{p}) = (-d)\frac{B_{\mathbf{m}}}{\mathbf{m}!} \prod_{i=1}^n \binom{m_i-1}{r_i-1} p_i^{m_i - r_i}$ in (20). Thus

$$d' q^{N-n+1} t_{\mathbf{r}}(q; \mathbf{p}) \equiv \sum_{\mathbf{m} \in M} \phi(\mathbf{m}, \mathbf{p}) \pmod q,$$

where $d' = d/\mathbf{r}! = d_{N,n}/\mathbf{r}!$.

Now let $J = \{ 1 \leq j \leq n \mid r_j = 1 \}$. For $T \subset J$, let $M(T)$ be a subset of M:

$$M(T) := \{ \mathbf{m} \in M \mid m_j = 1 \text{ for } j \in T \text{ and } m_j \neq 1 \text{ if } j \notin T \}.$$

We need to show

$$d' q^{N-n+1} s_{\mathbf{r}}(q; \mathbf{p}) \equiv \sum_{\mathbf{m} \in M(\emptyset)} \phi(\mathbf{m}, \mathbf{p}) \pmod q.$$

Note that M is the disjoint union of $M(T)$ for T running over subsets of J. In fact, $M(T)$ is nonempty only for T of even cardinality.

Now we use induction on $|J|$. If $|J| \leq 1$, then $M(\emptyset) = M$. Using Theorem 3.4, $t_{\mathbf{j}} = s_{\mathbf{j}}$ and the claim holds.

In general, let T be a nonempty subset of J of even cardinality $|T| = k$. Then it is easy to see that the partial sum of $\phi(\mathbf{m}, \mathbf{p})$ over $\mathbf{m} \in M(T)$ is, by induction assumption, congruent to

$$(-\frac{1}{2})^k d' q^{N-n+1} s_{r_{j_1}, \dots, r_{j_{n-k}}}(q; p^T_{j_1}, \dots, p^T_{j_{n-k-1}}) \equiv \sum_{\mathbf{m} \in M(T)} \phi(\mathbf{m}, \mathbf{p}) \pmod q,$$

where $\mathbf{j}_T$ and $\tilde{\mathbf{p}}_T$ are as defined in Theorem 3.4. This finishes the proof by the same theorem. □

6. Equidistribution of generalized Dedekind sums and exponential sums

Given $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n_{>0}$, let $N = |\mathbf{r}|$. Let $f_{\mathbf{r}}(\mathbf{p})$ be the right hand side of (3) considered as a Laurent polynomial in $\mathbf{p} = (p_1, \dots, p_{n-1})$ with integral coefficients. For $(q; \mathbf{p}) \in I_n$, we have by Theorem 1.2

$$\left\langle \frac{d_{N,n} q^{N-n}}{\mathbf{r}!} s_{\mathbf{r}}(q; \mathbf{p}) \right\rangle = \left\langle \frac{1}{q} f_{\mathbf{r}}(\mathbf{p}) \right\rangle, \tag{22}$$

where $\langle t \rangle = t - [t] \in [0, 1)$ denotes the fractional part of t and in the right side we take p_i^{-1} to be an inverse modulo q.

The goal of this section is to show the equidistribution of this sequence of numbers for varying $(q; \mathbf{p})$ with fixed $\mathbf{r}$. For $x \in \mathbb{R}_{>0}$, let $I_n(x)$ be the set of $(q; \mathbf{p}) \in I_n$ with $q < x$ and $1 \leq p_i < q$.

Following Weyl ([22]), we say a sequence $\{a_{(q;\mathbf{p})} \,|\, (q;\mathbf{p}) \in I_n\}$ in $[0,1)$ is *equidistributed*, if for any nonzero integer k

$$\lim_{x\to\infty} \frac{1}{|I_n(x)|} \sum_{(q;\mathbf{p})\in I_n(x)} \exp\left(2\pi i k \cdot a_{(q;\mathbf{p})}\right) = 0. \tag{23}$$

For a Laurent polynomial f in variables $x_1, x_2, \ldots$, we consider the following condition.

Basic Assumption (B): For some variable x_i, there exists only one term in f of highest degree > 0 in x_i.

