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INFINITELY MANY SOLUTIONS FOR FRACTIONAL SCHRÖDINGER EQUATION WITH SUPERQUADRATIC CONDITIONS OR COMBINED NONLINEARITIES

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Abstract. We obtain infinitely many solutions for a class of fractional Schrödinger equation, where the nonlinearity is superquadratic or involves a combination of superquadratic and subquadratic terms at infinity. By using some weaker conditions, our results extend and improve some existing results in the literature.

1. Introduction

In this paper, we consider the following fractional equation

$$(\mathcal{FS}) \qquad (-\Delta)^s u + V(x)u = f(x,u), \ x \in \mathbb{R}^N,$$

where $s \in]0,1[$, $N > 2s$, $(-\Delta)^s$ stands for the fractional Laplacian, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

Equation $(\mathcal{FS})$ arises in the study of the fractional Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + (-\Delta)^s \psi + V(x)\psi = f(x,\psi), \ x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

when looking for standing waves, that is, solutions with the form $\psi(x,t) = e^{i\omega t}u(x)$, where ω is a constant. This equation was introduced by Luskin [19,20] and comes from an expansion of the Feynman path integral and from Brownian-like to Levy-like quantum mechanical paths. In [11], the authors have proved that $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$ as $s \to 1$ (see Proposition 4.4 in [11]).

When $s = 1$, formally equation $(\mathcal{FS})$ reduces to the classical Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \ x \in \mathbb{R}^N. \tag{1.2}$$

Over the past decades, with the aid of critical point theory and variational methods, for various conditions on the potential $V(x)$ and the nonlinearity

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$f(x,u)$, the existence and multiplicity of nontrivial solutions for the equation (1.2) have been extensively investigated in the literature, see for example [5, 6, 22, 25, 29] but we do not even try to review the huge bibliography. Recently there has been an increasing interested in the study of equation $(\mathcal{FS})$, from a pure mathematical point of view as well as from concrete applications, since this equation naturally arises in several fields of research like phase transitions, finance, stratified materials, flame propagation, ultra-relativistic limits of quantum mechanics and water waves. For more detailed introductions and applications, we refer the reader to [10, 15, 28].

With the aid of variational methods, some authors have studied the existence and multiplicity of nontrivial solutions for equation $(\mathcal{FS})$ by assuming various conditions on the potential $V(x)$ and the nonlinearity $f(x,u)$, see for example [1–3, 7–9, 12–14, 16–18, 23, 24, 26, 27, 31–34] and the references cited therein. Felmer et al. [14] considered the existence and regularity of positive solution of $(\mathcal{FS})$ with $V(x) = 1$ and $s \in]0,1[$ when $f(x,u)$ has subscritical growth and satisfies the so-called global Ambrosetti-Rabinowitz ((AR) in short) superquadratic condition. That is, there exist constants $\nu > 2$ and $r > 0$ such that

$$0 < \nu F(x,u) \leq f(x,u)u, \ \forall x \in \mathbb{R}^N, \ |u| \geq r,$$

where $F(x,u) = \int_0^u f(x,t)dt$. Secchi [22] obtained the existence of ground state solution of $(\mathcal{FS})$ for $s \in]0,1[$ when $V(x) \to +\infty$ as $|x| \to +\infty$ and (AR) condition holds, by using the Ekeland's Variational Principle and the Mountain Pass Theorem. In [26], Teng used Variant Fountain Theorems and the $\mathbb{Z}_2$ version of Mountain Pass Theorem to establish the existence of infinitely many nontrivial high-energy or small energy solutions for $(\mathcal{FS})$. In this paper, the nonlinearity is allowed to be superquadratic and the potential $V(x)$ was assumed to satisfy the following conditions

(V_1)
$$\inf_{x\in\mathbb{R}^N} V(x) > 0;$$

(V_2) For any $M > 0$, there exists a constant $r > 0$ such that

$$\lim_{|y|\to\infty} meas\left\{x \in \mathbb{R}^N : |x-y| \leq r, V(x) \leq M\right\} = 0,$$

where $meas$ denotes the Lebesgue measure on the whole axis $\mathbb{R}^N$. J. Zhang and W. Jiang [33] dealt with the case that $f(x,u)$ is subquadratic and the potential V satisfies the following condition

(V_3)
$$V \geq 0, \exists b > 0 / V_b = \left\{x \in \mathbb{R}^N : V(x) \leq b\right\} \neq \phi, \ meas(V_b) < \infty,$$
$$\Omega = int V^{-1}(\{0\}) \neq \phi \text{ and } \overline{\Omega} = V^{-1}(\{0\});$$

and obtained the existence of solution for $(\mathcal{FS})$ by using a minimization theorem.

Motivated by the previous papers, in this paper, we will establish some new existence criteria to guarantee that problem $(\mathcal{FS})$ has infinitely many solutions under some assumptions on $f(x,u)$, which are different from the (AR) condition. We are also concerned with the multiplicity of solutions of $(\mathcal{FS})$ under combined nonlinearities cases. Firstly, we deal with the case that $f(x,u)$ is superquadratic at infinity. Consider the following hypotheses:

(V_1')
$$\inf_{x\in\mathbb{R}^N} V(x) > -\infty;$$

(H_1) There exist constants $a, b > 0$ and $2 < p < 2_s^*$ such that
$$|f(x,u)| \leq a\,|u| + b\,|u|^{p-1}\,,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(H_2) There exists a constant $r > 0$ such that $F(x,u) \geq 0$ for all $x \in \mathbb{R}^N$ and $|u| \geq r$, and
$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = +\infty \text{ for a.e. } x \in \mathbb{R}^N;$$

(H_3)
$$F(x,-u) = F(x,u),\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(H_4) There exist constants $C \geq 0$ and $\sigma > \frac{2_s^*}{2_s^*-2}$ such that
$$f(x,u)u - 2F(x,u) \geq 0,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$$
and
$$|F(x,u)|^\sigma \leq C\,|u|^{2\sigma}\,[f(x,u)u - 2F(x,u)],\ \forall x \in \mathbb{R}^N,\ |u| \geq r,$$
where $2_s^* = \frac{2N}{N-2s}$ is the so-called "fractional critical exponent" and r is the constant given in (H_2);

(H_4') There exist constants $\mu > 2$ and $\gamma > 0$ such that
$$\mu F(x,u) \leq f(x,u)u + \gamma u^2,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Our first main results read as follows.

