*J. Appl. Math. & Informatics Vol.* **38***(2020), No. 3 - 4, pp. 271 - 276*
https://doi.org/10.14317/jami.2020.271

# HILBERT BASIS THEOREM FOR RINGS WITH $*$-NOETHERIAN SPECTRUM[†]

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Abstract. Let $R$ be a commutative ring with identity, $R[X]$ the polynomial ring over $R$, $*$ a radical operation on $R$ and $\star$ a radical operation of finite character on $R[X]$. In this paper, we give Hilbert basis theorem for rings with $*$-Noetherian spectrum. More precisely, we show that if $(I^*R[X])^\star = (IR[X])^\star$ and $(I^*R[X])^\star \cap R = I^*$ for all ideals $I$ of $R$, then $R$ has $*$-Noetherian spectrum if and only if $R[X]$ has $\star$-Noetherian spectrum. This is a generalization of a well-known fact that $R$ has Noetherian spectrum if and only if $R[X]$ has Noetherian spectrum.

AMS Mathematics Subject Classification : 13A15, 13B25, 13E99.
*Key words and phrases* : $*$-Noetherian spectrum, $*$-finite ideal, radical operation, finite character.

## 1. Introduction

Throughout this paper, $R$ always denotes a commutative ring with identity and $\mathcal{I}(R)$ stands for the set of ideals of $R$. For an element $I \in \mathcal{I}(R)$, set $\sqrt{I} = \{a \in R \,|\, a^n \in I$ for some integer $n \geq 1\}$. Then it is easy to see that $\sqrt{I}$ is an ideal of $R$ containing $I$. We call $\sqrt{I}$ the *radical* of $I$. If $\sqrt{I} = I$, then $I$ is said to be a *radical ideal* of $R$. The radical theory is one of important topics in commutative algebra. For example, it is a powerful tool to explain Hilbert nullstellensatz and the prime spectrum with the Zariski topology (cf. [1, Section 1, Exercises 15, 16, 17 and 18] and [5, Theorem 33]).

In [2], Benhissi generalized the concept of the usual radical in commutative rings. Following [2, Definition 1.1], the mapping $* : \mathcal{I}(R) \to \mathcal{I}(R)$ given by

Received December 17, 2019. Revised January 16, 2020. Accepted January 20, 2020.
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[†] The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2017R1C1B1008085).

$I \mapsto I^*$ is called a *radical operation* on $R$ if it satisfies the following two conditions for all $I, J \in \mathcal{I}(R)$:

(1) $I \subseteq I^*$ and $(I^*)^* = I^*$; and
(2) $(I \cap J)^* = I^* \cap J^* = (IJ)^*$.

The usual radical and the trivial radical operation are simple examples of radical operations. (Recall that a mapping $* : \mathcal{I}(R) \to \mathcal{I}(R)$ defined by $I \mapsto I^* = R$ for all $I \in \mathcal{I}(R)$ is the *trivial radical operation* on $R$.) An element $I \in \mathcal{I}(R)$ is called a $*$-*ideal* of $R$ if $I^* = I$. A radical operation $*$ is said to be *of finite character* if for any $I \in \mathcal{I}(R)$, $I^* = \bigcup\{J^* \mid J$ is a finitely generated subideal of $I\}$.

Let $*$ be a radical operation on $R$. We say that an element $I \in \mathcal{I}(R)$ is a $*$-*finite ideal* of $R$ if $I^* = F^*$ for some finitely generated ideal $F$ of $R$; and $R$ has $*$-*Noetherian spectrum* if every ideal of $R$ is $*$-finite. It is easy to see that $I$ is a $*$-finite ideal of $R$ if and only if $I^*$ is a $*$-finite ideal of $R$; so $R$ has $*$-Noetherian spectrum if and only if every $*$-ideal of $R$ is $*$-finite. If $*$ is the usual radical, then the notion of $*$-finite ideals (resp., $*$-Noetherian spectrum) is precisely the same as that of radically finite ideals (resp., Noetherian spectrum). (Recall that an ideal $I$ of $R$ is *radically finite* if $\sqrt{I} = \sqrt{F}$ for some finitely generated ideal $F$ of $R$; and $R$ has *Noetherian spectrum* if each ideal of $R$ is radically finite.)

