J. Appl. Math. & Informatics Vol. **38**(2020), No. 3 - 4, pp. 239 - 247
https://doi.org/10.14317/jami.2020.239

MOMENT APPROACH TO THE TARGET CONTROL PROBLEM FOR LINEAR SYSTEM

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ABSTRACT. In this paper, we consider the target control problem for the linear systems by using the solution of the Hausdorff moment problem.

AMS Mathematics Subject Classification : 92B05, 34D20.
Key words and phrases : The target control problem, the Markov moment problem, the Hausdorff moment problem.

1. From beginning to obtaining the data

Consider the following linear continuous system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = C(t)\,x(t), \\ \text{s.t. } x(0) = x_0, \text{ and } r(\theta) = y(\theta), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $r(t) \in \mathbb{R}^p$ is the target signal, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C(t) \in \mathbb{R}^{p\times n}$. Given an initial condition $x_0 \in \mathbb{R}^n$, and a time θ, find one of the controls $|u(t)| \leq 1$ such that the trajectory from x_0 of the system (1) arrives to the target signal at time θ. This problem is called the target control (TC) problem.

Moment problem is related to operator theory and has many applications (see [1], [2], [4], and [5], etc). In this paper, we consider the target control problem for the linear systems by using the solution of the Hausdorff moment problem.

Let $\mathcal{C}_{0,L}$ be the set of all measurable functions on $[0,\theta]$ such that $0 \leq f(\tau) \leq L$ for all $\tau \in [0,\theta]$. Then the L-Markov moment problem (MMP) for the interval $[0,\theta]$ is stated as follows: Given a finite sequence of real numbers $c_0, c_1, \ldots, c_k$, find the set of functions $f \in \mathcal{C}_{0,L}$ such that

$$c_j = \int_0^\theta \tau^j f(\tau)\,\mathrm{d}\tau, \quad j = 0, 1, \ldots, k.$$

Received February 8, 2020. Revised February 26, 2020. Accepted March 10, 2020.
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Let $\mathcal{M}[0,\theta]$ be the set of all nonnegative measures on $[0,\theta]$. Then the Hausdorff moment problem (HMP) for an interval $[0,\theta]$ is stated as follows: Given a finite sequence of real numbers $s_0, s_1, \ldots, s_k$, find the set of measures $\sigma \in \mathcal{M}[0,\theta]$ such that

$$s_j = \int_0^\theta \tau^j \mathrm{d}\sigma(\tau), \quad j = 0, 1, \ldots, k.$$

Recall from [5] that there is a bijection between the set $\mathcal{C}_{0,L}$ and the set of measures $\sigma \in \mathcal{M}[0,\theta]$ satisfying $\int_0^\theta \mathrm{d}\sigma(\tau) = 1$. This bijection is given by

$$\int_0^\theta \frac{\mathrm{d}\sigma(\tau)}{\tau - z} = -\frac{1}{z} \exp\left(\frac{1}{L}\int_0^\theta \frac{f(\tau)\,\mathrm{d}\tau}{z-\tau}\right),$$

which determines the relation between $(c_j)_{j=0}^{k-1}$ and $(s_j)_{j=0}^{k}$: $s_0 = 1, s_1 = c_1$, and

$$s_k = \frac{1}{k!}\begin{vmatrix} c_1 & -1 & \cdots & 0 \\ 2c_2 & c_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (k-1)\,c_{k-1} & (k-2)\,c_{k-2} & \ddots & -(k-1) \\ kc_k & (k-1)\,c_{k-1} & \cdots & c_1 \end{vmatrix}, \quad k \geq 2. \tag{2}$$

By [5, Theorem 2.1], we know that the MMP is solvable with moments $(c_j)_{j=0}^{n-1}$ if and only if the HMP with $(s_j)_{j=0}^{n}$ is solvable. As usual, we let δ_{ij} be the Kronecker symbol.