Proposition 6.1. *Suppose $f(x_1,\ldots,x_{n-1}) \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, \ldots, x_{n-1}^{\pm}]$ satisfies the assumption* (B)*. Then the fractional parts of $f(\mathbf{p})/q$ for $(q;\mathbf{p}) \in I_n$ are equidistributed in the above sense of Weyl.*

We immediately have the following.

Theorem 6.2. *Let $\mathbf{r} = (r_1,\ldots,r_n) \in \mathbb{Z}^n_{>0}$ and suppose $|\mathbf{r}|$ is even. Then the fractional parts of the following generalized Dedekind sums for varying $(q;\mathbf{p}) \in I_n$*

$$\frac{d_{N,n} q^{N-n}}{\mathbf{r}!} s_{\mathbf{r}}(q;\mathbf{p})$$

are equidistributed in $[0,1)$.

The Laurent polynomial $f_{\mathbf{r}}$ of (3), which gives the fractional part of the generalized Dedekind sums $s_{\mathbf{r}}$, satisfies (B). For example, the term of the highest degree in x_1 is $x_1^{N-r_1} x_2^{-r_2} \cdots x_{n-1}^{-r_{n-1}}$ with some nonzero integral coefficient. Thus the theorem follows from the previous proposition.

The proof of Proposition 6.1 consists of three steps. At each step, we estimate the following exponential sums for q a prime, a prime power and any composite number, respectively. Then these together with the estimation of the order of $I_n(x)$ complete the proof. For a positive integer q, let $K(f,q)$ be the following exponential sum.

$$K(f,q) := \sum_{\mathbf{p}} \mathbf{e}_q\left(f\left(\mathbf{p}\right)\right), \tag{24}$$

where the summations are over $\mathbf{p} \in \mathbb{Z}^{n-1}$ with $1 \le p_j < q$ relatively prime to q $(1 \le j \le n-1)$ and $\mathbf{e}_q(x) := \exp(2\pi i x/q)$.

Proposition 6.3. *There exists a constant C_1 depending only on f such that we have for almost all prime q (hence for any prime q if we enlarge C_1),*

$$|K(f,q)| \le C_1 q^{(n-1)-\frac{1}{2}}.$$

Proof. Note the counting gives the trivial estimate $\le q^{n-1}$. When f is a Laurent polynomial in one variable, we apply (3.5.2) in [6] with $X_0 = \mathbb{P}^1$. One can put $C_1 = \nu_0(f) + \nu_\infty(f)$, where $\nu_z(f)$ is the order of the pole of f at z.

In general, suppose f satisfies (B) for x_1. In the sum over $p_1, \ldots, p_{n-1}$ of (24), we apply the above one-variable estimate for the sum over p_1 and apply the counting estimate for the sum over $p_2, \ldots, p_{n-1}$. $\square$

Proposition 6.4. *There exist a constant C_2 and integers $d > 0$, D depending only on f such that for any prime power q relatively prime to D, we have*

$$|K(f, q)| \leq C_2 q^{(n-1)-\frac{1}{3d}}.$$

Proof. Let $q = p^\alpha$ with p a prime and $\alpha \geq 2$. We assume f satisfies (B) for the first variable x_1. As before, for the exponential sum over $p_2, \ldots, p_{n-1}$ we apply the trivial estimate. For the exponential sum over p_1, we apply Lemmas 12.2 and 12.3 of [12] when α is even and odd, respectively. More precisely, suppose $\alpha = 2\beta$ is even. Then we have

$$K(f, p^{2\beta}) = \sum_{p_2, \ldots, p_{n-1}} p^\beta \sum_{p_1} \mathbf{e}_{p^{2\beta}} \left(f\left(p_1, p_2, \ldots, p_{n-1}\right)\right),$$

where the second sum is over the set A_1 of zeroes of $\partial_1 f(x_1, p_2, \ldots, p_{n-1})$ in $(\mathbb{Z}/p^\beta\mathbb{Z})^*$. By Corollary A.2 in Appendix A, we have $|A_1| \leq d^2 p^{\beta - \beta/d}$ where $d = \nu_0(\partial_1 f) + \nu_\infty(\partial_1 f)$ is the constant as in the proof of the last proposition. To apply the corollary, the coefficient D of the highest degree (with respect to x_1) term in $\partial_1 f$ should be relatively prime to p. This excludes only finitely many primes.

