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DEFORMATION OF LOCALLY FREE SHEAVES AND HITCHIN PAIRS OVER NODAL CURVE

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Abstract. In this article, we study the deformation theory of locally free sheaves and Hitchin pairs over a nodal curve. As a special case, the infinitesimal deformation of these objects gives the tangent space of the corresponding moduli spaces, which can be used to calculate the dimension of the corresponding moduli space. The deformation theory of locally free sheaves and Hitchin pairs over a nodal curve can be interpreted as the deformation theory of generalized parabolic bundles and generalized parabolic Hitchin pairs over the normalization of the nodal curve, respectively. This interpretation is given by the correspondence between locally free sheaves over a nodal curve and generalized parabolic bundles over its normalization.

1. Introduction

The moduli space of semistable locally free sheaves (coherent sheaves) and Hitchin pairs over a smooth curve is studied by many mathematicians and is by now well-understood. The moduli space of Hitchin pairs over a smooth curve was first constructed by Hitchin in [7] and generalized by Nitsure in [8]. Later on, Biswas and Ramanan [3] studied the infinitesimal deformation of Hitchin pairs. This deformation theory provides a way to study the tangent space of the moduli space of Hitchin pairs and the dimension of this moduli space.

In the last several decades, attention began to focus on the locally free sheaves and Hitchin pairs over a nodal curve. Bhosle has shown in [1] that there is a correspondence between locally free sheaves over a nodal curve and generalized parabolic bundles over its normalization. Later on, Bhosle proved that this correspondence can be extended to Hitchin pairs, more precisely, between Hitchin pairs over a nodal curve and generalized parabolic Hitchin pairs over its normalization [2]. Under this correspondence, studying the deformation

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theory of Hitchin pairs over a nodal curve is equivalent to study the deformation theory of the corresponding generalized parabolic Hitchin pairs over its normalization.

In this article, after providing necessary backgrounds in §2, we study the deformation theory of locally free sheaves over a nodal curve X in §3. To define the deformation theory, we consider the following short exact sequence

$$0 \longrightarrow J \longrightarrow C' \longrightarrow C \longrightarrow 0,$$

where $(C', \mathfrak{m}_{C'})$ and $(C, \mathfrak{m}_C)$ are two local Artin rings over a field k, and J is an ideal such that $\mathfrak{m}_{C'} J = 0$. Let X be a nodal curve over C and let X' be an extension of X flat over C', i.e., $X' \times_{\mathrm{Spec}\, C'} \mathrm{Spec}\, C = X$. Let $\mathcal{E}$ be a locally free sheaf on X. We say that a locally free sheaf $\mathcal{E}'$ over X' is a *deformation* of $\mathcal{E}$, if $\mathcal{E}' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \cong \mathcal{E}$. We review the necessary definitions for the deformation theory and pseudotorsor in §3. We describes the set of all deformations of $\mathcal{E}$ over the extension X' in terms of cohomology, which gives a way to calculate the dimension of the moduli space of locally free sheaves over a nodal curve.

Theorem. 3.1 *Let $\mathcal{E}$ be a locally free sheaf over a nodal curve X.*

(1) *The set of deformations $\mathcal{E}'$ over X' is a pseudotorsor under the action of the additive group $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$, where $R_{\mathcal{E}}$ is a sheaf over X.*
(2) *If an extension of $\mathcal{E}'$ over X' exists locally on X, then there is an obstruction $\phi \in H^1(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$, whose vanishing is necessary and sufficient for the global existence of $\mathcal{E}'$. If such a deformation $\mathcal{E}'$ over $\mathcal{E}$ exists, then the set of all such deformations is a torsor under $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$.*

In §4, we study the deformation theory of $\mathbb{L}$-twisted Hitchin pairs (E, Φ) over a nodal curve X, where $\mathbb{L}$ is a fixed line bundle over X. A *deformation* (E', Φ') of (E, Φ) is a $\mathbb{L}'$-twisted Hitchin pair over X' such that its restriction to X is (E, Φ), where $\mathbb{L}'$ is the line bundle corresponding to $\mathbb{L}$. We use two approaches to study this deformation theory.

Biswas and Ramanan studied the infinitesimal deformation theory of Hitchin pairs over smooth algebraic curves [3]. In our first approach, we generalize their approach to study the deformation of $\mathbb{L}$-twisted Hitchin pairs over a nodal curve. Note that in this approach, we generalize *infinitesimal deformation theory* to *deformation theory*, *Hitchin pairs* to *$\mathbb{L}$-twisted Hitchin pairs* and *smooth curves* to *nodal curves*.

We briefly state the deformation theory we study in the first approach. Let (E, Φ) be an $\mathbb{L}$-twisted Hitchin pair over a nodal curve X. Let ρ be the natural action of $\mathrm{End}(E)$ on itself. The deformation complex $C^\bullet_J$ is defined as follows

$$C^\bullet_J : C^0_J = \mathrm{End}(E) \otimes J \xrightarrow{e(\Phi)} C^1_J = \mathrm{End}(E) \otimes \mathbb{L} \otimes J,$$

where the map $e(\Phi)$ is given by

$$e(\Phi)(s) = -\rho(s)(\Phi).$$

We generalize the proof of [3, Theorem 2.3] and have the following theorem.

Theorem. 4.1 *The set of deformations of (E,Φ) is isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet_J)$, where $C^\bullet_J$ is the complex defined above.*

The second approach is based on the correspondence between twisted Hitchin pairs over a nodal curve and generalized parabolic Hitchin pairs over the corresponding normalization [2]. The second approach gives an alternative way to understand the deformation theory of Hitchin pairs over a nodal curve from the aspect of generalized parabolic Hitchin pairs. It is well-known that the normalization of a nodal curve is smooth. Therefore we can apply Biswas and Ramanan's deformation theory [3] to the $\widetilde{\mathbb{L}}$-twisted generalized parabolic Hitchin pair. Under the correspondence between $\mathbb{L}$-twisted Hitchin pairs over a nodal curve and $\widetilde{\mathbb{L}}$-twisted generalized parabolic Higgs bundles over its normalization, the deformation theory of a generalized parabolic Hitchin pair is exactly the deformation theory of the corresponding Hitchin pair over a nodal curve.