Theorem 1. *Let* $A = (\delta_{i,j+1})_{i,j=1}^{n}, B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$, *and* $C(\theta)$ *be invertible. And let*

$$\Phi(\theta) := e^{-A\theta} C^{-1}(\theta)\, r(\theta) = \begin{pmatrix} \Phi_1(\theta), & \Phi_2(\theta), & \cdots, & \Phi_n(\theta) \end{pmatrix}^{\mathrm{T}}.$$

Then the TC problem (1) *is solvable if and only if the Markov moment problem with* $L = 1$ *is solvable with entries* $c_i\ (i = 1, 2, \ldots n)$ *as the following*

$$c_i = \frac{(\Phi_i(\theta) - x_{0i})\, i! + (-1)^{i-1}\theta^i}{2(-1)^{i-1}\, i}. \tag{3}$$

Proof. Since $C(\theta)$ is invertible, we have

$$C^{-1}(\theta)\, r(\theta) = x(\theta) = e^{A\theta}\left(x_0 + \int_0^\theta e^{-A\tau} B u(\tau)\,\mathrm{d}\tau\right).$$

That is,

$$e^{-A\theta} C^{-1}(\theta)\, r(\theta) - x_0 = \int_0^\theta e^{-A\tau} B u(\tau)\,\mathrm{d}\tau.$$

Since $u(\tau) = 2f(\tau) - 1$,

$$\int_0^\theta e^{-A\tau} Bu(\tau)\,\mathrm{d}\tau = 2\int_0^\theta e^{-A\tau} Bf(\tau)\,\mathrm{d}\tau - \int_0^\theta e^{-A\tau} B\mathrm{d}\tau,$$

and

$$e^{-A\tau}B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\tau & 1 & 0 & \cdots & 0 & 0 \\ \frac{\tau^2}{2!} & -\tau & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ \frac{(-1)^{n-1}\tau^{n-1}}{(n-1)!} & \cdots & \cdots & \cdots & -\tau & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\tau \\ \frac{\tau^2}{2!} \\ \vdots \\ \frac{(-1)^{n-1}\tau^{n-1}}{(n-1)!} \end{bmatrix},$$

we obtain

$$\frac{(\Phi_i(\theta) - x_{0i})\,i! + (-1)^{i-1}\theta^i}{2(-1)^{i-1}i} = \int_0^\theta \tau^{i-1} f(\tau)\,\mathrm{d}\tau, \quad i = 1, 2, \ldots n.$$

Thus, if the Markov moment problem with $L = 1$ is solvable with entries c_i as in (3), then the TC problem (1) is solvable, i.e., the trajectory $x(t)$ satisfies $x(0) = x_0$, and $r(\theta) = y(\theta)$. The inverse implication can be reserved step by step. □

In [10], the authors introduced the method for obtaining the admissible control of the system (1). We summerize that as following algorithm.

I. Calculate all data c_i of (3).
II. Calculate s_i by (2).
III. Calculate H_i, u_i, v_i, T and $R_T(z)$ by the following relations

	if $n = 2k+1$	if $n = 2k$
H_1	$\begin{pmatrix} s_1 & \cdots & s_{k+1} \\ \vdots & \ddots & \vdots \\ s_{k+1} & \cdots & s_{2k+1} \end{pmatrix}$	$\begin{pmatrix} s_0 & \cdots & s_k \\ \vdots & \ddots & \vdots \\ s_k & \cdots & s_{2k} \end{pmatrix}$
H_2	$\begin{pmatrix} \theta s_0 - s_1 & \cdots & \theta s_k - s_{k+1} \\ \vdots & \ddots & \vdots \\ \theta s_k - s_{k+1} & \cdots & \theta s_{2k} - s_{2k+1} \end{pmatrix}$	$\begin{pmatrix} \theta s_1 - s_2 & \cdots & \theta s_k - s_{k+1} \\ \vdots & \ddots & \vdots \\ \theta s_k - s_{k+1} & \cdots & \theta s_{2k-1} - s_{2k} \end{pmatrix}$
u_1	$(-s_0, -s_1, \ldots, -s_k)^{\mathrm{T}}$	$(0, -s_0, \ldots, -s_{k-1})^{\mathrm{T}}$
T	$(\delta_{i,j+1})_{i,j=0}^k$	$(\delta_{i,j+1})_{i,j=0}^k$
u_2	$(\theta T - 1)\,u_1$	$(s_1 - \theta s_0, s_2 - \theta s_1, \ldots, s_k - \theta s_{k-1})^{\mathrm{T}}$
v_1	$(1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{k+1}$	$(1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{k+1}$
v_2	$(1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{k+1}$	$(1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{k}$
T_1	$(\delta_{i,j+1})_{i,j=0}^k$	$(\delta_{i,j+1})_{i,j=0}^k$
T_2	$(\delta_{i,j+1})_{i,j=0}^k$	$(\delta_{i,j+1})_{i,j=0}^k$
$R_T(z)$	$(I - zT)^{-1}$	$(I - zT)^{-1}$