Suppose $q = p^{2\beta+1}$ with $\beta \geq 1$. Then we have

$$K(f, p^{2\beta+1}) = \sum_{p_2, \ldots, p_{n-1}} p^\beta \sum_{p_1} \mathbf{e}_{p^{2\beta+1}} \left(f\left(p_1, \ldots, p_{n-1}\right)\right) G_p(p_1, \ldots, p_{n-1}),$$

where the second sum is over the same subset A_1 of $(\mathbb{Z}/p^\beta\mathbb{Z})^*$ as above and $G_p(p_1, \ldots, p_{n-1})$ is an exponential sum over $\mathbb{Z}/p\mathbb{Z}$. Since $|G_p| \leq p$ trivially, we obtain

$$|K(f, p^{2\beta+1})| \leq q^{n-2} p^{\beta+1} |A_1| \leq d^2 q^{(n-2)} p^{2\beta+1-\frac{\beta}{d}} \leq d^2 q^{(n-1)-\frac{1}{3d}}.$$

This completes the proof. $\square$

Proposition 6.5. *Let d and D be as in Proposition 6.4. Then for any $\epsilon > 0$ and any integer $q > 1$ relatively prime to D, we have*

$$|K(f, q)| \ll q^{(n-1)-\frac{1}{3d}+\epsilon}.$$

Proof. From the Chinese remainder theorem, we have

$$K(f, q_1 q_2) = K(f, q_1) K(f, q_2),$$

when $q_1, q_2 > 1$ are relatively prime integers. Hence from the last proposition, we have, for q prime to D,

$$|K(f, q)| \leq (C_2)^{\omega(q)}\, q^{(n-1)-\frac{1}{3d}}, \tag{25}$$

where $\omega(q)$ is the number of prime factors of q and C_2, d, D are as in Proposition 6.4.

For any $\epsilon > 0$, we have for sufficiently large q

$$C_2^{\omega(q)} \ll q^{\epsilon},$$

which follows from the well-known estimate: $\omega(q) \sim \frac{\log q}{\log\log q}$. This completes the proof. □

For $x > 1$ let $\varphi(x) := |(\mathbb{Z}/[x]\mathbb{Z})^*|$ be the Euler's φ-function.

Proposition 6.6. *For any $\epsilon > 0$, we have*

$$|I_n(x)| = \sum_{q<x} \varphi(q)^{n-1} \gg x^{n-\epsilon}.$$

Proof. It is known that for all but finitely many positive integers q,

$$\varphi(q) \geq \frac{q}{e^{\gamma}\log\log q}.$$

Since for any $\epsilon > 0$, there is a positive number C_ϵ such that

$$\log\log q \leq C_\epsilon q^{\epsilon},$$

we have that

$$\sum_{q<x} \varphi(q)^{n-1} \gg \sum_{q<x} \left(\frac{q}{e^{\gamma}\log\log q}\right)^{n-1} \gg \sum_{q<x} q^{(1-\epsilon)(n-1)} \gg x^{n-\epsilon}.$$

□

Now we come to the proof of Proposition 6.1. This will be done by combining previous estimates.

Proof of Proposition 6.1. To estimate $\sum_{0<q<x} |K(f,q)|$, we need to extend the result of Proposition 6.5 to arbitrary integer $q > 1$. For D, d given as in Proposition 6.4, we define a multiplicative arithmetic function η_D by

$$\eta_D(q) := \prod_{\substack{p|D \\ p:prime}} p^{\mathrm{ord}_p q}.$$

Since we have a trivial estimate $|K(f,q)| \leq q^{n-1}$, if we multiply the right hand side of inequalities in Propositions 6.4, 6.5 and (25) by $\eta_D(q)^{1/3d}$, then the inequalities hold for any $q > 1$. By (1.79) in [12], for sufficiently large x, we have

$$\sum_{q<x} \eta_D(q)^{1/3d} \leq x \prod_{p|D} \left(1 - p^{-1+\frac{1}{3d}}\right)^{-1}.$$

By the partial summation, we have

$$\sum_{q<x} q^{(n-1)-\frac{1}{3d}+\epsilon} \cdot \eta_D(q)^{1/3d} \ll x^{n-\frac{1}{3d}+\epsilon}.$$

Together with the last two propositions, this implies

$$\frac{1}{|I_n(x)|} \sum_{0<q<x} |K(f,q)| \to 0 \quad \text{as } x \to \infty.$$

This completes the proof of Proposition 6.1. □

7. Examples

In the following, we present the Laurent polynomials associated to some cases of generalized Dedekind sums for small indices (thus including small dimension). Note that the 2-dimensional case is thoroughly studied in [13]. The cases considered here are generalized Dedekind sums in 3-dimension (i.e., $n = 3$) and Dedekind-Zagier sums (i.e., $\mathbf{r} = (1, 1, \dots, 1)$) in [23].