Let $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ be a good generalized parabolic Hitchin pair over a smooth curve $\widetilde{X}$, which is considered as the normalization of a nodal curve X. The deformation complex $C^\bullet_{par,J}$ in this case is defined as follows:

$$C^\bullet_{par,J} : C^0_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes J \xrightarrow{e(\Phi_{\widetilde{E}})} C^1_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes \widetilde{L} \otimes J.$$

Proposition. 4.3 *The set of deformations of $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ is isomorphic to $\mathbb{H}^1(C^\bullet_{par,J})$, where $C^\bullet_{par,J}$ is the complex defined above.*

As we explained above, the correspondence gives the isomorphism

$$\mathbb{H}^1(C^\bullet_{par,J}) \cong \mathbb{H}^1(C^\bullet_J).$$

This is the second way to understand the deformation theory of Hitchin pairs over a nodal curve.

2. Background

2.1. Principal $\mathbb{L}$-twisted Higgs bundles over a nodal curve

Let X be an irreducible nodal curve over $\mathbb{C}$ and $\widetilde{X}$ the normalization of X. Denote by $\nu : \widetilde{X} \to X$ the normalization map. If $x \in X$ is a node, it has two preimages $\widetilde{x}_1, \widetilde{x}_2$ in $\widetilde{X}$ under the map ν.

Now we fix a line bundle $\mathbb{L}$ over X. Let G be a reductive linear algebraic group. An $\mathbb{L}$-*twisted principal G-Higgs bundle* over X is a pair (E,Φ) consisting of a principal G-bundle E over X and a section $\Phi : X \to \mathrm{ad}(E) \otimes \mathbb{L}$, where $\mathrm{ad}(E) = E \times_{\mathrm{Ad}} \mathfrak{g}$ is the adjoint representation of E and $\mathfrak{g}$ is the Lie algebra of G. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation. We say that the Higgs bundle (E,Φ) is *stable* (resp. *semistable*), if for any Φ-invariant subbundle F, we have

$$\frac{\deg F \times_\rho V}{\mathrm{rk} F} < \frac{\deg E \times_\rho V}{\mathrm{rk} E}, \text{ (resp. } \leq \text{).}$$

Consider the moduli problem (contravariant functor)

$$\widetilde{\mathcal{M}}(X, G, \mathbb{L}) : (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to \text{ Sets},$$

where $\mathrm{Sch}/\mathbb{C}$ is the category of schemes over $\mathbb{C}$ and "op" means the opposite category. Given $S \in \mathrm{Sch}/\mathbb{C}$, $\widetilde{\mathcal{M}}(X, G, \mathbb{L})(S)$ is the set of flat families of semistable $\mathbb{L}$-twisted principal G-Higgs bundles over the nodal curve X parametrized by S. The authors in [5] proved that the moduli problem $\widetilde{\mathcal{M}}(X, G, \mathbb{L})$ is represented by a projective scheme $\mathcal{M}(X, G, \mathbb{L})$, which is known as *the moduli space of semistable $\mathbb{L}$-twisted principal G-Higgs bundles.*

Theorem 2.1 (Theorem 1 in [5])**.** *The moduli space $\mathcal{M}(X, G, \mathbb{L})$ is a projective scheme which represents the moduli problem $\widetilde{\mathcal{M}}(X, G, \mathbb{L})$.*

In this paper, we are interested in the vector bundle, in other words, the $\mathrm{GL}(n, \mathbb{C})$-bundle. Instead of working on a principal $\mathrm{GL}(n, \mathbb{C})$-bundle, we consider the associated bundle $E \times_\rho V$ and the associated Higgs field Φ. In the rest of this paper, we use the same notation (E, Φ) for the associated $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle. We denote by $\mathcal{M}(X, n, d)$ the moduli space of semistable bundle with rank n, degree d over X.

2.2. Generalized parabolic Hitchin pairs

We review the definition and some properties of the generalized parabolic Hitchin pair in this subsection. Details can be found in [1, 2].

Let Y be an irreducible non-singular algebraic curve defined over an algebraically closed field k. Let $\mathbb{L}_Y$ be a fixed line bundle over Y. We fix s-many disjoint Cartier divisors D_i on Y, $1 \leq i \leq s$. Let $D = \sum_{i=1}^{s} D_i$. In this paper, we assume that D_i is the sum of two distinct points y_{i1}, y_{i2}, $1 \leq i \leq s$. Let E be a locally free sheaf over Y. Denote by n and d the rank and degree of E.

A *generalized parabolic $\mathbb{L}_Y$-twisted Hitchin pair* (GPH) of rank n and degree d on (Y, D) is a triple $(E, F(E), \Phi)$, where

(1) E is a locally free sheaf on Y with rank n and degree d.
(2) $F(E) = (F_1(E), \ldots, F_s(E))$ is an s-tuple such that $F_i(E) \subseteq E \otimes \mathcal{O}_{D_i}$.
(3) $\Phi : E \to E \otimes \mathbb{L}_Y$ is a homomorphism preserving the filtration, i.e., $\Phi(F_i(E)) \subseteq F_i(E) \otimes \mathbb{L}_Y$.

