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LOCAL EXPANSIVITY AND THE LIMIT SHADOWING PROPERTY†

KI-SHIK KOO

Abstract. Here, assume that X is a compact space. we show that if f is a locally nonexpansive map then $f : X \to X$ has the limit shadowing property if and only if $f : CR(f) \to CR(f)$ has the limit shadowing property. Also we prove that if f is a local expansion then f has the limit shadowing property.

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1. Preliminaries

Shadowing property has been the important subject of much interest in the study of stability theory in dynamical systems theory. There are various kinds of shadowing properties such as the shadowing, limit shadowing, two-sided limit shadowing and average limit shadowing property.

Our aim here is to study the relationships between the expansivity of distance and the limit shadowing property of continuous maps. A map has the limit shadowing property means that we can find a true orbit which is asymptotically near a limit pseudo-orbit.

Throughout this paper, let X be a compact metric space with a metric function d and $f : X \to X$ be a continuous map.

In this paper we show that if f is a locally nonexpansive map then $f : X \to X$ has the limit shadowing property if and only if $f : CR(f) \to CR(f)$ has the limit shadowing property [Theorem 2.7]. In [4] we showed that if f is a local expansion then f has the shadowing property. Also we prove that if f is a local expansion then f has the limit shadowing property [Theorem 2.9].

For a positive number δ, a sequence of points $\{x_i\}_{0 \le i \le b}$, $(0 < b \le \infty)$, is called a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for each i, $(0 \le i < b)$. A

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finite pseudo-orbit $\{x_0, x_1, \ldots, x_n\}$ is called a pseudo-orbit from x_0 to x_n. For $x, y \in X$, x is related to y (written $x \sim y$) if there are γ-pseudo-orbits of f from x to y and from y to x for every $\gamma > 0$. $x \in X$ is called a chain recurrent point of f if $x \sim x$. The set of all chain recurrent points of f, $CR(f)$, is called the chain recurrent set of f.

A sequence of points $\{x_i\}_{i \geq 0}$ is ϵ-shadowed if there is $y \in X$ satisfying that $d(f^i(y), x_i) < \epsilon$ holds for all integers $i \geq 0$. f has the shadowing property if for any $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed. We say that a sequence of points $\{x_i\}_{i \geq 0}$ is a limit pseudo-orbit of f if it satisfies $d(f(x_i), x_{i+1}) \to 0 \quad \text{as} \quad i \to \infty$. For a limit pseudo-orbit $\{x_i\}_{i \geq 0}$, $\omega(\{x_i\})$ is the set of limit points of $\{x_i\}_{i \geq 0}$. A sequence of points $\{x_i\}_{i \geq 0}$ is called limit shadowed by $x \in X$ if $d(f^i(x), x_{N+i}) \to 0$ for some integer $N \geq 0$ as $i \to \infty$. We say that f has the limit shadowing property if every limit pseudo-orbit of f is limit shadowed by some point in X.

The family of all nonempty closed subsets of X is denoted by $K(X)$. Recall that the Hausdorff metric h between A and B in $K(X)$ is defined by $h(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(x, A) : x \in B\}\}$, where $d(x, B) = \inf\{d(x, y) : y \in B\}$.

2. Basic results

Definition 2.1. A continuous surjective map $f : X \to X$ is called a local expansion if there exist numbers $\epsilon > 0$ and $0 < \lambda < 1$ such that $d(x, y) < \epsilon$ implies $h(f^{-1}(x), f^{-1}(y)) < \lambda d(x, y)$. A continuous map $f : X \to X$ is called locally nonexpansive if there is $\epsilon > 0$ satisfying that $d(x, y) < \epsilon$ implies $d(f(x), f(y)) \leq d(x, y)$.

Proposition 2.2. *Every sequence in $\omega(\{x_n\})$ is contained in a limit pseudo-orbit of f.*

Proof. Let $\{a_i\}_{i \geq 1}$ be a sequence in $\omega(\{x_n\})$. It suffices to show that for each i, there is a $\frac{1}{i}$-pseudo-orbit of f from a_i to a_{i+1}. Take $\gamma > 0$ such that $d(a_i, y) < \gamma$ implies $d(f(a_i), f(y)) < \frac{1}{2i}$. Since $\{x_n\}$ is a limit pseudo-orbit of f there is an integer $N > 0$ such that $\{x_n\}_{n \geq N}$ is a $\frac{1}{2i}$-pseudo-orbit of f. Choose $x_a, x_b \in \{x_n\}$ satisfying that $d(x_a, a_i) < \gamma$ and $d(x_b, a_{i+1}) < \frac{1}{2i}$, $(N < a < b)$. Then $\{a_i, x_{a+1}, x_{a+2}, \ldots, x_{b-1}, a_{i+1}\}$ is a $\frac{1}{i}$-pseudo-orbit of f from a_i to a_{i+1}. $\square$