Let $R[X]$ be the polynomial ring over $R$, $*$ a radical operation on $R$ and $\star$ a radical operation of finite character on $R[X]$. In [2, Theorem 4.5], the author studied Hilbert basis theorem for rings with $*$-Noetherian spectrum under some conditions. In fact, he showed that if $I^*R[X] = (IR[X])^\star$ for all $I \in \mathcal{I}(R)$, then $R$ has $*$-Noetherian spectrum if and only if $R[X]$ has $\star$-Noetherian spectrum. The purpose of this article is to prove Hilbert basis theorem for rings with $*$-Noetherian spectrum under weaker conditions. More precisely, we show that if $(I^*R[X])^\star = (IR[X])^\star$ and $(I^*R[X])^\star \cap R = I^*$ for all $I \in \mathcal{I}(R)$, then $R$ has $*$-Noetherian spectrum if and only if $R[X]$ has $\star$-Noetherian spectrum. Note that for an element $I \in \mathcal{I}(R)$, if $I^*R[X] = (IR[X])^\star$, then $(I^*R[X])^\star = (IR[X])^\star$ and $(I^*R[X])^\star \cap R = I^*$. Hence our result is a slight generalization of [2, Theorem 4.5].

For more on radical operations on commutative rings with identity, the readers can refer to [2], [3] and [4].

## 2. Main results

In this section, we give Hilbert basis theorem for rings with $*$-Noetherian spectrum. To do this, we need some lemmas.

**Lemma 1.** *Let $R$ be a commutative ring with identity, $*$ a radical operation of finite character on $R$ and $\mathcal{F}$ the set of $*$-ideals of $R$ which are not $*$-finite. If $\mathcal{F}$ is nonempty, then $\mathcal{F}$ contains maximal elements and any such maximal element of $\mathcal{F}$ is a prime ideal of $R$.*

*Proof.* This was shown in [2, Proposition 3.5]. □

Let $R$ be a commutative ring with identity, $*$ a radical operation on $R$ and $\star$ a radical operation on $R[X]$. It was shown in [2, Lemma 4.4] that if $(IR[X])^\star = I^*R[X]$ for all $I \in \mathcal{I}(R)$ and $A$ is a $\star$-ideal of $R[X]$, then $A \cap R$ is a $*$-ideal of $R$. We weaken the condition and prove the same result as follows.

**Lemma 2.** *Let $R$ be a commutative ring with identity, $*$ a radical operation on $R$ and $\star$ a radical operation on $R[X]$. Suppose that $(I^*R[X])^\star = (IR[X])^\star$ for all $I \in \mathcal{I}(R)$. If $A$ is a $\star$-ideal of $R[X]$, then $A \cap R$ is a $*$-ideal of $R$.*

*Proof.* The containment $A \cap R \subseteq (A \cap R)^*$ is obvious. For the reverse containment, we note that

$$\begin{aligned} (A \cap R)^* &\subseteq ((A \cap R)^* R[X])^\star \cap R \\ &= ((A \cap R)R[X])^\star \cap R \\ &\subseteq A^\star \cap R \\ &= A \cap R, \end{aligned}$$

where the first equality follows from the assumption and the second equality comes from the fact that $A$ is a $\star$-ideal of $R[X]$. Hence $(A \cap R)^* = A \cap R$. Thus $A \cap R$ is a $*$-ideal of $R$. $\square$

Let $R$ be a commutative ring with identity. For an element $f \in R[X]$, $c(f)$ stands for the ideal of $R$ generated by the coefficients of $f$ and is called the *content ideal* of $f$. Clearly, $c(f)$ is a finitely generated ideal of $R$.

**Lemma 3.** *Let $R$ be a commutative ring with identity, $I$ an ideal of $R$ and $\star$ a radical operation of finite character on $R[X]$. If $IR[X]$ is a $\star$-finite ideal of $R[X]$, then $(IR[X])^\star = (CR[X])^\star$ for some finitely generated subideal $C$ of $I$.*