Generally speaking, a (generalized) parabolic structure of a locally free sheaf should consist of a filtration and its weights. In the above definition of GPH, we only define the filtration (condition (2)). In fact, all of the weights are considered to be 1 in this paper. Thus we omit the condition about weights in the definition of GPH. Condition (3) is exactly the definition of homomorphism between (generalized) parabolic bundles (see [9]). Compared with the homomorphisms of holomorphic bundles, parabolic homomorphisms need to preserve the parabolic structures. More precisely, let $\mathrm{ParEnd}(E)$ be the set of parabolic homomorphisms of the generalized parabolic bundle E and let $E_{D_i} = E \otimes \mathcal{O}_{D_i}$. Define $P_{D_i}(E, E)$ to be the subspace of $\mathrm{End}(E_{D_i}, E_{D_i})$ consisting of maps preserving the filtration over D_i. We have

$$0 \to \mathrm{ParEnd}(E) \to \mathrm{End}(E) \to \mathrm{End}(E_D, E_D)/P_D(E, E) \to 0,$$

where $E_D = \bigoplus_{i=1}^s E_{D_i}$ and $P_D(E, E) = \bigoplus_{i=1}^s P_{D_i}(E, E)$.

A *generalized parabolic bundle* is a pair $(E, F(E))$ satisfying conditions (1) and (2) (see [1]). Let $f_i(E) = \dim F_i(E)$ be the dimension of the filtration. We define the *weight* $wt(E)$, and *parabolic degree* $\mathrm{par}\deg(E)$ of the locally free sheaf E as follows:

$$wt(E) = \sum_{i=1}^{s} f_i(E), \quad \mathrm{par}\deg(E) = d + wt(E).$$

The parabolic slope $\mathrm{par}\mu$ is defined by

$$\mathrm{par}\mu(E) = \frac{\mathrm{par}\deg(E)}{n}.$$

A parabolic bundle E' is a *parabolic subbundle* of E, if E' is a subbundle of E, and its filtration $F_i(E')$ satisfies $F_i(E') = F_i(E) \cap (E' \otimes \mathcal{O}_{D_j})$. It is called a Φ-*invariant subbundle*, if $\Phi(E') \subseteq E' \otimes \mathbb{L}_Y$.

A generalized parabolic bundle $(E, F(E))$ is *stable* (*resp. semistable*), if for every proper parabolic subbundle $E' \subseteq E$, we have

$$\mathrm{par}\mu(E') < \mathrm{par}\mu(E), \quad (\text{resp. } \leq) .$$

Denote by $\mathcal{M}_{par}(Y, n, d)$ the moduli space of isomorphism classes of semistable generalized parabolic bundles $(E, F(E))$ with rank n, parabolic degree d over the smooth curve Y. The existence of the moduli space $\mathcal{M}_{par}(Y, n, d)$ is given in [1, Theorem 1 and Theorem 3].

A GPH $(E, F(E), \Phi)$ is *stable* (resp. *semistable*), if for every proper Φ-invariant subbundle $E' \subseteq E$, we have

$$\mathrm{par}\mu(E') < \mathrm{par}\mu(E), \quad (\text{resp. } \leq) .$$

Denote by $\mathcal{M}_{par}(Y, n, d, \mathbb{L}_Y)$ the moduli space of semistable $\mathbb{L}_Y$-twisted generalized parabolic Hitchin pairs (GPH) $(E, F(E), \Phi)$ with rank n, parabolic degree d over the smooth curve Y.

The existence of the moduli space $\mathcal{M}_{par}(Y, n, d, \mathbb{L}_Y)$ of GPH is given by Bhosle [2].

Theorem 2.2 (Theorem 4.8 in [2]). *Let Y be a smooth algebraic curve of genus g. We fix a line bundle $\mathbb{L}_Y$. Let D_i, $1 \leq i \leq s$, be the sum of two distinct points in Y. There exists a moduli space $\mathcal{M}_{par}(Y, n, d, \mathbb{L}_Y)$ of semistable $\mathbb{L}_Y$-twisted GPH $(E, F(E), \Phi)$, where E is a holomorphic bundle of rank n, degree d with the following filtration*

$$E \otimes \mathcal{O}_{D_j} \supset F_j(E) \supset 0,$$

and $\Phi : E \to E \otimes \mathbb{L}_Y$ *is a homomorphism of parabolic bundles. The moduli space* $\mathcal{M}_{par}(Y, n, d, \mathbb{L}_Y)$ *is a projective scheme.*

It is well known that studying generalized parabolic Hitchin pairs is closely related to the study of Hitchin pairs overs a nodal curve [2]. Here is a brief review of this relation. Let X be an integral projective nodal curve and $\widetilde{X}$ its normalization. Let $\nu : \widetilde{X} \to X$ be the normalization map. Let $x_1, \ldots, x_s$ be the nodes of X. Let $D_i \subseteq \widetilde{X}$ be the preimage of x_i (as divisor). Clearly, D_i is the sum of two points. Let $\widetilde{\mathcal{O}}_{x_i}$ be the normalization of the local ring $\mathcal{O}_{x_i}$ at x_i. In this case, it is easy to check that $\dim(\widetilde{\mathcal{O}}_{x_i}/\mathcal{O}_{x_i}) = 1$. Given a line bundle $\mathbb{L}$ on X, define $\widetilde{\mathbb{L}} = \nu^*\mathbb{L}$. Let $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ be a GPH over $(\widetilde{X}, D = \sum_{i=1}^{s} D_i)$. We take

$$f_j(\widetilde{E}) = rk(\widetilde{E}).$$

A GPH $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ is *good*, if it satisfies the following conditions:

(1) the space $F_i(\widetilde{E})$ is an $\mathcal{O}_{x_i}$-sub-module of $\nu_*(\widetilde{E} \otimes \mathcal{O}_{D_j})$.

(2) we have $\nu_*(\widetilde{\Phi})(\nu_*(F_i(\widetilde{E}))) \subseteq \nu_*(F_i(\widetilde{E})) \otimes \mathbb{L}_{x_i}$, $1 \leq i \leq s$.