7.1. Three dimensional Dedekind sums

Let $(r_1, r_2, r_3) = (6, 4, 2)$. In this case, $n = 3$ and $N = 12$. We have $d_{12,3} = 2^{12} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$. Let

$$\begin{aligned} A_{1,2} &= 15202 p_1^{-6} p_2^{-4}, \quad A_{2,3} = 638484 p_1^6 p_2^{-4}, \quad A_{1,3} = 228030 p_1^{-6} p_2^8, \\ A_1 &= 382200 p_1^{-6} p_2^6 + 315315 p_1^{-6} p_2^4 + 143000 p_1^{-6} p_2^2 + 21021 p_1^{-6}, \\ A_2 &= 573300 p_1^4 p_2^{-4} + 189189 p_1^2 p_2^{-4} + 14300 p_2^{-4}, \\ A_3 &= 63063 p_1^2 + 28600 p_2^2. \end{aligned}$$

Then $f_{\mathbf{r}}$ is the sum of all the Laurent polynomials above. The Laurent polynomials supported on faces of the Newton polygon $\Delta_\infty(f_{\mathbf{r}})$ of $f_{\mathbf{r}}$ are (minus of) $A_{1,2}$, $A_{2,3}$, $A_{1,3}$, $A_{1,2} + A_2 + A_{2,3}$, $A_{1,3} + A_1 + A_{1,2}$ and $A_{2,3} + A_3 + A_{1,3}$. The Laurent polynomial $f_{\mathbf{r}}(p_1, p_2, \dots, p_{n-1})$ is easily checked to be nondegenerate with respect to its Newton polygon in the sense of Denef-Loeser (cf. [7]) and the associated exponential sum has actually a stronger Weil type bound.

7.2. Dedekind-Zagier sums

In [23], the Dedekind-Zagier sum $d(p; a_1, \dots, a_n)$ is investigated. It is a cotangent sum defined for a positive integer p and even number of integers $a_1, \dots, a_n$ prime to p. By [23, Theorem in p. 157], it is a special case of the higher dimensional generalized Dedekind sums considered here:

$$d(p; a_1, \dots, a_n) = 2^n p\, s_{1,\dots,1}(p; a'_1, \dots, a'_{n-1}), \tag{26}$$

where a'_i is an integer with $a_n a'_i \equiv -a_i \pmod p$ for $1 \le i \le n-1$. From (26) and Theorem 1.2, we immediately deduce the following.

Proposition 7.1. *Let $p, a_1, a_2, \dots, a_n$ be as above (with n even). And let $d = d_{n,n}$. Then we have*

$$\frac{d}{2^n} d(p; a_1, \dots, a_n) \in \mathbb{Z}.$$

Moreover,

$$\frac{d}{2^n} d(p; a_1, \dots, a_n) \equiv \sum_{\mathbf{m}} (-d) \frac{B_{\mathbf{m}}}{\mathbf{m}!} \prod_{i=1}^{n} a_i^{m_i - 1} \pmod p, \tag{27}$$

where the summation is over the set of n-tuples $\mathbf{m} = (m_1, \ldots, m_n)$ of non-negative even integers with $\sum_{i=1}^n m_i = n$.

Note that the result on the denominator of $d(p; a_1, \ldots, a_n)$ here is not sharp. After all, $d = d_{n,n}$ depends only on n, not on p. Since we consider Dedekind sums for varying p, we need a constant independent of p. A more precise result is given in [23, Theorem in p. 160]. It would be interesting to compare this bound with ours.

For small n, the $d_n = d_{n,n}$ are given by $d_2 = 2^2 \cdot 3,\ d_4 = 2^4 \cdot 3^2 \cdot 5,\ d_6 = 2^6 \cdot 3^3 \cdot 5 \cdot 7,\ d_8 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7,\ d_{10} = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11, \ldots$.