Theorem 1.1. *Assume that (V_1'), (V_2) and (H_1)-(H_4) hold. Then $(\mathcal{FS})$ has infinitely many nontrivial solutions.*

Theorem 1.2. *Assume that (V_1'), (V_2), (H_1)-(H_3) and (H_4') hold. Then $(\mathcal{FS})$ has infinitely many nontrivial solutions.*

Remark 1.3. Most of the above mentioned papers treat the superquadratic case under the (AR) condition. Without this condition, we do not know whether a Palais-Smale sequence is bounded. Note that the (AR) condition is not required in our Theorem 1.1. In fact, let
$$F(x,u) = a(x)\Big[(4\,|u|^2 - 1)\ln(\frac{1}{2} + |u|) - 2(\frac{1}{2} + |u|)^2 + 4\,|u| + \frac{1}{2} - \ln 2\Big],$$
where $a \in C(\mathbb{R}^N, \mathbb{R})$ is such that $0 < \inf_{x\in\mathbb{R}^N} a(x) \leq \sup_{x\in\mathbb{R}^N} a(x) < +\infty$. It is clear that $f(x,u) = \frac{\partial F}{\partial u}(x,u)$ satisfies (H_1)-(H_3). It remains to verify (H_4). A classical computation shows that
$$f(x,u)u - 2F(x,u) = a(x)\Big[(4\,|u|^2 - 1)\frac{2\,|u|}{2\,|u| + 1} - 2\,|u| + 2\ln(\frac{1}{2} + |u|) + 2\ln 2\Big].$$

It is easy to check that $f(x,u)u-2F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Moreover, for all $\sigma > \frac{2_s^*}{2_s^*-2}$, we have

$$\Big(\frac{F(x,u)}{|u|^2}\Big)^{2\sigma}[f(x,u)u-2F(x,u)]^{-1} \cong_\infty (4a(x))^{\sigma-1}\frac{\Big(\ln(\frac{1}{2}+|u|)\Big)^\sigma}{|u|^2},$$

which converges to 0 as $|u| \to \infty$, uniformly in $x \in \mathbb{R}^N$. Hence there exist two positive constants r, C such that

$$\Big(\frac{F(x,u)}{|u|^2}\Big)^{2\sigma} \leq C[f(x,u)u-2F(x,u)],\ \forall x \in \mathbb{R}^N,\ |u| \geq r.$$

Therefore (H_4) holds. However, we check easily that the (AR)-condition can not be satisfied for any $\nu > 2$. By Theorem 1.1, the corresponding fractional Schrödinger equation $(\mathcal{FS})$ possesses infinitely many nontrivial solutions.

Remark 1.4. Theorem 1.2 generalizes Theorem 1.3 in [13]. In fact, hypothesis (H_2) is weaker than hypothesis (f_5) in [13, Theorem 1.3].

Remark 1.5. From (V_1'), we know that there exists a positive constant v_0 such that $\inf_{x\in\mathbb{R}^N} V(x)+v_0 > 0$. Let $\overline{V}(x) = V(x)+v_0$ and $\overline{f}(x,u) = f(x,u)+v_0u$. Consider the following fractional Schrödinger equation

$$(\overline{\mathcal{FS}}) \qquad (-\Delta)^s u + \overline{V}(x)u = \overline{f}(x,u),\ x \in \mathbb{R}^N.$$

It is easy to check that the hypotheses (H_1)-(H_4), (H_4') still hold for $\overline{f}(x,u)$ provided that those hold for $f(x,u)$, and $\overline{V}$ satisfies the conditions (V_1), (V_2). Hence, in our above results, we will always assume without loss of generality that V satisfies (V_1) instead of (V_1').

Next, consider the equation $(\mathcal{FS})$ involving a combination of superquadratic and subquadratic terms at infinity. More precisely, assume that $f(x,u)$ is of the type $f(x,u) = g(x,u)+h(x,u)$, where $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions and take the following conditions.

(H_5) There exist constants $1 < \gamma < 2$, $1 < \sigma < 2$ and functions $c_0, a \in L^{\frac{2}{2-\gamma}}(\mathbb{R}^N, \mathbb{R}^+)$ and $b \in L^{\frac{2}{2-\sigma}}(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$c_0(x)\,|u|^\gamma \leq g(x,u)u,\ |g(x,u)| \leq a(x)\,|u|^{\gamma-1}+b(x)\,|u|^{\sigma-1},\ \text{a.e. } x \in \mathbb{R}^N, \forall u \in \mathbb{R};$$

(H_6) $H(x,u) = \int_0^u h(x,s)ds \geq 0$ and there exist $2 < \mu < \frac{2_s^*}{2}+1$, $c \in L^2(\mathbb{R}^N, \mathbb{R}^+)$ and $d \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$|h(x,u)| \leq c(x)+d(x)\,|u|^{\mu-1},\ \text{a.e. } x \in \mathbb{R}^N, \forall u \in \mathbb{R};$$

(H_7) There exist $\rho > 2$, $1 < \delta < 2$ and $\theta \in C(\mathbb{R}^N, \mathbb{R}^+) \bigcap L^{\frac{2}{2-\delta}}(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$\rho H(x,u) - h(x,u)u \leq \theta(x)\,|u|^\delta,\ \text{a.e. } x \in \mathbb{R}^N, \forall u \in \mathbb{R}.$$

Theorem 1.6. *Assume that* (V_1), (V_2), (H_3) *and* (H_5)-(H_7) *are satisfied. Then the equation* $(\mathcal{FS})$ *possesses infinitely many small energy solutions.*

Remark 1.7. Obviously, Theorem 1.6 generalizes Theorem 1.2 in [13], Theorem 1.1 in [33] and Theorem 1.2 in [34]. In fact, taking $h(x,u)=0$, Theorem 1.2 in [13] and Theorem 1.1 in [33] become special cases of Theorem 1.6. Similarly, taking $g(x,u)=0$, Theorem 1.2 in [34] becomes a special case of Theorem 1.6.

Example 1.8. Let $F(x,u)=G(x,u)+H(x,u)$ where

$$G(x,u)=\Big(\frac{1}{1+|x|^2}\Big)^{\frac{N}{3}}|u|^{\frac{4}{3}}+\Big(\frac{1}{1+|x|^2}\Big)^{\frac{N}{6}}|u|^{\frac{5}{3}},$$

$$H(x,u)=\Big(\frac{1}{1+|x|^2}\Big)^{N}\Big[|u|^{\frac{4}{3}}\ln(1+|u|)+|u|^3\Big].$$

By a classical computation, we check that (H_3), (H_5)-(H_7) are satisfied. Hence the corresponding equation $(\mathcal{FS})$ possesses infinitely many solutions.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of Theorems 1.1, 1.2. Section 4 is devoted to the proof of Theorem 1.6.