The good GPHs form a closed subscheme $\mathcal{M}_{par}^{good}(\widetilde{X}, n, d, \widetilde{\mathbb{L}})$ of $\mathcal{M}_{par}(\widetilde{X}, n, d, \widetilde{\mathbb{L}})$.

There is a one-to-one correspondence between good $\widetilde{\mathbb{L}}$-twisted GPHs over $\widetilde{X}$ and $\mathbb{L}$-twisted Hitchin pairs over the nodal curve X.

Proposition 2.3 (Proposition 2.8 in [2])**.** *With respect to the above notations, we have the following correspondences.*

(1) *An* $\widetilde{\mathbb{L}}$*-twisted good GPH* $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ *of rank* n*, degree* d *on* $\widetilde{X}$ *defines an* $\mathbb{L}$*-twisted Hitchin pair* (E, Φ) *of rank* n *and degree* d *on* X*.*

(2) *If* (E, Φ) *is an* $\mathbb{L}$*-twisted Hitchin pair on* X*, then* (E, Φ) *determines an* $\widetilde{\mathbb{L}}$*-twisted good GPH* $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ *on* $\widetilde{X}$*, where* $\widetilde{E} = \nu^*E$ *and* $\Phi_{\widetilde{E}} = \nu^*\Phi$*.*

This correspondence induces a birational morphism between $\mathcal{M}_{par}^{good}(\widetilde{X}, n, d, \widetilde{\mathbb{L}})$ and $\mathcal{M}(X, n, d, \mathbb{L})$.

Theorem 2.4 (Theorem 1.2 in [2])**.** *There exists a birational morphism*

$$\mathcal{M}_{par}^{good}(\widetilde{X}, n, d, \widetilde{\mathbb{L}}) \to \mathcal{M}(X, n, d, \mathbb{L})$$

from the moduli space of $\widetilde{\mathbb{L}}$*-twisted semistable good GPH on* $\widetilde{X}$ *to the moduli space of semistable* $\mathbb{L}$*-twisted Hitchin pairs on* X*.*

2.3. Infinitesimal deformation of Hitchin pairs over nonsingular algebraic curve

Let (E, Φ) be a $\mathbb{L}_Y$-twisted Hitchin pair over a nonsingular projective curve Y. An *infinitesimal deformation* of the Hitchin pair (E, Φ) is a pair (E', Φ') over $Y \times \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ with an isomorphism of the restriction to $Y \times \mathfrak{m}$, where $\mathfrak{m}$ is the closed point of Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Now we consider the $\mathbb{L}_Y$-twisted Hitchin pair $E[\varepsilon] = E \times \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$. The automorphisms of $E[\varepsilon]$ which induce identity over the closed point is $\operatorname{End}(E)$. Therefore for a section s of $\operatorname{End}(E)$, the corresponding automorphism of $E[\varepsilon]$ is denoted by $1 + s\varepsilon$. Moreover, if $v + w\varepsilon$ is a section of $(\operatorname{End}(E) \otimes \mathbb{L}_Y)[\varepsilon]$, we have

$$\rho(1 + s\varepsilon)(v + w\varepsilon) = v + w\varepsilon + \rho(s)(v)\varepsilon,$$

where ρ is the natural action of $\operatorname{End}(E)$ on E. The deformation complex $C^\bullet$ is defined as follows:

$$C^\bullet : C^0 = \operatorname{End}(E) \xrightarrow{e(\Phi)} C^1 = \operatorname{End}(E) \otimes \mathbb{L}_Y,$$

where the map $e(\Phi)$ is given by

$$e(\Phi)(s) = -\rho(s)(\Phi).$$

The authors in [3] used this complex to calculate the space of infinitesimal deformations of the Hitchin pair (E, Φ) over Y.

Theorem 2.5 (Theorem 2.3 in [3])**.** *The space of infinitesimal deformations of a given $\mathbb{L}_Y$-twisted Hitchin pair (E, Φ) over Y is isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet)$ of the complex $C^\bullet$.*

3. Deformation of locally free sheaves over nodal curve

In this section, we want to study the (infinitesimal) deformation theory of locally free sheaves over a nodal curve X, which will give us a way to calculate the tangent space of $\mathcal{M}(X, n, d)$. We first review the definition of deformation theory from [6, Chapter 6].

Let C', C be two local Artin rings over a field k with maximal ideals $\mathfrak{m}_{C'}$, $\mathfrak{m}_C$ respectively satisfying the following exact sequence

$$0 \longrightarrow J \longrightarrow C' \longrightarrow C \longrightarrow 0, \tag{1}$$

where J is an ideal such that $\mathfrak{m}_{C'} J = 0$. Thus we can consider J as a k-vector space, where k is the residue field of C with characteristic zero.

Let X be a scheme over C and let X' be an extension of X flat over C'. In other words, X' is a flat family over $\operatorname{Spec} C'$ and there is a closed embedding $X \hookrightarrow X'$ such that $X' \times_{\operatorname{Spec} C'} \operatorname{Spec} C = X$. We fix a locally free sheaf $\mathcal{E}$ over X. In this section, we will consider the deformation theory over the sequence (1). We say that a locally free sheaf $\mathcal{E}'$ over X' is a *deformation* of $\mathcal{E}$, if $\mathcal{E}' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \cong \mathcal{E}$. If we work on the following exact sequence

$$0 \longrightarrow (\varepsilon) \cong k \longrightarrow k[\varepsilon]/(\varepsilon^2) \longrightarrow k \longrightarrow 0, \tag{2}$$

where k is a field with character 0, we say that $\mathcal{E}'$ is an *infinitesimal deformation* of $\mathcal{E}$.