Lemma 2.3. *$\omega(\{x_n\}) \subset CR(f)$.*

Proof. Let $a \in \omega(\{x_n\})$ and $\gamma > 0$ be given. Let $\delta > 0$ be a number with $\delta < \frac{\gamma}{2}$ such that $d(a, z) < \delta$ implies $d(f(a), f(z)) < \frac{\gamma}{2}$. There is an integer $N > 0$ such that $\{x_n\}_{n \geq N}$ is a $\frac{\gamma}{2}$-pseudo-orbit of f. Choose $x_u, x_v \in \{x_n\}$ such that $d(a, x_u) < \delta$ and $d(a, x_v) < \delta$ with $N < u < v$. Then $\{a, x_{u+1}, \ldots, x_{v-1}, a\}$ is a γ-pseudo-orbit of f from a to a. $\square$

Corollary 2.4. *For a limit pseudo-orbit $\{x_n\}$ of f, $d(x_n, CR(f)) \to 0$ as $n \to \infty$.*

Proof. It follows from the above result. □

Theorem 2.5. *Let $f : CR(f) \to CR(f)$ have the limit shadowing property. Then $f : X \to X$ has the limit shadowing property.*

Proof. Let $\{x_i\}_{i\geq 0}$ be a limit pseudo-orbit of f. By the uniform continuity of f, for each positive integer a, there is $\delta_a > 0$ with $\delta_a < \frac{1}{4a}$ satisfying that $d(u, v) < \delta_a$ implies $d(f(u), f(v)) < \frac{1}{2a}$ for all $u, v \in X$.

Since $d(x_i, CR(f)) \to 0$ and $\{x_i\}$ is a limit pseudo-orbit of f, we can obtain an increasing sequence of positive integers $\{l_a\}_{a>0}$ satisfying that, for each $i \geq l_a$, $\{x_i\}_{i\geq l_a}$ is a δ_a-pseudo-orbit of f and $d(x_i, CR(f)) < \delta_a$. Here we may assume that $l_a < l_{a+1}$.

For each integer $a > 0$ and k with $l_a \leq k < l_{a+1}$, since $d(x_k, CR(f)) < \delta_a$ we can choose $y_k \in CR(f)$ such that $d(x_k, y_k) < \delta_a$. So for each integer k with $l_a \leq k < l_{a+1}$, we have

$$\begin{aligned} d(f(y_k), y_{k+1}) &\leq d(f(y_k), f(x_k)) + d(f(x_k), x_{k+1}) + d(x_{k+1}, y_{k+1}) \\ &< \frac{1}{2a} + \delta_a + \delta_a < \frac{1}{a}. \end{aligned}$$

This shows that $\{y_i\}_{i\geq l_1}$ is a limit pseudo-orbit of f in $CR(f)$. Since $f : CR(f) \to CR(f)$ has the limit shadowing property, there is a point y in $CR(f)$ such that $d(f^i(y), y_{N+i}) \to 0$ for some integer $N \geq l_1$.

By the fact that $d(x_i, y_i) < \delta_a < \frac{1}{4a}$ for $l_a \leq i < l_{a+1}$, we have $d(x_i, y_i) \to 0$ as $i \to \infty$. Therefore we have $d(f^i(y), x_{N+i}) \to 0$ as $i \to \infty$. This shows that $f : X \to X$ has the limit shadowing property and this completes the proof. □

Lemma 2.6. *Let $f : X \to X$ be a locally nonexpansive map. For any closed invariant subset M of X ($f(M) = M$), if a sequence in M is limit shadowed by some point in X then the sequence is limit shadowed by a point in M.*

Proof. Let $\epsilon > 0$ be a number as in the definition of locally nonexpansive map f. Suppose $\{x_i\}_{i\geq 0}$ is a sequence in M and $d(f^i(a), x_{N+i}) \to 0$ for some $a \in X$ and integer $N \geq 0$. Put $a_i = x_{N+i}$, $(i \geq 0)$.

