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TRIDIAGONALITY OF J-NORMAL AND J-CONJUGATE NORMAL HESSENBERG MATRICES

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ABSTRACT. In this parer we express the sufficient conditions under which it is proved that a J-normal irreducible Hessenberg matrix is tridiagonal and it is also proved that a similar statement exists for J-conjugate normal matrices.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. All definitions and concepts in this paper are taken from [1], [2], [3]. Suppose that $J = I_r \bigoplus(-I_{n-r})$, in which $0 < r \leq n$. A matrix $A \in M_n(\mathbb{C})$ is said to be J-normal if $AA^{[*]} = A^{[*]}A$, where $A^{[*]} = JA^*J$ denotes the J-adjoint of A. Also, recall that a matrix $A = [a_{ij}] \in M_n$ is said to be in upper Hessenberg form or to be an upper Hessenberg matrix if $a_{ij} = 0$ for $i > j + 1$ and is said to be tridiagonal if $a_{ij} = 0$ for $|i - j| > 1$. In other words, a matrix which is both upper and lower Hessenberg is called tridiagonal.

In 1995, L. Elsner and Kh. D. Ikramov proved the following theorem in [4]. This theorem states sufficient condition for the tridiagonality of a normal Hessenberg matrix:

Theorem 1.1. *Let in Eq. (1.1), A be an irreducible upper Hessenberg and normal matrix. If a leading principal submatrix B of order m, $(2 \leq m < n)$ is also normal, then A is actually tridiagonal.*

Then in 2008, a similar theorem was proved for conjugate normal matrices([5]). Now, in this paper we first recall the definitions of J-normal and J-conjugate normal matrices and then examine some sufficient conditions for the tridiagonality of these two class matrices and the results will be mentioned.

In the following, we mention the symbols and relations that are required in the main theorem and its proof.

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let A be a $n \times n$ matrix with real or complex elements and assume that B is $m \times m$ leading principal submatrix of A, $(3 \leq m < n)$. A can be represented as follows:

$$A = \begin{pmatrix} B & C \\ D^* & E \end{pmatrix}. \tag{1.1}$$

Assume that B_{m+1} is a matrix obtained by adding one row and one column to the end of B and B_{m-1} is matrix obtained by deleting the last row and the last column of B. Throughout this paper, wherever we state that "the matrix A is J_A-normal" or "the matrix B is J_B-normal" or "the matrix B_{m-1} is J_{m-1}-normal" or "the matrix B_{m+1} is J_{m+1}-normal", in each of these sentences respectively, the definition of J is as follows:

$$J_A = I_{m-2} \bigoplus (-I_{n-m+2}), \qquad J_B = I_{m-2} \bigoplus (-I_2)$$
$$J_{m-1} = I_{m-2} \bigoplus (-I_1), \qquad J_{m+1} = I_{m-2} \bigoplus (-I_3)$$

For the submatrices with size less than $m-1$, the matrix J will be considered the identity matrix of the same size as it. Thus, anywhere from J-normality is discussed, based on the desired matrix, one of these five cases, is in mind. Note that throughout this paper:

$$A^{[*]} = J_A A^* J_A,\ B^{[*]} = J_B B^* J_B,\ B_{m+1}^{[*]} = J_{m+1} B_{m+1}^* J_{m+1}, \cdots .$$

Now, assume that A is a J_A-normal matrix with the leading principal J_B-normal submatrix B. Then,

$$J_A = I_{m-2} \bigoplus -I_{n-m+2} = I_{m-2} \bigoplus -I_2 \bigoplus -I_{n-m} = J_B \bigoplus -I_{n-m}.$$

So we have

$$A^{[*]} = J_A A^* J_A = \begin{pmatrix} J_B B^* J_B & -J_B D \\ -C^* J_B & E^* \end{pmatrix}.$$

We know from J_A-normality of A that $AA^{[*]} = A^{[*]}A$, thus

$$\begin{pmatrix} BJ_B B^* J_B - CC^* J_B & -BJ_B D + CE^* \\ D^* J_B B^* J_B - EC^* J_B & -D^* J_B D + EE^* \end{pmatrix}$$
$$= \begin{pmatrix} J_B B^* J_B B - J_B DD^* & J_B B^* J_B C - J_B DE \\ -C^* J_B B + E^* D^* & -C^* J_B C + E^* E \end{pmatrix}. \tag{1.2}$$

2. TRIDIAGONALITY OF A J-NORMAL HESSENBERG MATRICES

The aim of this section is to express a sufficient condition under which the J-normal and irreducible Hessenberg matrix is tridiagonal. In fact, we prove the following theorem.

Theorem 2.1. *Let in Eq. (1.1), $A = \begin{pmatrix} a_{i,j} \end{pmatrix} \in M_n(\mathbb{C})$ be irreducible upper Hessenberg and B is its leading principal submatrix of order m, $(3 \leq m < n)$. If A is J_A-normal and B is J_B-normal, then A is actually tridiagonal.*