Let $\widetilde{X}$ be the normalization of X. Denote by $\pi : \widetilde{X} \to X$ the natural projection map. We first work on this problem in the affine case. Let $X =$

Spec A be an affine space over Spec C and $\widetilde{X} = \text{Spec } \widetilde{A}$ its normalization. We have a short exact sequence

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow R \longrightarrow 0, \tag{3}$$

where R is an A-module. Let E be a fixed A-module. We have the following exact sequence

$$0 \longrightarrow E \longrightarrow \pi_* \widetilde{E} \longrightarrow R_E \longrightarrow 0, \tag{4}$$

where $\widetilde{E} = \pi^* E = E \otimes_A \widetilde{A}$ and $R_E = E \otimes_A R$. Note that $\widetilde{E}$ is exactly the bundle corresponding to E in Proposition 2.3. The parabolic structure comes from R_E. More precisely, we have

$$0 \longrightarrow E \longrightarrow \pi_* \widetilde{E} \longrightarrow \pi_* \left(\sum_i \frac{\widetilde{E} \otimes \mathcal{O}_{D_i}}{F_j(E)} \right) \longrightarrow 0,$$

where the sum runs over all nodes x_i of X and D_i is the preimage of the node x_i in $\widetilde{X}$.

We fix an extension $X' = \text{Spec } A'$ of X. Exact sequences (1) and (3) then provide the following 3×3 commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C A & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C \widetilde{A} & \longrightarrow & \widetilde{A}' & \longrightarrow & \widetilde{A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C R & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{A}' = \widetilde{A} \otimes_A A'$. Given an A-module E, let $\widetilde{E}' := E \otimes_A \widetilde{A}'$. We want to classify deformations E' of E over A'. In other words, we want to find all A'-modules E' such that $\widetilde{E}' \otimes_{\widetilde{A}'} A' = E'$ and E' satisfies the following 3×3 commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C E & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow j & & \\
0 & \longrightarrow & J \otimes_C \pi_* \widetilde{E} & \longrightarrow & \pi_* \widetilde{E}' & \xrightarrow{p} & \pi_* \widetilde{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C R_E & \longrightarrow & R'_E & \longrightarrow & R_E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Before we state the result, we want to give the definition about *torsor* and *pseudotorsor* [6]. Let G be a group acting on a set S. We say that S is a *torsor* under the action of G, if it satisfies the following two conditions:

(1) For every $s \in S$, the induced mapping $g \mapsto g(s)$ is a bijective map from G to S,
(2) the set S is nonempty.

We say that S is a *pseudotorsor*, if it satisfies condition (1) above.

Theorem 3.1. *With the same notation as above, let $\mathcal{E}$ be a locally free sheaf over a nodal curve X.*

(1) *The set of deformations $\mathcal{E}'$ of $\mathcal{E}$ over X' is a pseudotorsor under the action of the additive group $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$.*
(2) *If an extension $\mathcal{E}'$ of $\mathcal{E}$ over X' exists locally on X, then there is an obstruction $\phi \in H^1(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$, whose vanishing is necessary and sufficient for the global existence of $\mathcal{E}'$. If such a deformation $\mathcal{E}'$ of $\mathcal{E}$ exists, then the set of all such deformations is a torsor under $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$.*

Proof. We first consider this problem in the affine case and we will use the second 3×3 commutative diagram for E. Let E'_1 and E'_2 be two possible choices for E'. Let $x_1 \in E'_1$ and $x_2 \in E'_2$ be two elements with the same image $x \in R_E$. Note that the choice of x_1, x_2 is not unique but determined only up to some element in $J \otimes_C E$. The element $x_1 - x_2$ is also a well-defined element in $\pi_* \widetilde{E}'$. Note that this also gives a well-defined element in $J \otimes_C \pi_* \widetilde{E}$, and we use the same notation $x_1 - x_2$ for the element in $J \otimes_C \pi_* \widetilde{E}$. Denote by $\varpi(x)$ the image of $x_1 - x_2$ in $J \otimes_C R_E$. Thus $x \in E$ gives us a well-defined element in $J \otimes_C R_E$. Denote by $\varpi : E \to J \otimes_C R_E$ the map sending x to the corresponding element in $J \otimes_C R_E$. It is easy to check that this map ϖ is A-linear. Therefore we get a map $\varpi \in \mathrm{Hom}_A(E, J \otimes_C R_E)$.

Now given E'_1 and a map $\varpi \in \mathrm{Hom}_A(E, J \otimes_C R_E)$, we define another module E'_2 fitting into the 3×3 diagram. Note that E' and R'_E determine each other uniquely. Therefore it is equivalent to construct $(R'_E)_2$ for E'_2. Let $(R'_E)_2$ be the set of $x_2 \in \pi_* \widetilde{E}'$ such that there exists an element $x \in E$ such that $j(x) = p(x_2)$, and for any lifting x_1 of x to E'_1, the image of $x_2 - x_1 \in J \otimes_C R_E$ equals $\varpi(x)$. It is easy to check that E'_2 is a well-defined element fitting into the diagram.

Finally, we have to check that this action is a group action. Let E'_1, E'_2, E'_3 be three choices of E'. The map ϖ_1 is defined by E'_1, E'_2, ϖ_2 is defined by E'_2, E'_3 and ϖ_3 is defined by E'_1, E'_3, then $\varpi_3 = \varpi_1 + \varpi_2$. Thus the operation $\varpi(E'_1) = E'_2$ is a group action with the additive group $\mathrm{Hom}_A(E, J \otimes_C R_E)$. This additive group $\mathrm{Hom}_A(E, J \otimes_C R_E)$ is exactly $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$. It is easy to check that if the pseudotorsor exists locally in the affine chart, it can be globalized naturally. This finishes the proof of part (1) of the theorem.