Let $\mathrm{Todd}^N_{\mathrm{ev}}(x_1, \ldots, x_k)$ be the totally even part of N-th Todd polynomial in k variables (i.e., sum of terms which are of even degree in each variable). Then (27) can be written as

$$\frac{d}{2^n} d(p; a_1, \ldots, a_n) \equiv (-d) \frac{\mathrm{Todd}^n_{\mathrm{ev}}(a_1, \ldots, a_n)}{a_1 \cdots a_n} \pmod{p}.$$

For example if $n = 4$, then $d_{n,n} = 720$ and

$$(-720)\, \mathrm{Todd}^4_{\mathrm{ev}}(a_1, \ldots, a_4) = \sum_{i=1}^4 a_i^4 - 5 \sum_{1 \le i < j \le 4} a_i^2 a_j^2.$$

Appendix A. Number of congruence solutions modulo a prime power

In this appendix, we prove a simple estimate of the number of solutions of a polynomial congruence equation modulo a prime power.

Proposition A.1. *Let $f \in \mathbb{Z}[x]$ be an polynomial of degree $d > 0$ and let p be a prime which does not divide the coefficient of x^d. If $r \in \mathbb{Z}/p\mathbb{Z}$ is a root of $f(x) \equiv 0 \mod p$ of multiplicity m, then for $n \ge 2$ we have*

$$|\{z \in \mathbb{Z}/p^n\mathbb{Z} \mid f(z) \equiv 0 \mod p^n,\ z \equiv r \mod p\}| \ \le\ kp^{n - \lceil \frac{n}{m} \rceil},$$

where $1 \le k \le m$ is an integer depending on f and r, and $\lceil a \rceil$ denotes the smallest integer greater than or equal to a.

Proof. When $m = 1$, this is a usual version of Hensel's lemma. We assume $m \ge 2$. And by replacing $f(x)$ by $f(x + \tilde{r})$ ($\tilde{r} \in \mathbb{Z}$ being a lift of r), we may assume $r = 0$. By the polynomial form of Hensel's lemma, there exists a decomposition $f(x) = g(x)h(x)$ in $\mathbb{Z}_p[x]$ lifting $f(x) = x^m \bar{h}(x)$ in $\mathbb{Z}/p\mathbb{Z}[x]$. In other words, there exists polynomials $g(x), h(x) \in \mathbb{Z}_p[x]$ with $f(x) = g(x)h(x)$ such that $g(x)$ is monic, relatively prime to $h(x)$ and $g(x) \equiv x^m \mod p$.

Let $g(x) = g_1(x)^{m_1} \cdots g_k(x)^{m_k}$ be the decomposition as a product of irreducible polynomials in $\mathbb{Q}_p[x]$ (we can take them in $\mathbb{Z}_p[x]$) and for each i, let $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,d_i}$ be the roots of $g_i(x)$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ ($d_i = \deg\ g_i$). Recall that the nonarchimedean norm $|\ |_p$ on $\mathbb{Q}_p$ is canonically extended to $\overline{\mathbb{Q}_p}$ and the same is true for $\mathrm{ord}_p = -\log_p |\cdot|_p$. For $\alpha \in \mathbb{Z}_p$ with $\alpha \equiv 0 \mod p$, we have $|h(\alpha)|_p = 1$ and

$$|f(\alpha)|_p = \prod_{i=1}^{k} |(\alpha - \alpha_{i,1}) \cdots (\alpha - \alpha_{i,d_i})|_p^{m_i} = \prod_{i=1}^{k} |\alpha - \alpha_i|_p^{d_i m_i},$$

where we have put $\alpha_i = \alpha_{i,1}$. Hence if $\mathrm{ord}_p f(\alpha) \geq n$, then there exists $1 \leq i \leq k$ with

$$\mathrm{ord}_p(\alpha - \alpha_i) \geq \frac{n}{d_1 m_1 + \cdots + d_k m_k} = \frac{n}{m}.$$

This determines α modulo $p^{\lceil \frac{n}{m} \rceil}$. Hence the set A of $\alpha \in p\mathbb{Z}_p$ satisfying the above inequality is stable under the translation action of $p^n\mathbb{Z}_p$ and $|A/p^n\mathbb{Z}_p| \leq p^{n-\lceil \frac{n}{m} \rceil}$. □

Corollary A.2. *Let $f \in \mathbb{Z}[x]$ be an polynomial of degree $d > 0$ and let p be a prime which does not divide the coefficient of x^d. Then we have for $n \geq 2$*

$$|\{z \in \mathbb{Z}/p^n\mathbb{Z} \mid f(z) \equiv 0 \mod p^n\}| \ \leq \ clp^{n-\lceil \frac{n}{l} \rceil} \ \leq \ d^2 p^{n-\lceil \frac{n}{d} \rceil},$$

where c is the number of distinct roots of $f(x) \equiv 0 \mod p$ and l is the maximum of multiplicity of these roots.