Since $d(f^i(a), a_i) \to 0$, for each integer $n > 0$, we can choose an integer $i_n > 0$ satisfying that $d(f^i(a), a_i) < \frac{\epsilon}{2n}$ for $i \geq i_n$. We may assume that $i_n < i_{n+1}$. Since f is locally nonexpansive, for $i \geq 0$, we have

$$\begin{aligned} d(f^i(a_{i_n}), a_{i_n+i}) &\leq d(f^i(a_{i_n}), f^i(f^{i_n}(a))) + d(f^i(f^{i_n}(a)), a_{i_n+i}) \\ &< d(a_{i_n}, f^{i_n}(a)) + d(f^i(f^{i_n}(a)), a_{i_n+i}) \\ &< \frac{\epsilon}{2n} + \frac{\epsilon}{2n} = \frac{\epsilon}{n}. \end{aligned}$$

For each n, take $y_n \in M$ with $f^{i_n}(y_n) = a_{i_n}$. Suppose $y_{n_k} \to y \in M$ for some subsequence $\{y_{n_k}\}$ of $\{y_n\}$. We claim that $d(f^i(y), a_i) \to 0$ as $i \to \infty$. Assume, on the contrary. Then, for some $\gamma > 0$ and a sequence $\{m_j\}$ of positive integers, $d(f^{m_j}(y), a_{m_j}) \geq \gamma$ holds. Let choose $y_{n_l} \in \{y_{n_k}\}$ such that

$d(y, y_{n_l}) < \min\{\frac{\gamma}{2}, \epsilon\}$ and $\frac{\epsilon}{n_l} < \frac{\gamma}{2}$. Then, for $i \geq 0$, $d(f^{n_l+i}(y_{n_l}), a_{n_l+i}) = d(f^i(a_{n_l}), a_{n_l+i}) < \frac{\epsilon}{n_l}$. holds. Choose $m_L \in \{m_j\}$ with $m_L > n_l$. Then we have

$$\begin{aligned} d(f^{m_L}(y), a_{m_L}) &\leq d(f^{m_L}(y), f^{m_L}(y_{n_l})) + d(f^{m_L}(y_{n_l}), a_{m_L}) \\ &\leq d(y, y_{n_l}) + d(f^{m_L - n_l}(a_{n_l}), a_{n_l + (m_L - n_l)}) \\ &< \frac{\gamma}{2} + \frac{\epsilon}{n_l} < \gamma, \end{aligned}$$

which is absurd. Hence $d(f^i(y), a_i) \to 0$, $(y \in M)$. This completes the proof. □

Theorem 2.7. *Let $f : X \to X$ be a locally nonexpansive map. Then the followings are equvalent;*
(1) $f : X \to X$ has the limit shadowing property.
(2) $f : CR(f) \to CR(f)$ has the limit shadowing property.

Proof. (1) $\to$ (2). $CR(f)$ is a closed invariant subset of X ([4]). So this is the consequence of the above lemma by replacing closed invariant set M to $CR(f)$ and also replacing sequence in M to limit pseudo-orbit of f in $CR(f)$.

(2) $\to$ (1). This follows from Theorem 2.5. □

Theorem 2.8 ([4])**.** *If $f : X \to X$ is a local expansion then f has the shadowing property.*

If $h(A, B) < \delta$ for $A, B \in K(X)$ then for any point $a \in A$, there is $b \in B$ with $d(a, b) < \delta$.

Theorem 2.9. *If $f : X \to X$ is a local expansion then f has the limit shadowing property.*

Proof. Let ϵ, λ be positive numbers as given in the definition of the local expansion f and $\{a_i\}_{i\geq 0}$ be a limit pseudo-orbit of f. Let N be a positive integer such that $d(f^i(a), a_{i+1}) < \epsilon$ for all $i \geq N$. Set $a_{N+i} = x_i$ for all integers $i \geq 0$. For each integer $a > 0$, there exists an integer $n_a > 0$ satisfying that $d(f(x_i), x_{i+1}) < \frac{\epsilon}{a}$, for all integers $i \geq n_a$. We may assume that $a < n_a$ and $n_a < n_b$ for $a < b$.

For each integer $i \geq 0$, by using induction, we will construct a sequence of points $\{p_i^j\}_{j\geq 0}$, $p_i^0 = x_i$, satisfying that

$$f(p_i^j) = p_{i+1}^{j-1} \text{ and } d(p_i^j, p_i^{j+1}) < \lambda d(p_{i+1}^{j-1}, p_{i+1}^j) \text{ for } j \geq 1. \tag{1}$$

To do this, first, let $x_i = p_i^0$ for all integers $i \geq 0$.

For $k = 0$, $\{x_0\} = \{p_0^0\}$. Next, $k > 0$ be a fixed integer. Assume that, for every integer $i \leq k$, there is a sequence of points $\{p_i^j\}_{0\leq j\leq k-i}$ satisfying (1).