*Proof.* Suppose that $IR[X]$ is a $\star$-finite ideal of $R[X]$. Then $(IR[X])^\star = F^\star$ for some finitely generated ideal $F$ of $R[X]$. Write $F = (f_1, \ldots, f_n)$ for some $f_1, \ldots, f_n \in R[X]$. Since $\star$ is of finite character, $(IR[X])^\star = \bigcup\{J^\star \mid J$ is a finitely generated subideal of $IR[X]\}$; so for each $k \in \{1, \ldots, n\}$, there exists a finitely generated subideal $J_k$ of $IR[X]$ such that $f_k \in (J_k)^\star$. Therefore we have

$$(IR[X])^\star = F^\star \subseteq ((J_1)^\star + \cdots + (J_n)^\star)^\star = (J_1 + \cdots + J_n)^\star \subseteq (IR[X])^\star,$$

where the second equality comes from [2, Lemma 1.2(iv)] and two containments follow from [2, Lemma 1.2(iii)]. Hence $(IR[X])^\star = (J_1 + \cdots + J_n)^\star$. Let $J = J_1 + \cdots + J_n$. Then $J$ is a finitely generated subideal of $IR[X]$ and $(IR[X])^\star = J^\star$. Write $J = (g_1, \ldots, g_m)$ for some $g_1, \ldots, g_m \in IR[X]$. Then we obtain

$$J \subseteq (c(g_1) + \cdots + c(g_m))R[X] \subseteq IR[X].$$

Let $C = c(g_1) + \cdots + c(g_m)$. Then $C$ is a finitely generated subideal of $I$ and $(IR[X])^\star = (CR[X])^\star$. Thus the proof is complete. $\square$

Now, we are ready to give Hilbert basis theorem for rings with $*$-Noetherian spectrum which is the main result in this article.

**Theorem 4.** *Let $R$ be a commutative ring with identity, $*$ a radical operation on $R$ and $\star$ a radical operation of finite character on $R[X]$. Suppose that for all $I \in \mathcal{I}(R)$, $(I^*R[X])^\star = (IR[X])^\star$ and $(I^*R[X])^\star \cap R = I^*$. Then the following statements are equivalent.*

(1) *$R$ has $*$-Noetherian spectrum.*
(2) *$R[X]$ has $\star$-Noetherian spectrum.*

*Proof.* (1) $\Rightarrow$ (2) Suppose to the contrary that $R[X]$ does not have $\star$-Noetherian spectrum, and let $\mathcal{F}$ be the set of $\star$-ideals of $R[X]$ which are not $\star$-finite. Then $\mathcal{F}$ is a nonempty set; so by Lemma 1, there exists a maximal element in $\mathcal{F}$. Let $P$ be such a maximal element. Then $P$ is a $\star$-ideal of $R[X]$ which is not $\star$-finite. By Lemma 2, $P \cap R$ is a $*$-ideal of $R$. Since $R$ has $*$-Noetherian spectrum, $P \cap R = J^*$ for some finitely generated ideal $J$ of $R$. Let $M = ((P \cap R)R[X])^\star$. Then by the assumption, we get

$$M = (J^*R[X])^\star = (JR[X])^\star.$$

Note that $JR[X]$ is a finitely generated ideal of $R[X]$; so $M$ is a $\star$-finite ideal of $R[X]$. Hence $M \subsetneq P$.

Let $f$ be an element of least degree in $P \setminus M$. Let $n$ be the degree of $f$ and let $a$ be the coefficient of $X^n$ in $f$. Then $n \geq 1$ and $a \notin P$. (To see this, if $n = 0$, then $f = a \in P \cap R \subsetneq M$. This contradicts the choice of $f$. Hence $n \geq 1$. Next, suppose to the contrary that $a \in P$. Then $f - aX^n \in P$. Since $\deg(f - aX^n) < \deg(f)$, $f - aX^n \in M$ by the minimality of $n$. Also, note that $a \in P \cap R$; so $aX^n \in (P \cap R)R[X] \subseteq M$. Therefore $f = (f - aX^n) + aX^n \in M$, which is a contradiction. Hence $a \notin P$.) Now, we note that $P \subsetneq (P + (a))^\star$; so by the maximality of $P$, $(P + (a))^\star$ is a $\star$-finite $\star$-ideal of $R[X]$. Therefore there exists a finitely generated ideal $F$ of $R[X]$ such that $(P + (a))^\star = F^\star$. Since $\star$ is of finite character, we may assume that $F = B + (a)$ for some finitely generated subideal $B$ of $P$. Hence we obtain