2. Preliminaries

In the sequel, s will denote a fixed number, $s\in]0,1[$, we denote by $\|\cdot\|_q$ the usual norm of the space $L^q(\mathbb{R}^N)$. In terms of finite differences, the nonhomogeneous Sobolev space can be defined as follows

$$H^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\right\}.$$

This space is endowed with the natural norm

$$\|u\|_{H^s}=\left(\int_{\mathbb{R}^N}|u|^2\,dx+\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}},$$

while

$$|u|_{H^s}=\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}$$

is the Gagliardo (semi) norm. The space $H^s(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}^N}u(x)v(x)dx+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy.$$

The space $H^s(\mathbb{R}^N)$ can be described by means of the Fourier transform. Indeed, it is defined by

$$H^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):\int_{\mathbb{R}^N}(1+|\xi|^2)^s\,|\hat{u}(\xi)|^{2s}\,d\xi<\infty\right\}$$

and the norm can be equivalently written by

$$\|u\|_{H^s} = \Big(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} |u(x)|^2 \, dx \Big)^{\frac{1}{2}}.$$

Furthermore, we know that $\|\cdot\|_{H^s}$ is equivalent to the norm

$$\|u\|_s = \Big(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u(x)\right|^2 dx + \int_{\mathbb{R}^N} |u(x)|^2 \, dx \Big)^{\frac{1}{2}}.$$

Lemma 2.1 ([11])**.** *Let $s \in]0,1[$ such that $2s < N$. Then there exists a positive constant $C = C(N,s)$ such that for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\|u\|_{2_s^*} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

Consequently, the space $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$. Moreover, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is locally compact whenever $q \in [2, 2_s^*[$.

In this section, we assume that V satisfies (V_1) and we consider the subspace

$$E = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) |u(x)|^2 < \infty \right\}.$$

Then E is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi) \hat{v}(\xi) d\xi + \int_{\mathbb{R}^N} V(x) u(x) v(x) dx$$

and the associated norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) |u(x)|^2 \, dx \Big)^{\frac{1}{2}}.$$

Furthermore, we know that $\|\cdot\|$ is equivalent to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} [\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + V(x) |u(x)|^2] dx \right)^{\frac{1}{2}}$$

and the corresponding inner product is

$$\langle u, v \rangle = \int_{\mathbb{R}^N} [(-\Delta)^{\frac{s}{2}} u(x) (-\Delta)^{\frac{s}{2}} v(x) + V(x) u(x) v(x)] dx.$$

In view of the assumptions (V_1) and (V_2), some compactness embedding can be deduced as follows.

Lemma 2.2 ([26])**.** *The Hilbert space E is continuously embedded into $L^q(\mathbb{R}^N)$ for $2 \leq q \leq 2_s^*$ and compactly embedded into $L^q(\mathbb{R}^N)$ for $2 \leq q < 2_s^*$.*

It follows directly from Lemma 2.2 that there are constants $\eta_q > 0$ such that

$$\|u\|_q \leq \eta_q \|u\|, \ \forall u \in E, \ \forall q \in [2, 2_s^*]. \quad (2.1)$$

Definition. We say $u \in E$ is a weak solution of $(\mathcal{FS})$ if

$$\int_{\mathbb{R}^N} |\xi|^{2s}\, \hat{u}(\xi)\hat{v}(\xi)d\xi + \int_{\mathbb{R}^N} V(x)u(x)v(x)dx = \int_{\mathbb{R}^N} f(x,u)dx,\ \forall v \in E.$$

In order to prove our main results, we shall use the following critical point theorems.

Definition. Let E be an infinite dimensional Banach space. We say that $\psi \in C^1(E,\mathbb{R})$ satisfies the

a) Palais-Smale condition at level c (we denote $(PS)_c$ condition in short) if any sequence $(u_n) \subset E$ satisfying

$$\psi(u_n) \to c \text{ and } \psi'(u_n) \to 0 \text{ as } n \to \infty$$

possesses a convergent subsequence,

b) Cerami's condition at level c (we denote $(C)_c$ condition in short) if any sequence $(u_n) \subset E$ satisfying

$$\psi(u_n) \to c,\ \|\psi'(u_n)\|\,(\|u_n\| + 1) \to 0 \text{ as } n \to \infty$$

possesses a convergent subsequence.

Lemma 2.3 (Symmetric Mountain Pass Theorem, [21])**.** *Let $E = Y \oplus Z$ be an infinite dimensional space where Y is finite dimensional. If $\psi \in C^1(E,\mathbb{R})$ satisfies the $(PS)_c$-condition for all level $c > 0$ and*

(1) $$\psi(0) = 0,\ \psi(-u) = \psi(u),\ \forall u \in E;$$

(2) $$\psi_{|\partial B_\rho \cap Z} \geq \alpha \text{ for some } \rho, \alpha > 0;$$

(3) *for any finite dimensional subspace $\tilde{E} \subset E$, there is a positive constant $R = R(\tilde{E})$ such that $\psi(u) \leq 0$ on $\tilde{E} \setminus B_R$.*

Then ψ possesses an unbounded sequence of critical values.

Remark 2.4. As shown in [4], a deformation lemma can be proved with $(C)_c$-condition replacing the $(PS)_c$-condition, and it turns out that Lemma 2.3 still holds true with the $(C)_c$-condition instead of the $(PS)_c$-condition.

Now, let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$, where X_j is a finite dimensional subspace of E. For each $k \in \mathbb{N}$, let $Y_k = \oplus_{j=0}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. The functional $\psi \in C^1(E,\mathbb{R})$ is said to satisfy the $(PS)^*$ condition if for any sequence (u_j) for which $(\psi(u_j))$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \to \infty$ and $(\psi_{|Y_{k_j}})'(u_j) \to 0$ as $j \to \infty$, has a subsequence converging to a critical point of ψ.

Lemma 2.5 (Dual Fountain Theorem, [30])**.** *Suppose that the functional $\psi \in C^1(E,\mathbb{R})$ is even and satisfies the $(PS)^*$ condition. Assume that for each sufficiently large integer k, there exist $0 < r_k < \rho_k$ such that*

(a) $$a_k = \inf_{u\in Z_k, \|u\|=\rho_k} \psi(u) \geq 0;$$

(b) $$b_k = \max_{u \in Y_k, \|u\| = r_k} \psi(u) < 0;$$

(c) $$d_k = \inf_{u \in Z_k, \|u\| \le \rho_k} \psi(u) \to 0 \ as\ k \to \infty.$$

Hence ψ has a sequence of negative critical values converging to zero.