To prove (2), we assume that the deformation $\mathcal{E}'$ of $\mathcal{E}$ exists locally. In other words, there exists an open affine covering $\mathcal{X} = (X_i)_{i\in\mathcal{I}}$ of X, where $\mathcal{I}$ is the index set, such that on each local chart X_i, there exists a deformation $\mathcal{E}'_i$ of $\mathcal{E}_i = \mathcal{E}|_{X_i}$. Let $X'_i := X_i \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, C'$ be the local chart of X'. We first focus on the intersection $X'_{ij} = X'_i \cap X'_j$. There are two possible extensions $\mathcal{E}'_i$ and $\mathcal{E}'_j$ of $\mathcal{E}_{ij}$ on the intersection $X'_{ij} = X'_i \cap X'_j$. By part (1), these two extensions define an element $\varpi_{ij} \in H^0(X_{ij}, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$. On the intersection $X'_{ijk} = X'_i \cap X'_j \cap X'_k$ of three affine open sets, there are three deformations $\mathcal{E}'_i$, $\mathcal{E}'_j$ and $\mathcal{E}'_k$. The differences define the elements ϖ_{ij}, ϖ_{ik} and ϖ_{jk} in $H^0(X_{ij}, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$ such that $\varpi_{ik} = \varpi_{ij} + \varpi_{jk}$. Clearly, (ϖ_{ij}) is a 1-cocycle for the covering $\mathcal{X}$ and the sheaf $\mathcal{E}^* \otimes J \otimes_C R_E$. If $(\mathcal{E}'^0_i)_{i\in\mathcal{I}}$ is another choice of local deformations. Similarly, this choice defines $\varpi^0_{ij} \in H^0(X_{ij}, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$ such that (ϖ^0_{ij}) is a 1-cocycle. Also note that these two deformations $\mathcal{E}'_i$ and $\mathcal{E}'^0_i$ give us a well defined element $\alpha_i \in H^0(X_i, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$ such that $\alpha_i - \alpha_j = \varpi_{ij} - \varpi^0_{ij}$. Therefore the cohomology class $\alpha = (\alpha_i)$ is well-defined. This cohomology class α is the obstruction to the existence of a global deformation $\mathcal{E}'$ of $\mathcal{E}$ over X'. It is easy to check that a global deformation $\mathcal{E}'$ exists if and only if $\alpha = 0$. This finishes the proof of part (2). $\square$

Example 3.2. In this example, we consider the infinitesimal deformation of a rank n, degree 0 locally free sheaf E on a nodal curve X over $\mathbb{C}$ with a single node. Let $J = (\varepsilon) \cong \mathbb{C}$, $C' = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ and $C = \mathbb{C}$. We use the exact sequence (2). In this case, we have

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_*\widetilde{E} & \longrightarrow & \pi_*\widetilde{E}' & \longrightarrow & \pi_*\widetilde{E} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R_E & \longrightarrow & R'_E & \longrightarrow & R_E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $R_E \cong \mathbb{C}$. (If X has s nodes $x_1, \ldots, x_s$, then $R_E \cong \sum_{i=1}^{s} \mathbb{C}_{x_i}$.) Thus, if $H^1(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}})$ vanishes, we have $H^0(X, \mathcal{E}^* \otimes J \otimes_C R_{\mathcal{E}}) = H^0(X, \mathcal{E}^*)$. It is easy to check that $\dim H^0(X, \mathcal{E}^*) = n^2(g_X - 1) + 1$. This number is the dimension of the tangent space of the moduli space $\mathcal{M}(X, n, 0)$ at the smooth point E, more precisely, the dimension of $\mathcal{M}(X, n, 0)$.

Another interpretation of $\dim H^0(X, \mathcal{E}^*) = n^2(g_X - 1) + 1$ comes from the moduli space of generalized parabolic bundle $\mathcal{M}_{par}(\widetilde{X}, n, 0)$, where $\widetilde{X}$ is a normalization of X. By Theorem 1 in [1], we know the dimension of $\mathcal{M}_{par}(\widetilde{X}, n, 0)$ is $n^2(g_{\widetilde{X}} - 1) + 1 + n^2$, where the term n^2 is the dimension of the flag variety for the corresponding parabolic structure of $\widetilde{E}$. This flag variety is exactly the Grassmanian $Gr(2n,n)$, i.e., n-dimensional subspace of a $2n$-dimensional vector space. Note that $g_X = g_{\widetilde{X}} + 1$. Thus we have

$$\dim \mathcal{M}_{par}(\widetilde{X}, n, 0) = n^2(g_{\widetilde{X}} - 1) + 1 + n^2 = n^2(g_X - 1) + 1 = \dim \mathcal{M}(X, n, 0).$$

In fact, the above equality is not a coincidence. Proposition 2.3 implies an one-to-one correspondence between generalized parabolic bundles and bundles over nodal curve. Thus the dimension of the moduli spaces $\mathcal{M}_{par}(\widetilde{X}, n, 0)$ and $\mathcal{M}(X, n, 0)$ are the same as expected.

4. Deformation of Hitchin pairs over a nodal curve

In this section, we study the deformation of Hitchin pairs over a nodal curve X. We use two approaches to study this problem: the first one is to generalize Biswas and Ramanan's approach [3] to study the deformation of L-twisted Hitchin pairs over a nodal curve; the second one is to use the correspondence between Hitchin pairs over a nodal curve and generalized parabolic Hitchin pairs over its normalization to study this problem. The second approach means that studying the deformation of Hitchin pairs over a nodal curve is equivalent to study the deformation of the corresponding GPH over its normalization.

We want to remind the reader that Yokogawa studied the infinitesimal deformation theory for parabolic bundles [9]. Together with Biswas and Ramanan's work, the deformation theory of parabolic Higgs bundles is studied in a similar way in [4]. Note that the definition of the parabolic bundle is different from that of the generalized parabolic bundle. The usual parabolic structure depends on a fixed reduced effective divisor D and involves a filtration over each point x in the divisor D, while the generalized parabolic structure defines a filtration over each divisor D_i, $1 \leq i \leq s$, which can be a single point or a collection of points. In the case of a nodal curve X, the divisor D_i is the preimage of the node x_i in the normalization $\widetilde{X}$, which is the sum of two points. Although the definition of the parabolic structure is slightly different, the approach to calculate deformations can be applied to the generalized parabolic Hitchin pair.