Appendix B. Iterated constant term and multidimensional residue in general coefficient

The goal of this appendix is to introduce the notion of rational functions with coefficient of a general commutative ring R and to check the definability of the iterated residue of a rational function in several variables in the sense of Parshin ([17]). As the result here is straightforward, we will briefly state the idea how it works.

Definition B.1. A rational function in variables $x_1, \ldots, x_n$ with coefficients in R is an element of the localized ring of $R[x_1, \ldots, x_n]$ by the multiplicative system of nonzero divisor polynomials. We denote the ring of rational functions by $R(x_1, x_2, \ldots, x_n)$.

In one variable case, the Cauchy integral formula applied to a Laurent series $f(x) = \sum_n a_n x^n \in R[[x]][x^{-1}]$ computes the constant term

$$\mathrm{CT}(f) := a_0 = \mathrm{Res}_{x=0}\, f(x) \frac{dx}{x}.$$

Without further assumption, the coefficient of degree n is identified as

$$a_n := \mathrm{Res}_{x=0}\, x^{-n} f(x) \frac{dx}{x}.$$

If R is a field, every rational function has a Laurent series at 0. However, for a rational function with coefficients of a general commutative ring, it is not always the case. We say a rational function is *admissible* if it has a Laurent series. For an admissible function, the constant term is well-defined by the Cauchy integral formula as above.

Here we give a necessary and sufficient condition for a rational function in a variable to be admissible.

Lemma B.2. *A nonzero rational function* $f(x) = \frac{g(x)}{h(x)}$ *is admissible if and only if it has a factorization*

$$f(x) = \frac{1}{x^n}\frac{a(x)}{u(x)}$$

such that $a(x), u(x) \in R[x]$ *with* $a(0) \neq 0$ *and* $u(0) \in R^\times$.

This condition can be stated to a ratio $\frac{g(x)}{h(x)}$ of two power series $g(x), h(x) \in R[[x]]$. It is equivalent to saying $u(x) \in R[[x]]^\times$. Conversely, the previous lemma explains why any rational function with coefficient in a field admits Laurent series expansion.

When there are many variables we need to consider the Parshin's residue. For a formal distribution $\phi(x_1, x_2, \ldots, x_n) = \sum_{I\in\mathbb{Z}^n} a_I x^I \in R[[x_1^\pm, \ldots, x_n^\pm]]$, one can apply the multivariable version of the Cauchy integral formula so as to define the constant term:

$$\mathrm{CT}(\phi) = a_{0\ldots0} = \mathrm{Res}_0\, \phi(x_1, \ldots, x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

As in the 1 variable case, we need to associate a formal distribution to a rational function $f(x_1, \ldots, x_n)$:

$$\iota : f(x_1, \ldots, x_n) \mapsto \sum_{I\in\mathbb{Z}^n} a_I x^I \in R[[x_1^\pm, \ldots, x_n^\pm]].$$

Then the constant term is well-defined as stated above (depending on ι).

$$\mathrm{CT}_\iota(f) = a_{0\ldots0} = \mathrm{Res}_0\, \iota\left(f(x_1, \ldots, x_n)\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

Such an embedding is defined for a complete flag in $\mathbb{A}_k^n$. Equivalently, one may replace the complete flag with a regular sequence of linear forms $(f_1, \ldots, f_n)$ in $R[x_1, \ldots, x_n]$ relative to R (i.e., $M_i := (f_1, \ldots, f_i)/(f_1, \ldots, f_{i-1}) \simeq R$ for $i = 1, \ldots, n$). By *Parshin point* at 0, we mean a complete flag given by a regular sequence of linear forms. When $R = k$ a field, this defines a tower of local fields. This coincides with the notion of higher dimensional local fields.

From now on, consider a Parshin point at 0 given as a regular sequence of coordinate functions: $\mathfrak{A} = (x_1, \ldots, x_n)$.