IV. Calculate U_{11}, U_{12}, U_{21}, and U_{22} by the following relations

	if n is odd	if n is even
$U_{11}(z)$	$1 - zu_2^* R_{T_1^*}(z) H_2^{-1} v_1$	$1 - zu_1^* R_{T_1^*}(z) H_1^{-1} v_1$
$U_{12}(z)$	$u_1^* R_{T_1^*}(z) H_1^{-1} u_1$	$M - zu_1^* R_{T_1^*}(z) H_1^{-1} v_1 M + zu_1^* R_{T_1^*}(z) H_1^{-1} u_1$
$U_{21}(z)$	$-(\theta - z) zv_1^* R_{T_1^*}(z) H_2^{-1} v_1$	$-zv_1^* R_{T_1^*}(z) H_1^{-1} v_1$
$U_{22}(z)$	$1 + zv_1^* R_{T_1^*}(z) H_1^{-1} u_1$	$1 - zv_1^* R_{T_1^*}(z) H_1^{-1} v_1 M + zv_1^* R_{T_1^*}(z) H_1^{-1} u_1$
M		$\left(1 + \theta\left(u_1^* H_1^{-1} v_1 - u_2^* H_2^{-1} v_2\right)\right)\left(\theta v_1^* H_1^{-1} v_1\right)^{-1}$

V. Let $z = t + i\epsilon$, and calculate $-zs(z)$ by the following relations

$$-zs(z) = \frac{U_{11}(\theta - (t + i\epsilon))(F + iG + i\pi) + U_{12}}{U_{21}(\theta - (t + i\epsilon))(F + iG + i\pi) + U_{22}},$$

where

$$F = \frac{1}{2} \ln \frac{(\theta - t)^2 + \epsilon^2}{t^2 + \epsilon^2}, \quad G = \arctan \frac{\theta \epsilon}{t^2 - Tt + \epsilon^2}.$$

VI. Let $\epsilon = 0$, and calculate the real part X and the imaginary part Y of $-zs(z)$.

VII. Calculate X and Y, and obtain

$$u(t) = -\frac{2}{\pi} \arg \frac{Y}{X} - 1.$$

Theorem 2. *The TC problem (if $n = 1$)*

$$\dot{x} = u(t), \ |u| \leq 1, \ x(0) = x_0, \ \text{s.t. } x(\theta) = x_f, \tag{4}$$

is admissible if and only if $|x_0 - x_f| \leq \theta$. In this case, $u(t) = -\frac{2}{\pi} \arg \frac{Y}{X} - 1$, where

$$\begin{aligned} X &= \frac{1}{8}\Big\{t(t-\theta)^3(\theta - x_0 + x_f)^2(2t - \theta - x_0 + x_f)F^2 \\ &\quad + \frac{1}{2}(t-\theta)\left(\theta^2 - (x_0 - x_f)^2\right)\left(\theta^2 - (x_0 - x_f)^2 + 8t(t-\theta)\right)F \\ &\quad + \pi^2 t(t-\theta)^3(\theta - x_0 + x_f)^2(2t - \theta - x_0 + x_f) \\ &\quad + (\theta + x_0 - x_f)^2(2t - \theta + x_0 - x_f)\Big\}, \quad \text{with } F = \ln \frac{\theta - t}{t}, \\ Y &= \frac{1}{16}\pi(t-\theta)(\theta - x_0 + x_f)^2(\theta + x_0 - x_f)^2 \leq 0. \end{aligned}$$

Proof. By the algorithm. □

Corollary 3. *The TC problem (if $n = 1$)*

$$\dot{x} = ax + u(t), \ a < 0, |u| \leq 1, \ x(0) = x_0, \ \text{s.t. } x(\theta) = x_f, \tag{5}$$

is admissible if and only if $\left|x_0 - x_f e^{-a\theta}\right| \leq \theta$. In this case, $u(t) = \left(-\frac{2}{\pi} \arg \frac{Y}{X} - 1\right) e^{at}$, where X and Y are as in Theorem 2.