3. Proof of Theorems 1.1, 1.2

Consider the functional ψ associated to the equation $(\mathcal{FS})$

$$\begin{aligned}\psi(u) &= \frac{1}{2}\Big(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 + \int_{\mathbb{R}^N} V(x) |u(x)|^2 \, dx\Big) - \int_{\mathbb{R}^N} F(x,u) dx \\ &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(x,u) dx\end{aligned}$$

defined on the space E introduced in Section 2. Under the assumptions of Theorems 1.1, 1.2, it is well-known that $\psi \in C^1(E, \mathbb{R})$ and critical points of ψ are solutions of $(\mathcal{FS})$.

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of E, we set

$$Y_m = span\{e_1, \dots, e_m\}, \ Z_m = \overline{span\{e_{m+1}, \dots\}}, \ m \in \mathbb{N}.$$

Then $E = Y_m \oplus Z_m$.

Lemma 3.1. *Assume that (V_1), (V_2) and (H_1) are satisfied. Then there exist positive constants m_0, α, ρ such that*

$$\psi_{|\partial B_\rho \cap Z_{m_0}} \ge \alpha.$$

Proof. Note that by (H_1), we have

$$|F(x,u)| \le \frac{a}{2} |u|^2 + \frac{b}{p} |u|^p, \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.1}$$

For any $m \in \mathbb{N}$, let

$$l_2(m) = \sup_{u \in Z_m \setminus \{0\}} \frac{\|u\|_2}{\|u\|} \text{ and } l_p(m) = \sup_{u \in Z_m \setminus \{0\}} \frac{\|u\|_p}{\|u\|}. \tag{3.2}$$

It is clear that $l_2(m+1) \le l_2(m)$, so $l_2(m) \to l \ge 0$ as $m \to \infty$. For any $m \in \mathbb{N}$, there exists $u_m \in Z_m$ such that $\|u_m\| = 1$ and $\|u_m\|_2 \ge \frac{1}{2} l_2(m)$. By the definition of Z_m, $u_m \rightharpoonup 0$ in E. By Lemma 2.2, we can assume that $u_m \to 0$ in $L^2(\mathbb{R}^N)$. Hence, we have $l = 0$, that is $l_2(m) \to 0$ as $m \to \infty$. Similarly $l_p(m) \to 0$ as $m \to \infty$. Therefore, we can choose a larger integer m_0 such that

$$\|u\|_2^2 \le \frac{1}{2a} \|u\|^2, \ \|u\|_p^p \le \frac{p}{4b} \|u\|^p, \ \forall u \in Z_{m_0}. \tag{3.3}$$

Then by (3.1) and (3.3), we have

$$\begin{aligned}\psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} (\frac{a}{2}|u|^2 + \frac{b}{p}|u|^p)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \frac{b}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|^2 - \frac{1}{4}\|u\|^p \\ &= \frac{1}{4}(\|u\|^2 - \|u\|^p) \\ &= \frac{2^{p-2}-1}{2^{p+2}} = \alpha,\ \forall u \in Z_{m_0},\ \|u\| = \frac{1}{2} = \rho,\end{aligned}$$

which finish the proof of Lemma 3.1. □

To apply Lemma 2.3, we will take $E = Y \oplus Z$ with $Y = Y_{m_0}$ and $Z = Z_{m_0}$, where m_0 is introduced in Lemma 3.1.

Lemma 3.2. *Assume that* (V_1), (V_2), (H_1) *and* (H_2) *are satisfied. Then for any finite dimensional subspace* $\tilde{E} \subset E$, *there is a constant* $R = R(\tilde{E}) > 0$ *such that*

$$\psi(u) \leq 0,\ \forall u \in \tilde{E},\ \|u\| \geq R. \tag{3.4}$$

Proof. In order to prove (3.4), we only need to prove

$$\psi(u) \to -\infty \text{ as } \|u\| \to \infty,\ u \in \tilde{E}. \tag{3.5}$$

Assume by contradiction that there exists a sequence $(u_n) \subset \tilde{E}$ with $\|u_n\| \to \infty$ as $n \to \infty$ and $\psi(u_n) \geq -M$ for some constant $M > 0$, $\forall n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Going to a subsequence if necessary, we can assume that $v_n \rightharpoonup v$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v$ in E, thus $\|v\| = 1$, and up to a subsequence, Lemma 2.2 implies that $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$. Let

$$\Lambda_n(c,d) = \left\{x \in \mathbb{R}^N : c \leq |u_n(x)| < d\right\},\ 0 \leq c < d$$

and

$$\Lambda = \left\{x \in \mathbb{R}^N : v(x) \neq 0\right\}.$$

For any $x \in \Lambda$, we have $\lim_{n\to\infty} |u_n(x)| = \lim_{n\to\infty} \|u_n\| |v_n(x)| = +\infty$. Hence $x \in \Lambda_n(r,\infty)$ for n large enough, where r is the constant given in (H_2). Property (3.1), Lemma 2.2, assumption (H_2) and Fatou's lemma imply

$$\begin{aligned}0 = \lim_{n\to\infty} \frac{-M}{\|u_n\|^2} &\leq \frac{\psi(u_n)}{\|u_n\|^2} \\ &= \lim_{n\to\infty}[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2 dx] \qquad (3.6)\end{aligned}$$

$$= \lim_{n\to\infty}[\frac{1}{2} - \int_{\Lambda_n(0,r)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx - \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx]$$
$$\le \limsup_{n\to\infty}[\frac{1}{2} + (\frac{a}{2} + \frac{b}{p}r^{p-2})\int_{\mathbb{R}^N}|v_n|^2\,dx - \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx]$$
$$\le \frac{1}{2} + (\frac{a}{2} + \frac{b}{p}r^{p-2})\eta_2^2 - \liminf_{n\to\infty}\int_{\mathbb{R}^N} \frac{F(x,u_n)}{|u_n|^2}\chi_{\Lambda_n(r,\infty)}|v_n|^2\,dx$$
$$\le \frac{1}{2} + (\frac{a}{2} + \frac{b}{p}r^{p-2})\eta_2^2 - \int_{\mathbb{R}^N}\liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^2}\chi_{\Lambda_n(r,\infty)}|v_n|^2\,dx$$
$$= -\infty,$$

where η_2 is given in (2.1). It is a contradiction. Hence (3.5) is satisfied and the proof of Lemma 3.2 is finished. □

3.1. Proof of Theorem 1.1

By (H_3) and Lemmas 3.1, 3.2, ψ satisfies the conditions (1), (2) and (3) of Lemma 2.3. It remains to prove the Cerami's condition.