For the integer $k+1$, note that $x_{k+1} = p_{k+1}^0$. Since $d(f(x_k), x_{k+1}) < \epsilon$ we have

$$h(f^{-1}(f(x_k)), f^{-1}(x_{k+1})) < \lambda d(f(x_k), x_{k+1}).$$

For the point x_k in $f^{-1}(f(x_k))$, there is a point p_k^1 in $f^{-1}(x_{k+1})$ such that

$$d(x_k, p_k^1) = d(p_k^0, p_k^1) < \lambda d(f(x_k), x_{k+1}) < \lambda\epsilon < \epsilon.$$

Also, $d(f^{-1}(p_k^0), f^{-1}(p_k^1)) < \lambda d(p_k^0, p_k^1)$ implies that, for $p_{k-1}^1 \in f^{-1}(p_k^0)$, there is $p_{k-1}^2 \in f^{-1}(p_k^1)$ such that

$$d(p_{k-1}^1, p_{k-1}^2) < \lambda d(p_k^0, p_k^1).$$

Again, since $d(p_{k-1}^1, p_{k-1}^2) < \lambda d(p_k^0, p_k^1) < \epsilon$, we have

$$h(f^{-1}(p_{k-1}^1), f^{-1}(p_{k-1}^2)) < \lambda d(p_{k-1}^1, p_{k-1}^2).$$

Thus, for $p_{k-2}^2 \in f^{-1}(p_{k-1}^1)$, there is $p_{k-2}^3 \in f^{-1}(p_{k-1}^2)$ such that

$$d(p_{k-2}^2, p_{k-2}^3) < \lambda d(p_{k-1}^1, p_{k-1}^2).$$

Using this method consecutively, we obtain a sequence of points $\{p_i^j\}_{0\le j\le (k+1)-i}$ for all integers $i \le k+1$. This shows that for every integer $i \ge 0$ there exists a sequence of points $\{p_i^j\}_{j\ge 0}$ satisfying (1) as required.

For any integer $n > 0$ and $i > 0$, we have

$$\begin{aligned} d(p_i^{n-1}, p_i^n) &< \lambda d(p_{i+1}^{n-2}, p_{i+1}^{n-1}) < \lambda^2 d(p_{i+2}^{n-3}, p_{i+2}^{n-2}) \\ &< \cdots \\ &< \lambda^{n-1} d(p_{i+n-1}^0, p_{i+n-1}^1) < \lambda^n d(f(x_{i+n-1}), x_{i+n}). \end{aligned}$$

Thus, if $i \ge n_a$,

$$d(p_i^{n-1}, p_i^n) < \lambda^n d(f(x_{i+n-1}), x_{i+n}) < \lambda^n \cdot \frac{\epsilon}{a}$$

holds. This means that the sequence $\{p_i^j\}_{j\ge 0}$ is a cauchy sequence and we let $p_i^j \to b_i$ as $j \to \infty$. By the continity of f and (1) we have $f(b_i) = b_{i+1}$. Also, we have, for $i \ge n_a$,

$$\begin{aligned} d(x_i, p_i^n) &\le d(x_i, p_i^1) + d(p_i^1, p_i^2) + \cdots + d(p_i^{n-1}, p_i^n) \\ &< (\lambda + \lambda^2 + \cdots + \lambda^n) \cdot \frac{\epsilon}{a} \\ &< \frac{\lambda}{1-\lambda} \cdot \frac{\epsilon}{a}. \end{aligned}$$

Therefore, we get that

$$d(x_i, b_i) \le \frac{\lambda}{1-\lambda} \cdot \frac{\epsilon}{a}.$$

This shows that $d(x_i, f^i(b_0)) \to 0$ as $i \to \infty$. Thus, we have $d(f^i(b_0), a_{N+i}) \to 0$ as $i \to \infty$. So we concllude that f has the limit shadowing property and the proof of this result is complete. $\square$

Example 2.10. (1) Let Σ_2^+ denote the set of all sequences $x = (x_0, x_1, x_2, \dots)$ where each x_k is in $\{0, 1\}$. We define a metric d on Σ_2^+ by setting

$$d(x, y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^k}.$$

The shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$ defined by $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$ is a local expansion since $h(\sigma^{-1}(x), \sigma^{-1}(y)) = \frac{1}{2}d(x, y)$. Hence this shift map has the limit shadowing property.

(2) Let I be the unit interval. The family of asymmetric tent maps $\{f_s : I \to I : \ 0 < s < 1\}$ defined by

$$\begin{aligned} f_s(x) &= \frac{1}{s}x, & \text{if } \ 0 \leq x \leq s, \\ &= -\frac{1}{1-s}x + \frac{1}{1-s}, & \text{if } \ s \leq x \leq 1 \end{aligned}$$

are local expansions. So they have the limit shadowing property.

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Ki-Shik Koo received M.Sc. and Ph.D. at Chungnam National University. Since 1984 he has been at Daejeon University. His research interests include topological dynamics.

Department of Computer and Information Security, Daejeon University 34520, Korea.
`e-mail: kskoo@dju.kr`