$$(P + (a))^\star = (B + (a))^\star.$$

Let $g \in P$. If $g \in M$, then $ag \in ((f) + M)^\star$; so we next suppose that $g \in P \setminus M$. Let $m$ be the degree of $g$. Then $m \geq n$ by the minimality of $n$; so by a routine iterative calculation, we can find suitable elements $q, r \in R[X]$ such that $a^m g = fq + r$ and $\deg(r) < \deg(f)$. Therefore $r = a^m g - fq \in P$. By the minimality of $n$, $r \in M$. Also, note that $(ag)^m = fqg^{m-1} + rg^{m-1} \in ((f) + M)^\star$; so $ag \in \sqrt{((f) + M)^\star} = ((f) + M)^\star$ [2, Lemma 1.2(vi)]. Hence $aP \subseteq ((f) + M)^\star$. Now, we obtain

$$\begin{aligned} P^2 &\subseteq P(P + (a))^\star \\ &= P(B + (a))^\star \\ &\subseteq (P(B + (a)))^\star \\ &\subseteq (B + aP)^\star \\ &\subseteq (B + ((f) + M)^\star)^\star \end{aligned}$$

$$\begin{aligned} &= (B+(f)+M)^{\star} \\ &= (B+(f)+(JR[X])^{\star})^{\star} \\ &= (B+(f)+JR[X])^{\star}, \end{aligned}$$

where the second inclusion follows from [2, Lemma 1.2(v)] and the second and the final equalities come from [2, Lemma 1.2(iv)].

Let $h \in P$. Then $h^2 \in (B+(f)+JR[X])^{\star}$; so $h \in \sqrt{(B+(f)+JR[X])^{\star}} = (B+(f)+JR[X])^{\star}$ [2, Lemma 1.2(vi)]. Therefore $P \subseteq (B+(f)+JR[X])^{\star}$. Note that $J \subsetneq P$; so $JR[X] \subseteq P$. Hence we obtain

$$P = (B+(f)+JR[X])^{\star}.$$

Note that $B+(f)+JR[X]$ is a finitely generated ideal of $R[X]$; so $P$ is an $\star$-finite ideal of $R[X]$. This is a contradiction to the fact that $P$ is not a $\star$-finite ideal of $R[X]$. Thus $R[X]$ has $\star$-Noetherian spectrum.

(2) $\Rightarrow$ (1) Let $I$ be an ideal of $R$. Then $IR[X]$ is an ideal of $R[X]$. Since $R[X]$ has $\star$-Noetherian spectrum, $IR[X]$ is a $\star$-finite ideal of $R[X]$. Since $\star$ is of finite character, Lemma 3 guarantees the existence of a finitely generated subideal $C$ of $I$ such that $(IR[X])^{\star} = (CR[X])^{\star}$. Therefore by the assumption, we obtain

$$\begin{aligned} I^* &= (I^*R[X])^{\star} \cap R \\ &= (IR[X])^{\star} \cap R \\ &= (CR[X])^{\star} \cap R \\ &= (C^*R[X])^{\star} \cap R \\ &= C^*. \end{aligned}$$

Hence $I$ is a $*$-finite ideal of $R$. Thus $R$ has $*$-Noetherian spectrum. □

Let $R$ be a commutative ring with identity. Then the usual radical is a radical operation of finite character on $R$ and $R[X]$. Also, for any $I \in \mathcal{I}(R)$, it is easy to check that $\sqrt{\sqrt{I}R[X]} = \sqrt{IR[X]}$ and $\sqrt{\sqrt{I}R[X]} \cap R = \sqrt{I}$. Hence by Theorem 4, we recover

**Corollary 5.** ([6, Theorem 2.5]) *Let $R$ be a commutative ring with identity. Then $R$ has Noetherian spectrum if and only if $R[X]$ has Noetherian spectrum.*

It is natural to ask whether $*$-Noetherian spectrum is preserved under the power series ring extension. We are closing this article by answering this question. In [7, Example], Ribenboim constructed an example of a commutative ring $R$ with identity such that $R$ has Noetherian spectrum but the power series ring $R[\![X]\!]$ does not have Noetherian spectrum. Hence the power series ring over a ring with $*$-Noetherian spectrum does not generally have $*$-Noetherian spectrum. This is the case when $*$ is the usual radical.

**Acknowledgement:** We would like to thank the referee for several valuable suggestions.

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