4.1. First approach

With the same notation as in §3.1, let C', C be two local Artin rings satisfying the following exact sequence

$$0 \longrightarrow J \longrightarrow C' \longrightarrow C \longrightarrow 0.$$

We can consider J as a k-vector space, where k is the residue field of C. Let X be a nodal curve over C and let X' be an extension of X flat over C'. Note that

$$X' \times_{\text{Spec } C'} \text{Spec } C = X.$$

We fix a line bundle $\mathbb{L}$ over X together with the corresponding line bundle $\mathbb{L}'$ over X'. Let (E, Φ) be a $\mathbb{L}$-twisted Hitchin pair over X. A *deformation* (E', Φ')

of (E, Φ) is a $\mathbb{L}'$-twisted Hitchin pair over X' such that its restriction to X is (E, Φ). Note that Φ can be considered as a section of $\mathrm{End}(E) \otimes \mathbb{L}$.

Let us consider a special case. Let $C' = C[J] := C \oplus J$. The algebra structure of C' is given as follows:

$$(m, n)(p, q) = (mn, mq + np).$$

Clearly, J is a nilpotent ideal in C'. With the same notation as above, let $E' = E \times \mathrm{Spec}\ k[J]$. For a section s of $\mathrm{End}(E) \otimes J$, the corresponding automorphism of E' is denoted by $1 + s$. Moreover, if $v + w$ is a section of $\mathrm{End}(E') \otimes \mathbb{L}'$, we have

$$\rho(1 + s)(v + w) = v + w + \rho(s)(v),$$

where ρ is the natural action of $\mathrm{End}(E)$ on itself. The deformation complex $C_J^\bullet$ is defined as follows:

$$C_J^\bullet : C_J^0 = \mathrm{End}(E) \otimes J \xrightarrow{e(\Phi)} C_J^1 = \mathrm{End}(E) \otimes \mathbb{L} \otimes J,$$

where the map $e(\Phi)$ is given by

$$e(\Phi)(s) = -\rho(s)(\Phi).$$

Theorem 4.1. *Let (E, Φ) be a $\mathbb{L}$-twisted Hitchin pair over a nodal curve X. The set of deformations of (E, Φ) is isomorphic to $\mathbb{H}^1(C_J^\bullet)$, where $C_J^\bullet$ is the complex defined above.*

Proof. The proof of this theorem is similar to that of Theorem 2.3 in [3]. We only give the construction of the deformation of (E, Φ) from an element in $\mathbb{H}^1(C_J^\bullet)$.

Let $\mathcal{U} = \{U_i = \mathrm{Spec}(A_i)\}_{i \in I}$ be an open covering of X by affine schemes, where I is the index set. Set

$$\mathrm{End}(E) \otimes J|_{U_i} = C_i^0, \quad \mathrm{End}(E) \otimes \mathbb{L} \otimes J|_{U_i} = C_i^1,$$

where C_i^0 and C_i^1 are A_i-modules. Similarly, modules C_{ij}^0 (resp. C_{ij}^1) are restrictions of C_J^0 (resp. C_J^1) to $U_{ij} = U_i \bigcap U_j$. We consider the following Čech resolution of $C_J^\bullet$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_J^0 & \xrightarrow{e(\Phi)} & C_J^1 & \longrightarrow & 0 \\
 & & \downarrow d_0^0 & & \downarrow d_0^1 & & \\
0 & \longrightarrow & \sum C_i^0 & \xrightarrow{e(\Phi)} & \sum C_i^1 & \longrightarrow & 0 \\
 & & \downarrow d_1^0 & & \downarrow d_1^1 & & \\
0 & \longrightarrow & \sum C_{ij}^0 & \xrightarrow{e(\Phi)} & C_{ij}^1 & \longrightarrow & 0 \\
 & & \downarrow d_2^0 & & \downarrow d_2^1 & & \\
 & & \vdots & & \vdots & &
\end{array}$$

The first hypercohomology group $\mathbb{H}^1(C^\bullet_J)$ can be calculated from the above diagram. Let Z be the set of pairs (s_{ij}, t_i), where $s_{ij} \in C^0_{ij}$ and $t_i \in C^1_i$ satisfying the following conditions:

(1) $s_{ij} + s_{jk} = s_{ik}$ as elements of C^0_{ijk}.
(2) $t_i - t_j = e(\Phi)(s_{ij})$ as elements of C^1_{ij}.

Let B be the subset of Z consisting of elements $(s_i - s_j, e(\Phi)(s_i))$, where $s_i \in C^0_i$. The hypercohomology group $\mathbb{H}^1(C^\bullet_J)$ is Z/B.

Given an element $(s_{ij}, t_i) \in Z$, we shall construct a $\mathbb{L}$-twisted Hitchin pair (E', Φ') on X' such that $E'|_X \cong E$ and $\Phi'|_X \cong \Phi$.

For each $U_i[J]$, there is a natural projection $\pi : U_i[J] \to U_i$. Take the sheaf $E'_i = \pi^*(E|_{U_i})$. By the first condition of Z, we can identify the restrictions of E'_i and E'_j to $U_{ij}[J]$ by the isomorphism $1 + s_{ij}$ of E'_{ij}. Therefore we get a well-defined quasi-coherent sheaf E' on X'.

On each affine set $U_i[J]$, we have $\Phi_i + t_i : \mathrm{End}(E'_i) \otimes \mathbb{L}$. It is easy to check

$$e(\Phi_i + t_i)(1 + s_{ij}) = \Phi_j + t_j$$

by the second condition of Z. Therefore $\{\Phi_i + t_i\}$ can be glued together to give a global homomorphism $\Phi' : E' \to E' \otimes \mathbb{L}'$. In conclusion, for each element in Z, we can construct a deformation of (E, Φ).