When $R = k$ a field, for this Parshin point, we have a unique embedding

$$\iota : k(x_1, \ldots, x_n) \hookrightarrow k((x_1))((x_2))\cdots((x_n)) \left(\subset k[[x_1^\pm, \ldots, x_n^\pm]]\right)$$

by iterating completions at each stage:

$$k(x_1, \ldots, x_i) \overset{\iota_i}{\hookrightarrow} k(x_1, \ldots, x_{i-1})((x_i)).$$

Then ι_i yields the constant map $\mathrm{CT}_{x_i=0}$ with respect to x_i:

$$\begin{aligned}\mathrm{CT}_{x_i} f(x_1,\ldots,x_i) &:= \mathrm{CT}_{x_i}\, \iota_i\,(f\,(x_1,\ldots,x_i)) \\ &= \mathrm{Res}_{x_i=0}\, \iota(f)\frac{dx_i}{x_i} = a_0(x_1,\ldots,x_{i-1}),\end{aligned}$$

where $\iota_i f(x_1,\ldots,x_i) = \sum_{k=-N}^{\infty} a_k(x_1,\ldots,x_{i-1})x_i^k$.

The *iterated constant term* with respect to $\mathfrak{A}$ is defined as the iteration of CT_{x_i}:

$$\begin{aligned}&\mathrm{iCT}_{\mathfrak{A}}(f(x_1,x_2,\ldots,x_n)) \\ := \;& \mathrm{CT}_{x_1}\circ \mathrm{CT}_{x_1}\circ\cdots\circ \mathrm{CT}_{x_n} f(x_1,x_2,\ldots,x_n) \\ = \;& \mathrm{Res}_{x_1=0}\circ\cdots\circ \mathrm{Res}_{x_n=0} f(x_1,\ldots,x_n)\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}.\end{aligned} \tag{28}$$

The iterated constant term is strongly dependent on the order in the flag. For example, we may consider the two embeddings of $k(x,y)$ into $k[[x^{\pm},y^{\pm}]]$:

$$\begin{aligned}\iota_{x,y} &: k(x,y)\to k((x))((y)),\\ \iota_{y,x} &: k(x,y)\to k((y))((x)).\end{aligned}$$

Then $\mathrm{iCT}_{x,y}\left(\frac{1}{x-y}\right) = 1$ but $\mathrm{iCT}_{y,x}\left(\frac{1}{x-y}\right) = -1$.

Now consider

$$r(x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}$$

a rational function in coefficient of a commutative ring R.

Definition B.3. For a Parshin point $\mathfrak{A}$ at 0, a rational function is called $\mathfrak{A}$-admissible, if it has a well-defined iterated residue (or equivalently iterated constant term) with respect to $\mathfrak{A}$.

The $\mathfrak{A}$-admissibility can be checked by iterating the condition of Lemma B.2. One can write $g(x_1,\ldots,x_n)$ in a unique way:

$$\begin{aligned}&g(x_1,\ldots,x_n)\\ &= a_0 + a_1(x_1)x_1 + a_2(x_1,x_2)x_2 + a_3(x_1,x_2,x_3)x_3 + \cdots + a_n(x_1,\ldots,x_n)x_n\end{aligned} \tag{29}$$

for $a_i(x_1,\ldots,x_i)\in R[x_1,\ldots,x_i]$.

Theorem B.4. *Let $r(x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)} \in R(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$ be as above. Then $r(x_1,\ldots,x_n)$ is $\mathfrak{A}$-admissible if and only if for the smallest i such that $a_i(x_1,\ldots,x_i)$ is not zero-divisor,*

$$a_0, a_1(x_1)x_1,\ldots,a_{i-1}(x_1,\ldots,x_{i-1})x_{i-1}$$

are nilpotent and $\frac{1}{a_i(x_1,\ldots,x_i)}$ is $\mathfrak{A}$-admissible.