Lemma 3.3. *Assume that (V_1), (V_2), (H_1), (H_2) and (H_4) are satisfied. Then ψ satisfies the $(C)_c$-condition for any level $c > 0$.*

Proof. Let c be a positive real number and $(u_n) \subset E$ be a $(C)_c$-sequence, that is

$$\psi(u_n) \to c \text{ and } \|\psi'(u_n)\|(1 + \|u_n\|) \to 0 \text{ as } n \to \infty.$$

Assume by contradiction that (u_n) is not bounded, then up to a subsequence, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Taking a subsequence if necessary, then $v_n \rightharpoonup v$ in E and Lemma 2.2 implies that $v_n \to v$ in $L^q(\mathbb{R}^N)$ for $q = 2, p, 2\sigma' = \frac{2\sigma}{\sigma-1}$ and $v_n \to v$ a.e. on $\mathbb{R}^N$.

If $v \neq 0$, Hölder's inequality implies as above

$$0 = \lim_{n\to\infty}\frac{\psi(u_n)}{\|u_n\|^2} = \lim_{n\to\infty}[\frac{1}{2} - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx] \le -\infty,$$

which is a contradiction. So (u_n) is bounded.

If $v = 0$, on one hand, since $\psi(u_n) \to c$ and $\|u_n\| \to \infty$, then it is easy to see that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx \ge \frac{1}{2}. \tag{3.7}$$

On the other hand, for the constant r given in (H_4), (3.1) implies

$$\begin{aligned}\int_{\Lambda_n(0,r)}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx &\le (\frac{a}{2} + \frac{b}{p}r^{p-2})\int_{\Lambda_n(0,r)}|v_n|^2\,dx \\ &\le (\frac{a}{2} + \frac{b}{p}r^{p-2})\|v_n\|_2^2 \to 0.\end{aligned} \tag{3.8}$$

Now, there exists a positive constant c_1 such that for all integer n, one has

$$\int_{\mathbb{R}^N} [\frac{1}{2} f(x,u_n)u_n - F(x,u_n)]dx = \psi(u_n) - \frac{1}{2}\psi'(u_n)u_n \le c_1$$

which with Hölder's inequality and assumption (H_4) implies

$$\begin{aligned} &\int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx \\ &\le (\int_{\Lambda_n(r,\infty)} (\frac{F(x,u_n)}{|u_n|^2})^{\sigma} dx)^{\frac{1}{\sigma}} (\int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'} \, dx)^{\frac{1}{\sigma'}} \\ &\le (2C)^{\frac{1}{\sigma}} (\int_{\Lambda_n(r,\infty)} [\frac{1}{2} f(x,u_n)u_n - F(x,u_n)]dx)^{\frac{1}{\sigma}} (\int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'} \, dx)^{\frac{1}{\sigma'}} \\ &\le (2C(c_1))^{\frac{1}{\sigma}} \|v_n\|_{2\sigma'}^2 \to 0. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx = &\int_{\Lambda_n(0,r)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx + \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx \\ &\to 0 \end{aligned}$$

which contradicts (3.7). Hence (u_n) is bounded.

Up to a subsequence, Lemma 2.2 implies that $u_n \to u$ in both $L^2(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. It follows from (H_1) and Hölder's inequality that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(x,u_n)(u_n - u)dx \right| &\le \int_{\mathbb{R}^N} (a|u_n| + b|u_n|^{p-1}) |u_n - u| \, dx \\ &\le a \int_{\mathbb{R}^N} |u_n| |u_n - u| \, dx + b \int_{\mathbb{R}^N} |u_n|^{p-1} |u_n - u| \, dx \\ &\le a \|u_n\|_2 \|u_n - u\|_2 + b \|u_n\|_p^{p-1} \|u_n - u\|_p \to 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \psi'(u_n)(u_n - u) \\ &= \lim_{n\to\infty} \langle u_n, u_n - u \rangle - \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n)(u_n - u)dx \\ &= \lim_{n\to\infty} \|u_n\|^2 - \|u\|^2 . \end{aligned}$$

That is $\lim_{n\to\infty} \|u_n\|^2 = \|u\|^2$, which with $u_n \rightharpoonup u$ in E implies

$$\|u_n - u\|^2 = \langle u_n - u, u_n - u \rangle \to 0.$$

Hence (u_n) possesses a convergent subsequence in E. Thus ψ satisfies the $(C)_c$-condition. The proof of Lemma 3.3 is completed. □

Consequently, Lemma 2.3 with Remark 2.4 imply that the functional ψ possesses an unbounded sequence of critical points. Therefore, the fractional

Schrödinger equation $(\mathcal{FS})$ possesses infinitely many solutions. The proof of Theorems 1.1 is finished.

3.2. Proof of Theorem 1.2

Lemma 3.4. *Assume that* (V_1), (V_2) *and* (H_4') *are satisfied. Then for all positive constant* c, ψ *satisfies the* $(C)_c$*-condition.*

Proof. Let c be a positive real number and $(u_n) \subset E$ be a $(C)_c$-sequence, that is

$$\psi(u_n) \to c \text{ and } \|\psi'(u_n)\| (1 + \|u_n\|) \to 0 \text{ as } n \to \infty.$$

Assume by contradiction that (u_n) is not bounded, then up to a subsequence, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By (H_4'), there exists a positive constant c_2 such that for all integer n, one has

$$\begin{aligned} c_2 &\geq \psi(u_n) - \frac{1}{\mu}\psi'(u_n)u_n \\ &= \frac{\mu-2}{2\mu}\|u_n\|^2 + \int_{\mathbb{R}^N} [\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)]dx \\ &\geq \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\gamma}{\mu}\|u_n\|_2^2 . \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|v_n\|_2^2 \geq \frac{\mu-2}{2\gamma}. \tag{3.10}$$

Since $\|v_n\| = 1$, then passing to a subsequence, $v_n \rightharpoonup v$ in E and Lemma 2.2 implies that $v_n \to v$ in $L^2(\mathbb{R}^N)$, which with (3.10) implies that $v \neq 0$. Similar to (3.6), we get a contradiction. Therefore (u_n) is bounded. We conclude as in the proof of Lemma 3.3 that (u_n) possesses a convergent subsequence. Hence ψ satisfies the $(C)_c$-condition. The proof of Lemma 3.4 is completed. □

Similarly to the proof of Theorem 1.1, we deduce that the functional ψ possesses an unbounded sequence of critical points and the proof of Theorem 1.2 is finished.