Let (s_{ij}, t_i) be an element in B. In other words, $s_{ij} = s_i - s_j$ and $t_i = e(\Phi)(s_i)$. The identification of $E'_i \cong E'_j$ on $U_{ij}[J]$ is given by the isomorphism

$$1 + s_{ij} = 1 + (s_i - s_j).$$

Consider the following diagram:

$$\begin{array}{ccc} E'_{ij} & \xrightarrow{1+s_i} & E'_{ij} \\ \big\downarrow{\scriptstyle 1+s_{ij}} & & \big\downarrow{\scriptstyle \mathrm{Id}} \\ E'_{ij} & \xrightarrow{1+s_j} & E'_{ij} \end{array}$$

The commutativity of the above diagram implies that E' is trivial. Similarly, we have

$$e(\Phi_i + t_i)(1 + s_i) = \Phi_i.$$

Therefore the associated Hitchin pair (E', Φ') is isomorphic to $(\pi^* E, \pi^* \Phi)$. The above construction gives us a well-defined map from $\mathbb{H}^1(C^\bullet_J)$ to the set of deformations of (E, Φ).

Note that given a deformation (E', Φ') of (E, Φ), we can define an element (s_{ij}, t_i) by restricting to the open sets $U_i[J]$ for $i \in I$. It is easy to check that the element (s_{ij}, t_i) is a well-defined element in $\mathbb{H}^1(C^\bullet_J)$. Thus we construct a map from the set of deformations of (E, Φ) to $\mathbb{H}^1(C^\bullet_J)$.

It is easy to check that the above two maps are inverse to each other. Thus the set of deformations of (E, Φ) is isomorphic to $\mathbb{H}^1(C^\bullet_J)$. $\square$

Remark 4.2. The above proof works for either a singular (nodal) curve or a smooth curve. It can be also applied to a general scheme X. More generally, the above proof can be generalized for an algebraic space or a Deligne-Mumford stack. Note that if we are working on an algebraic space or a Deligne-Mumford stack, the covering $\mathcal{U} = \{U_i = \mathrm{Spec}(A_i)\}$ that we took in the proof should be an étale covering. Thus in the case of algebraic space or stack, the hypercohomology group we calculate is in fact the étale cohomology.

4.2. Second approach

By Theorem 2.4, we have a birational morphism between the moduli space $\mathcal{M}(X, n, d, \mathbb{L})$ and the moduli space $\mathcal{M}^{good}_{par}(\widetilde{X}, n, d, \widetilde{\mathbb{L}})$ of good GPH, which is induced by the correspondence in Proposition 2.3. Thus studying the deformation theory of $\mathbb{L}$-twisted Hitchin pairs (E, Φ) over a nodal curve X is equivalent to study the deformation theory of the corresponding $\widetilde{\mathbb{L}}$-twisted good GPH $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ over $\widetilde{X}$.

Let $\mathrm{ParEnd}(\widetilde{E})$ be the set of parabolic homomorphisms of the generalized parabolic bundle $\widetilde{E}$. As we discussed in §2.2, we have the following exact sequence

$$0 \to \mathrm{ParEnd}(\widetilde{E}) \to \mathrm{End}(\widetilde{E}) \to \mathrm{End}(E_D, E_D)/P_D(E, E) \to 0.$$

With respect to the notation in §4.1, the deformation complex $C^\bullet_{par,J}$ in the parabolic case is defined as follows:

$$C^\bullet_{par,J} : C^0_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes J \xrightarrow{e(\Phi_{\widetilde{E}})} C^1_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes \widetilde{L} \otimes J.$$

Proposition 4.3. *Let $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ be a good generalized parabolic Hitchin pair over $\widetilde{X}$. The set of deformations of $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$ is isomorphic to the hypercohomology group $\mathbb{H}^1(C^\bullet_{par,J})$, where $C^\bullet_{par,J}$ is the complex defined above.*

The proof of this proposition is similar to that of Theorem 4.1. The only difference is that, in the parabolic case, we have to work on the parabolic endomorphisms $\mathrm{ParEnd}(\widetilde{E})$ of $\widetilde{E}$ instead of the endomorphisms $\mathrm{End}(\widetilde{E})$ of $\widetilde{E}$.

Now we will explain why the set of deformations $\mathbb{H}^1(C^\bullet_{par,J})$ is isomorphic to the set of deformations $\mathbb{H}^1(C^\bullet_J(\widetilde{E}))$. The set of parabolic homomorphisms $\mathrm{ParEnd}(\widetilde{E})$ is exactly the homomorphisms $\mathrm{End}(E)$ over the nodal curve, which is implied in [1, Section 1, 4]. Therefore the following two complexes are isomorphic

$$C^\bullet_{par,J} : C^0_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes J \xrightarrow{e(\Phi')} C^1_{par,J} = \mathrm{ParEnd}(\widetilde{E}) \otimes \widetilde{L} \otimes J,$$

$$C^\bullet_J : C^0_J = \mathrm{End}(E) \otimes J \xrightarrow{e(\Phi)} C^1_J = \mathrm{End}(E) \otimes L \otimes J.$$

The isomorphism of complexes gives us the isomorphism of the hypercohomology groups

$$\mathbb{H}^1(C^\bullet_{par,J}(\widetilde{E})) \cong \mathbb{H}^1(C^\bullet_J(E)).$$

In conclusion, the set of deformations $\mathbb{H}^1(C^\bullet_J(E))$ of a locally free sheaf E over a nodal curve X can be calculated by the set of deformations $\mathbb{H}^1(C^\bullet_{par,J}(\widetilde{E}))$ of the corresponding good generalized parabolic Hitchin pair $(\widetilde{E}, F(\widetilde{E}), \Phi_{\widetilde{E}})$. This is the second approach to calculate $\mathbb{H}^1(C^\bullet_J(\widetilde{E}))$.

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