Proof. Let $z = xe^{-at}$, then by Theorem 2, we know that

$$\dot{z} = u(t)e^{-at} := \tilde{u}(t), |\tilde{u}| \leq 1, \ z(0) = z_0, \ \text{s.t. } z(\theta) = z_f,$$

is admissible if and only if $|z_0 - z_f| \le \theta$, that is, $\left|x_0 - x_f e^{-a\theta}\right| \le \theta$. In this case, $\tilde{u}(t) = -\frac{2}{\pi}\arg\frac{Y}{X} - 1$, that is, $u(t) = \left(-\frac{2}{\pi}\arg\frac{Y}{X} - 1\right)e^{at}$. □

2. Examples

In this section, we give some interesting examples, for $n = 2$ and $n = 3$.

Example 1. We consider

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \\ y(t) = \begin{pmatrix} 1 & \sin\frac{\pi t}{3} \\ 0 & 1 \end{pmatrix} x(t), \end{cases} \tag{6}$$

with initial state vector point $x(0) = (x_{01}, x_{02})^{\mathrm{T}}$ and output terminal point $y(3) = (2,3)^{\mathrm{T}}$, i.e., $\theta = 3$. In this case, we can find the admissible region

$$\mathrm{R}_{ad} = \left\{ (x_{01}, x_{02})^{\mathrm{T}} \middle| \frac{1}{4}x_{01}^2 - \frac{5}{2}x_{01} - \frac{5}{4} \le x_{02} \le -\frac{1}{4}x_{01}^2 - \frac{1}{2}x_{01} + \frac{5}{4} \right\}.$$

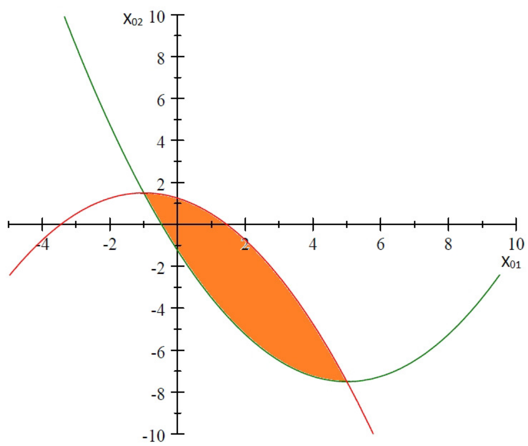

Fig. 1. The admissible region R_{ad} of system (6).
Green line is $x_{02} = \frac{1}{4}x_{01}^2 - \frac{5}{2}x_{01} - \frac{5}{4}$, red line is $x_{02} = -\frac{1}{4}x_{01}^2 - \frac{1}{2}x_{01} + \frac{5}{4}$.

In particular, we take $x_{01} = 0, x_{02} = 1$. Then $(0,1)^{\mathrm{T}}$ is in the above admissible region and according to the algorithm, we finally obtain

$$\begin{aligned} X &= \frac{1}{81}t(t-3)(t(t-3)(20t-59)(20t-9))\left(\ln\frac{3-t}{t}\right)^2 \\ &\quad + \frac{1}{81}t(t-3)\left(1776t^2 - 480t^3 - 1062t + 81\right)\left(\ln\frac{3-t}{t}\right) \\ &\quad + \frac{1}{81}t(t-3)\left(\pi^2 t(t-3)(20t-59)(20t-9) + 9(4t-1)(4t-3)\right), \\ Y &= \pi t(t-3). \end{aligned}$$

The numerical roots of X in $[0,3]$ are $t_1 \approx 0.0064797, t_2 \approx 0.459614$. Hence the control is given by the following

$$u(t) = \begin{cases} -\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right) - \pi\right) - 1, & \text{if } 0 \le t \le 0.0064797, \\ -\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)\right) - 1, & \text{if } 0.0064797 \le t \le 0.459614, \\ -\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right) - \pi\right) - 1. & \text{if } 0.459614 \le t \le 3, \end{cases}$$

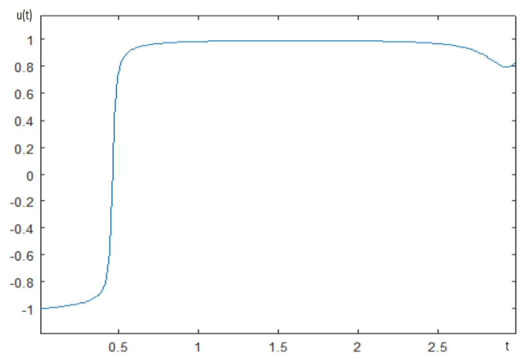

Fig. 2. The plot of $u(t)$ for system (6)

The plots of state vector and output curve of system (6) are as following.