4. Proof of Theorem 1.6

Consider the functional ψ associated to the equation $(\mathcal{FS})$

$$\begin{aligned} \psi(u) &= \frac{1}{2}\Big(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 + \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx\Big) - \int_{\mathbb{R}^N} F(x,u)dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \end{aligned}$$

defined in Section 3. Here, $F(x,u) = G(x,u) + H(x,u)$ with $G(x,u) = \int_0^u g(x,s)ds$.

Lemma 4.1. *Assume that* (V_1), (V_2), (H_5) *and* (H_6) *are satisfied. If* $u_n \rightharpoonup u$ *in* E, *then*

$$f(\cdot, u_n) \to f(\cdot, u) \text{ in } L^2(\mathbb{R}^N). \tag{4.1}$$

Proof. Arguing indirectly, by Lemma 2.2, there exists a subsequence u_{n_j} such that

$$u_{n_j} \to u \text{ in both } L^2(\mathbb{R}^N) \text{ and } L^{2(\mu-1)}(\mathbb{R}^N) \text{ and } u_{n_j} \to u \text{ a.e. in } \mathbb{R}^N \tag{4.2}$$

as $j \to \infty$ and

$$\int_{\mathbb{R}^N} \left| f(x, u_{n_j}(x)) - f(x, u(x)) \right|^2 dx \geq \epsilon_0, \ \forall j \in \mathbb{N}, \tag{4.3}$$

for some positive constant ϵ_0. By (4.2) and up to a subsequence if necessary, we can assume that

$$\sum_{j=1}^{\infty} \left\| u_{n_j} - u \right\|_{L^2} < \infty \text{ and } \sum_{j=1}^{\infty} \left\| u_{n_j} - u \right\|_{L^{2(\mu-1)}} < \infty.$$

Let $w(x) = \sum_{j=1}^{\infty} \left| u_{n_j}(x) - u(x) \right|$ for all $x \in \mathbb{R}^N$. Then

$$w \in L^2(\mathbb{R}^N) \bigcap L^{2(\mu-1)}(\mathbb{R}^N).$$

By (H_5) and (H_6), there holds for all $j \in \mathbb{N}$ and $x \in \mathbb{R}^N$

$$\begin{aligned}
&\left| f(x, u_{n_j}) - f(x, u) \right|^2 \leq \left| f(x, u_{n_j}) \right| + |f(x, u)|^2 \\
&\leq \Big[\left| g(x, u_{n_j}) \right| + \left| h(x, u_{n_j}) \right| + |g(x, u)| + |h(x, u)| \Big]^2 \\
&\leq \Big[a \left| u_{n_j} \right|^{\gamma-1} + b \left| u_{n_j} \right|^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} \\
&\quad + 2c + d \left| u_{n_j} \right|^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
&\leq \Big[a(\left| u_{n_j} - u \right| + |u|)^{\gamma-1} + b(\left| u_{n_j} - u \right| + |u|)^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} \\
&\quad + 2c + d(\left| u_{n_j} - u \right| + |u|)^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
&\leq \Big[a(w + |u|)^{\gamma-1} + b(w + |u|)^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} \\
&\quad + 2c + d(w + |u|)^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
&\leq c_3 \Big[a^2 w^{2(\gamma-1)} + a^2 |u|^{2(\gamma-1)} + b^2 w^{2(\sigma-1)} + b^2 |u|^{2(\sigma-1)} \\
&\quad + c^2 + d^2 w^{2(\mu-1)} + d^2 |u|^{2(\mu-1)} \Big] = k(x),
\end{aligned} \tag{4.4}$$

where c_3 is a positive constant. It is easy to see that $k \in L^1(\mathbb{R}^N)$. Hence, combining (4.2) and (4.4), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} \left| f(x, u_{n_j}(x)) - f(x, u(x)) \right|^2 dx = 0,$$

which contradicts with (4.3). Hence (4.1) is true. □

It is well known that under the assumptions of Theorem 1.6, $\psi \in C^1(E,\mathbb{R})$. Moreover, $\psi : E \to E^*$ is compact and critical points of ψ on E are classical solutions for equation $(\mathcal{FS})$.

In the next, we shall prove our Theorem 1.6 by applying Lemma 2.5. Choose a completely orthonormal basis (e_j) of E and define $X_j = \mathbb{R}e_j$, then Y_k and Z_k can be defined as in Section 2. By (H_3), $\psi \in C^1(E,\mathbb{R})$ is even. In the following, we will check that all the conditions of Lemma 2.4 are satisfied.

Lemma 4.2. *Assume that (V_1), (V_2), (H_5) and (H_7) are satisfied. Then ψ satisfies the $(PS)^*$-condition.*

Proof. Let (u_j) be a $(PS)^*$-sequence, that is, $(\psi(u_j))$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \to \infty$ and $(\psi_{|Y_{k_j}})'(u_j) \to 0$ as $j \to \infty$. We will show that (u_j) is bounded in E. By virtue of (2.1), (H_5) and (H_7), there exists a positive constant M such that

$$\begin{aligned}
\rho M + M\|u_j\| &\geq \rho\psi(u_j) - \psi'(u_j)u_j \\
&= (\frac{\rho}{2}-1)\|u_j\|^2 + \int_{\mathbb{R}^N}[f(x,u_j)u_j - \rho F(x,u_j)]dx \\
&= (\frac{\rho}{2}-1)\|u_j\|^2 + \int_{\mathbb{R}^N}[g(x,u_j)u_j - \rho G(x,u_j)]dx \\
&\quad + \int_{\mathbb{R}^N}[h(x,u_j)u_j - \rho H(x,u_j)]dx \\
&\geq (\frac{\rho}{2}-1)\|u_j\|^2 - \int_{\mathbb{R}^N}[a(x)|u_j|^\gamma + b(x)|u_j|^\sigma]dx \\
&\quad - \rho\int_{\mathbb{R}^N}[\frac{a(x)}{\gamma}|u_j|^\gamma + \frac{b(x)}{\sigma}|u_j|^\sigma]dx - \int_{\mathbb{R}^N}\theta(x)|u_j|^\delta dx \\
&\geq (\frac{\rho}{2}-1)\|u_j\|^2 - (1+\frac{\rho}{\gamma})\|a\|_{\frac{2}{2-\gamma}}\|u_j\|_2^\gamma \\
&\quad - (1+\frac{\rho}{\sigma})\|b\|_{\frac{2}{2-\sigma}}\|u_j\|_2^\sigma - \|\theta\|_{\frac{2}{2-\delta}}\|u_j\|_2^\delta \\
&\geq (\frac{\rho}{2}-1)\|u_j\|^2 - (1+\frac{\rho}{\gamma})\eta_2^\gamma\|a\|_{\frac{2}{2-\gamma}}\|u_j\|^\gamma \\
&\quad - (1+\frac{\rho}{\sigma})\eta_2^\sigma\|b\|_{\frac{2}{2-\sigma}}\|u_j\|^\sigma - \|\theta\|_{\frac{2}{2-\delta}}\eta_2^\delta\|u_j\|^\delta .
\end{aligned}$$

Since $\rho > 2$ and $\gamma, \sigma, \delta < 2$, it follows that (u_j) is bounded in E.