References

[1] T. M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J. **17** (1950), 147–157. http://projecteuclid.org/euclid.dmj/1077476005

[2] M. Brion and M. Vergne, *Lattice points in simple polytopes*, J. Amer. Math. Soc. **10** (1997), no. 2, 371–392. https://doi.org/10.1090/S0894-0347-97-00229-4

[3] L. Carlitz, *Some theorems on generalized Dedekind sums*, Pacific J. Math. **3** (1953), 513–522. http://projecteuclid.org/euclid.pjm/1103051325

[4] R. Chapman, *Reciprocity laws for generalized higher dimensional Dedekind sums*, Acta Arith. **93** (2000), no. 2, 189–199. https://doi.org/10.4064/aa-93-2-189-199

[5] R. Dedekind, *Erläuterungen zu zwei Fragmenten von Riemann*, Riemann's Gesammelte Mathematische Werke, 2nd edition, 1892.

[6] P. Deligne, *Applications de la formule des traces aux sommes trigonométriques*, in Cohomologie étale, 168–232, Lecture Notes in Math., **569**, Springer, Berlin, 1977.

[7] J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math. **106** (1991), no. 2, 275–294. https://doi.org/10.1007/BF01243914

[8] S. Garoufalidis and J. E. Pommersheim, *Values of zeta functions at negative integers, Dedekind sums and toric geometry*, J. Amer. Math. Soc. **14** (2001), no. 1, 1–23. https://doi.org/10.1090/S0894-0347-00-00352-0

[9] G. van der Geer, *Hilbert Modular Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **16**, Springer-Verlag, Berlin, 1988. https://doi.org/10.1007/978-3-642-61553-5

[10] D. Hickerson, *Continued fractions and density results for Dedekind sums*, J. Reine Angew. Math. **290** (1977), 113–116. https://doi.org/10.1515/crll.1977.290.113

[11] S. Hu and D. Solomon, *Properties of higher-dimensional Shintani generating functions and cocycles on* $\mathrm{PGL}_3(\mathbf{Q})$, Proc. London Math. Soc. (3) **82** (2001), no. 1, 64–88. https://doi.org/10.1112/S0024611500012612

[12] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, **53**, American Mathematical Society, Providence, RI, 2004. https://doi.org/10.1090/coll/053

[13] B. Jun and J. Lee, *Equidistribution of generalized Dedekind sums and exponential sums*, J. Number Theory **137** (2014), 67–92. https://doi.org/10.1016/j.jnt.2013.10.020

[14] ______, *Special values of partial zeta functions of real quadratic fields at nonpositive integers and the Euler-Maclaurin formula*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 7935–7964. https://doi.org/10.1090/tran/6679

[15] C. Meyer, *Die Berechnung der Klassenzahl Abelscher Körper über quadratischen Zahlkörpern*, Akademie-Verlag, Berlin, 1957.

[16] G. Myerson, *Dedekind sums and uniform distribution*, J. Number Theory **28** (1988), no. 3, 233–239. https://doi.org/10.1016/0022-314X(88)90039-X

[17] A. N. Paršin, *On the arithmetic of two-dimensional schemes. I. Distributions and residues*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 4, 736–773, 949.

[18] J. E. Pommersheim, *Barvinok's algorithm and the Todd class of a toric variety*, J. Pure Appl. Algebra **117/118** (1997), 519–533. https://doi.org/10.1016/S0022-4049(97)00025-X

[19] H. Rademacher and E. Grosswald, *Dedekind Sums*, The Mathematical Association of America, Washington, DC, 1972.

[20] C. L. Siegel, *Bernoullische Polynome und quadratische Zahlkörper*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1968), 7–38.

[21] I. Vardi, *A relation between Dedekind sums and Kloosterman sums*, Duke Math. J. **55** (1987), no. 1, 189–197. https://doi.org/10.1215/S0012-7094-87-05510-4

[22] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), no. 3, 313–352.

[23] D. Zagier, *Higher dimensional Dedekind sums*, Math. Ann. **202** (1973), 149–172. https://doi.org/10.1007/BF01351173

[24] ———, *Valeurs des fonctions zêta des corps quadratiques réels aux entiers négatifs*, in Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976), 135–151. Astérisque 41–42, Soc. Math. France, Paris, 1977.

Hi-joon Chae
Department of Mathematics Education
Hongik University
Seoul 04066, Korea
Email address: hchae@hongik.ac.kr

Byungheup Jun
Department of Mathematical Sciences
Ulsan National Institute of Science and Technology
Ulsan 44919, Korea
Email address: bhjun@unist.ac.kr

Jungyun Lee
Department of Mathematics Education
Kangwon National University
Chuncheon 24341, Korea
Email address: lee9311@kangwon.ac.kr