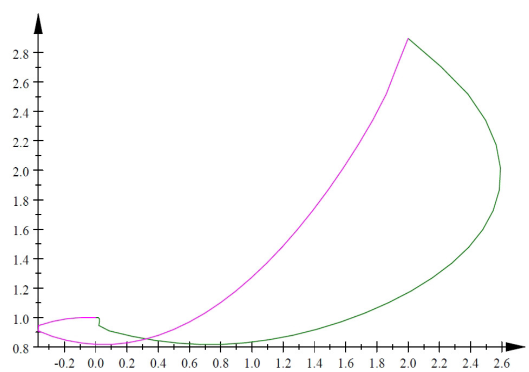

Fig. 3. The state vector (pink) and output curve (green) of system (6)

Example 2. We consider

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t), \\ y(t) = \begin{pmatrix} 1 & \sin\frac{\pi t}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t), \end{cases} \tag{7}$$

with initial state vector point $x(0) = \left(0, \frac{1}{2}, 0\right)^{\mathrm{T}}$ and output terminal point $y(3) = (1,2,3)^{\mathrm{T}}$, i.e., $\theta = 3$. In this case, we can obtain

$$\begin{aligned}
X &= \frac{1}{4}t^2(3t-5)\left(-8t+3t^2+2\right)(t-3)^3\left(\ln\frac{3-t}{t}\right)^2 \\
&+\frac{1}{5}t(t-3)\left(-60t+101t^2-54t^3+9t^4+5\right)\left(\ln\frac{3-t}{t}\right) \\
&+\left(\frac{1}{4}\pi^2t^2(3t-5)\left(-8t+3t^2+2\right)(t-3)^3+\frac{1}{25}t(3t-4)\left(-10t+3t^2+5\right)\right), \\
Y &= \pi t(t-3).
\end{aligned}$$

The numerical roots of X in $[0,3]$ are $t_1 \approx 0.005692, t_2 \approx 0.289924, t_3 \approx 1.66483, t_4 \approx 2.34597$, and $t_5 \approx 2.9001$. Hence the control is given by the following

$$u(t) = \begin{cases}
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)-\pi\right)-1, & \text{if } 0 \le t \le 0.005692, \\
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)\right)-1, & \text{if } 0.005692 \le t \le 0.289924, \\
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)-\pi\right)-1, & \text{if } 0.289924 \le t \le 1.66483, \\
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)\right)-1, & \text{if } 1.66483 \le t \le 2.34597, \\
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)-\pi\right)-1, & \text{if } 2.34597 \le t \le 2.9001, \\
-\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)\right)-1. & \text{if } 2.9001 \le t \le 3,
\end{cases}$$

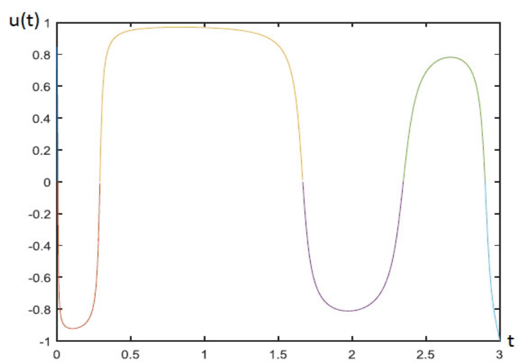

Fig. 4. The plot of $u(t)$

The plots of state vector and output vector of system (7) are as following.

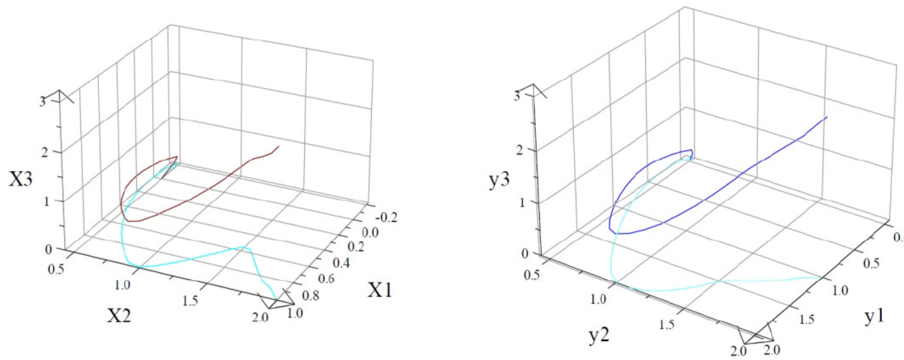

Fig. 5. The plots of state vector and output vector of system (7)

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