From the reflexivity of E and up to a subsequence if necessary, we may assume that $u_j \rightharpoonup u$ in E, for some $u \in E$. Now, we have

$$\|u_j - u\|^2 = (\psi'(u_j)-\psi'(u))(u_j-u) + \int_{\mathbb{R}^N}(f(x,u_j)-f(x,u))(u_j-u)dx. \tag{4.5}$$

It is clear that

$$(\psi'(u_j) - \psi'(u))(u_j - u) \to 0 \text{ as } j \to \infty. \tag{4.6}$$

By Hölder's inequality, (2.1) and Lemma 4.1, one has

$$\tag{4.7} \begin{aligned} \left|\int_{\mathbb{R}^N}(f(x,u_j)-f(x,u))(u_j-u)dx\right| &\leq \|f(\cdot,u_j)-f(\cdot,u)\|_2\,\|u_j-u\|_2 \\ &\leq \eta_2\,\|f(\cdot,u_j)-f(\cdot,u)\|_2\,\|u_j-u\| \\ &\to 0 \text{ as } j\to\infty. \end{aligned}$$

Combining (4.5)-(4.7), we deduce that $u_j \to u$ in E and the proof of Lemma 4.2 is completed. □

Lemma 4.3. *Assume that* (V_1), (V_2), (H_5) *and* (H_6) *are satisfied. Then for any sufficiently large* $k\in\mathbb{N}$, *there exist* $\rho_k>0$ *such that*

$$\tag{4.8} a_k = \inf_{u\in Z_k,\|u\|=\rho_k}\psi(u)\geq 0.$$

Proof. Let $l_2(k)$ be defined in Lemma 3.1. By the Mean Value Theorem, (H_5), (H_6) and (2.1), we have for any $u\in Z_k$

$$\tag{4.9} \begin{aligned} \psi(u) &= \frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N}F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N}[\frac{1}{\gamma}a\,|u|^\gamma+\frac{1}{\sigma}b\,|u|^\sigma]dx-\int_{\mathbb{R}^N}[c\,|u|+\frac{1}{\mu}d\,|u|^\mu]dx \\ &\geq \frac{1}{2}\|u\|^2-\frac{1}{\gamma}\|a\|_{\frac{2}{2-\gamma}}\|u\|_2^\gamma-\frac{1}{\sigma}\|b\|_{\frac{2}{2-\sigma}}\|u\|_2^\sigma \\ &\quad -\|c\|_2\,\|u\|_2-\frac{1}{\mu}\|d\|_\infty\,\|u\|_\mu^\mu \\ &\geq \frac{1}{2}\|u\|^2-\frac{1}{\gamma}l_2^\gamma(k)\,\|a\|_{\frac{2}{2-\gamma}}\|u\|^\gamma-\frac{1}{\sigma}l_2^\sigma(k)\,\|b\|_{\frac{2}{2-\sigma}}\|u\|^\sigma \\ &\quad -l_2(k)\,\|c\|_2\,\|u\|-\frac{1}{\mu}\eta_\mu^\mu\,\|d\|_\infty\,\|u\|^\mu. \end{aligned}$$

In view of (4.9), $\mu>2$ and $\gamma,\sigma>1$, one has

$$\tag{4.10} \psi(u)\geq\frac{1}{4}\|u\|^2-\Big(\frac{1}{\gamma}l_2^\gamma(k)\,\|a\|_{\frac{2}{2-\gamma}}+\frac{1}{\sigma}l_2^\sigma(k)\,\|b\|_{\frac{2}{2-\sigma}}+l_2(k)\,\|c\|_2\Big)\|u\|$$

for $\|u\|\leq\inf\left\{1,\left(\frac{\mu}{4\|d\|_\infty\eta_\mu^\mu}\right)^{\frac{1}{\mu-2}}\right\}$. Let $\rho_k=8\Big(\frac{1}{\gamma}l_2^\gamma(k)\,\|a\|_{\frac{2}{2-\gamma}}+\frac{1}{\sigma}l_2^\sigma(k)\,\|b\|_{\frac{2}{2-\sigma}}+l_2(k)\,\|c\|_2\Big)$, it is easy to see that $\rho_k\to 0$ as $k\to\infty$. Thus, for sufficiently large integer k, (4.10) implies

$$a_k\geq\frac{1}{8}\rho_k^2>0.$$

The proof of Lemma 4.3 is completed. □

Lemma 4.4. *Assume that* (V_1), (V_2), (H_5) *and* (H_6) *are satisfied. Then*

$$\tag{4.11} d_k=\inf_{u\in Z_k,\|u\|\leq\rho_k}\psi(u)\to 0 \text{ as } k\to\infty.$$

Proof. By (4.10), for any $u \in Z_k$, we have

$$\psi(u) \geq -\Big(\frac{1}{\gamma} l_2^{\gamma}(k)\, \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^{\sigma}(k)\, \|b\|_{\frac{2}{2-\sigma}} + l_2(k)\, \|c\|_2\Big)\, \|u\|\,. \tag{4.12}$$

Therefore, we get with $\|u\| \leq \rho_k$

$$0 \geq d_k \geq -\Big(\frac{1}{\gamma} l_2^{\gamma}(k)\, \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^{\sigma}(k)\, \|b\|_{\frac{2}{2-\sigma}} + l_2(k)\, \|c\|_2\Big)\rho_k. \tag{4.13}$$

Since $l_2(k), \rho_k \to 0$ as $k \to \infty$, one has $d_k \to 0$ as $k \to \infty$. The proof of Lemma 4.4 is completed. □

Lemma 4.5. *Assume that* (V_1), (V_2), (H_5) *and* (H_6) *are satisfied. Then, for all integer* k, *there exists* $0 < r_k < \rho_k$ *such that*

$$b_k = \inf_{u\in Y_k, \|u\|=r_k} \psi(u) < 0, \ \forall k \in \mathbb{N}. \tag{4.14}$$

Proof. Firstly, we claim that there exists $\epsilon > 0$ such that

$$meas(\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \epsilon\, \|u\|^{\gamma}\}) \geq \epsilon, \ \forall u \in Y_k \setminus \{0\}\,. \tag{4.15}$$

If not, there exists a sequence $(u_n) \subset Y_k$ with $\|u_n\| = 1$ such that

$$meas(\Big\{x \in \mathbb{R}^N : c_0(x)\, |u_n(x)|^{\gamma} \geq \frac{1}{n}\Big\}) \leq \frac{1}{n}. \tag{4.16}$$

Since $\dim Y_k < \infty$, it follows from the compactness of the unit sphere of Y_k that there exists a subsequence, say (u_n) such that (u_n) converges to some $u \in Y_k$. Hence, we have $\|u\| = 1$. Since all norms are equivalent in the finite-dimensional space Y_k, we have $u_n \to u$ in $L^2(\mathbb{R}^N)$. By the Hölder's inequality, one has

$$\int_{\mathbb{R}^N} c_0(x)\, |u_n - u|^{\gamma}\, dx \leq \|c_0\|_{\frac{2}{2-\gamma}} \Big(\int_{\mathbb{R}^N} |u_n - u|^2\, dx\Big)^{\frac{\gamma}{2}} \to 0 \text{ as } n \to \infty. \tag{4.17}$$

Thus, there exists $\epsilon_0 > 0$ such that

$$meas(\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \epsilon_0\}) \geq \epsilon_0. \tag{4.18}$$

In fact, if not, we have for all $n \in \mathbb{N}$

$$meas(\Big\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \frac{1}{n}\Big\}) \leq \frac{1}{n}.$$

Let $n \in \mathbb{N}$, then for all integer $m \geq n$

$$meas(\Big\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \frac{1}{n}\Big\}) \leq meas(\Big\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \frac{1}{m}\Big\}) \leq \frac{1}{m}$$

which implies

$$meas(\Big\{x \in \mathbb{R}^N : c_0(x)\, |u(x)|^{\gamma} \geq \frac{1}{n}\Big\}) = 0.$$

So

$$\begin{aligned}\int_{\mathbb{R}^N} c_0(x)\,|u|^{\gamma+2}\,dx &= \int_{\left\{x\in\mathbb{R}^N : c_0(x)|u(x)|^{\gamma}\le\frac{1}{n}\right\}} c_0(x)\,|u|^{\gamma+2}\,dx \\ &\le \frac{1}{n}\int_{\mathbb{R}^N}|u|^2\,dx \le \frac{\eta_2^2}{n}\,\|u\|^2 = \frac{\eta_2^2}{n} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence $u=0$, which contradicts $\|u\|=1$. Therefore (4.18) holds. Thus, define

$$\Omega_0 = \left\{x\in\mathbb{R}^N : c_0(x)\,|u(x)|^{\gamma} \ge \epsilon_0\right\},\ \Omega_n = \left\{x\in\mathbb{R}^N : c_0(x)\,|u_n(x)|^{\gamma} \le \frac{1}{n}\right\}.$$

Combining (4.16) and (4.18), we obtain

$$\begin{aligned}meas(\Omega_0\bigcap\Omega_n) &= meas(\Omega_0\setminus(\Omega_n^c\bigcap\Omega_0)) \\ &\ge meas(\Omega_0) - meas(\Omega_n^c\bigcap\Omega_0) \ge \epsilon_0 - \frac{1}{n},\ \forall n\in\mathbb{N}.\end{aligned}$$

Let n be large enough such that $\epsilon_0 - \frac{1}{n} \ge \frac{1}{2}\epsilon_0$ and $\frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n} \ge \frac{\epsilon_0}{2^{\gamma}}$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} c_0(x)\,|u_n-u|^{\gamma}\,dx &\ge \int_{\Omega_0\bigcap\Omega_n} c_0(x)\,|u_n-u|^{\gamma}\,dx \\ &\ge (\frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n})meas(\Omega_0\bigcap\Omega_n) \\ &\ge \frac{\epsilon_0^2}{2^{\gamma+1}}\end{aligned}$$

for all large integer n, which is a contradiction with (4.17). Therefore (4.15) holds.

For the ϵ given in (4.15), let

$$\Omega_u = \left\{x\in\mathbb{R}^N : c_0(x)\,|u(x)|^{\gamma} \ge \epsilon\,\|u\|^{\gamma}\right\}),\ \forall u\in Y_k\setminus\{0\}. \tag{4.19}$$

By (4.15), we obtain

$$meas(\Omega_u) \ge \epsilon,\ \forall u\in Y_k\setminus\{0\}. \tag{4.20}$$

For any $u\in Y_k$, by the Mean Value Theorem, (H_5), (H_6), (4.19) and (4.20), one has

$$\begin{aligned}\psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\le \frac{1}{2}\|u\|^2 - \frac{1}{\gamma}\int_{\mathbb{R}^N} c_0(x)\,|u|^{\gamma}\,dx - \int_{\mathbb{R}^N} H(x,u)dx \\ &\le \frac{1}{2}\|u\|^2 - \frac{1}{\gamma}\int_{\Omega_u} c_0(x)\,|u|^{\gamma}\,dx \\ &\le \frac{1}{2}\|u\|^2 - \frac{\epsilon}{\gamma}\|u\|^{\gamma}\,meas(\Omega_u) \\ &\le \frac{1}{2}\|u\|^2 - \frac{\epsilon^2}{\gamma}\|u\|^{\gamma}.\end{aligned}$$

Choose $0 < r_k < \inf\left\{\rho_k, (\frac{\epsilon^2}{\gamma})^{\frac{1}{2-\gamma}}\right\}$. Direct computation shows that

$$b_k = \inf_{u\in Y_k, \|u\|=r_k} \psi(u) \le \frac{1}{2}r_k^2 - r_k^{2-\gamma} r_k^\gamma = -\frac{1}{2}r_k^2 < 0.$$

The proof of Lemma 4.5 is completed. □

Lemmas 4.2-4.5 imply that all the conditions of Lemma 2.5 are satisfied. Thus, by Lemma 2.5, ψ has infinitely many nontrivial critical points, that is, equation $(\mathcal{FS})$ possesses infinitely many solutions.

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