Proof. Let $l = n - m$, then in Eq. (1.1), D is an $m \times l$ matrix with the only nonzero entry in its $(m, 1)$ position. By using J_A and J_B-normality of A and B respectively, and by comparing the entries (i, i), for $i = 1, ..., m-2$, we have:

$$(AA^{[*]})_{i,i} = \sum_{j=1}^{m-2} a_{ij}\overline{a_{ij}} - \sum_{j=m-1}^{n} a_{ij}\overline{a_{ij}} = \sum_{j=1}^{m-2} |a_{ij}|^2 - \sum_{j=m-1}^{n} |a_{ij}|^2 =$$

$$(A^{[*]}A)_{i,i} = \sum_{j=1}^{m-2} \overline{a_{ji}}a_{ji} - \sum_{j=m-1}^{n} \overline{a_{ji}}a_{ji} = \sum_{j=1}^{m-2} |a_{ji}|^2 - \sum_{j=m-1}^{n} |a_{ji}|^2$$

and similarly for B

$$(BB^{[*]})_{i,i} = \sum_{j=1}^{m-2} |a_{ij}|^2 - \sum_{j=m-1}^{m} |a_{ij}|^2 =$$

$$(B^{[*]}B)_{i,i} = \sum_{j=1}^{m-2} |a_{ji}|^2 - \sum_{j=m-1}^{m} |a_{ji}|^2$$

and for $i = m-1, m$, we have the same equations that have been multiplied by a negative. But according to the Hessenberg structure of A, by removing zero sentences and by comparing the remaining sentences of the two above equations, it is seen that $\sum_{j=m+1}^{n} |a_{i,j}|^2 = 0$, for $i = 1, ..., m-1$. Thus $|a_{i,j}| = 0$, for $j = m+1, ..., n$, i.e.

$$|a_{1,m+1}| = ... = |a_{1,n}| = 0, \ldots, |a_{m-1,m+1}| = ... = |a_{m-1,n}| = 0. \tag{2.1}$$

Thus, we conclude that in both C and D matrices the zero rows have the same number, for the 1 to m-1 rows. Analyzing the zero-nonzero patterns of the first $m-1$ and the last $l-1$ rows of A, we see for $i = 1, ...m-1$ and $j = m+2, ..., n$:

$$(AA^{[*]})_{i,j} = -\sum_{t=1}^{m-2} a_{i,t} \times \overline{a_{j,t}} + \sum_{t=m-1}^{n} a_{i,t} \times \overline{a_{j,t}}$$

But with regard to Hessenberg structure of A and according to Eq. (2.1), respectively, we have the following relations:

$$a_{j,1} = a_{j,2} = ... = a_{j,j-2} = 0, \qquad a_{i,j-1} = a_{i,j} = ... = a_{i,n} = 0.$$

Thus

$$\left(AA^{[*]} \right)_{i,j} = 0, \qquad 1 \le i \le m-1, \qquad m+2 \le j \le n. \tag{2.2}$$

In particular equality (2.2) holds for $i = m-1$ and $j = m+2, ..., n$. I.e.,

$$\begin{pmatrix} A^{[*]}A \\ eq. \end{pmatrix}_{m-1,j} = 0, \qquad j = m+2, ..., n.$$

On the other hand,

$$\left(A^{[*]}A \right)_{m-1,j} = \overline{a_{m,m-1}}a_{m,j}, \qquad j = m+2, ..., n.$$

This is because:

$$(A^{[*]}A)_{m-1,j} = -\sum_{t=1}^{m-2} \overline{a_{t,m-1}} \times a_{t,j} + \sum_{t=m-1}^{n} \overline{a_{t,m-1}} \times a_{t,j}.$$

But by considering the Hessenberg structure of A and according to Eq. (2.1) , respectively, $a_{m+1,m-1} = ... = a_{n,m-1} = 0$ and $a_{1,j} = ... = a_{m-1,j} = 0$ and the only remaining is $\overline{a_{m,m-1}} \times a_{mj}$. Since the entry $a_{m,m+1}$ is nonzero, we have

$$a_{mj} = 0, \qquad j = m+2, ..., n.$$

Now a comparison of the norms of the m-th row and the m-th column in A and B yields

$$(AA^{[*]})_{m,m} = -\sum_{j=1}^{m-2} |a_{m,j}|^2 + \sum_{j=m-1}^{n} |a_{m,j}|^2 =$$
$$(A^{[*]}A)_{m,m} = -\sum_{j=1}^{m-2} |a_{j,m}|^2 + \sum_{j=m-1}^{n} |a_{j,m}|^2$$

and similarly for B

$$(BB^{[*]})_{m,m} = -\sum_{j=1}^{m-2} |a_{m,j}|^2 + \sum_{j=m-1}^{m} |a_{m,j}|^2 =$$
$$(B^{[*]}B)_{m,m} = -\sum_{j=1}^{m-2} |a_{j,m}|^2 + \sum_{j=m-1}^{m} |a_{j,m}|^2.$$

Because of Hessenberg structure of A, $a_{m,1} = ... = a_{m,m-2} = 0$ and $a_{m+2,m} = ... = a_{n,m} = 0$ and because of Eq. (2.1), $a_{m,m+2} = ... = a_{m,n} = 0$. Now comparison of the non-zero sentences of the two above equality, it is seen that

$$|a_{m+1,m}| = |a_{m,m+1}| \equiv s. \tag{2.3}$$

Thus, the block C has the only nonzero entry in the position $(m, 1)$ as does the block D, and both C^*C and D^*D and therefore C^*J_BC and D^*J_BD are $l \times l$ matrices with the only nonzero entry in position $(1, 1)$.

If $m = n - 1$, then A is a bordering J_A-normal matrix for the $m \times m$ matrix B. Assume that $m < n - 1$. If, instead of A, we consider its principal $(m + 1) \times (m + 1)$ submatrix B_{m+1} bordering B, it is easily seen to be J_{m+1}-normality of B_{m+1}. Indeed, by considering the zero-nonzero patterns of entries and relation Eq. (2.3), we earn

$$(B_{m+1}B_{m+1}^{[*]})_{m+1,m+1} = -\sum_{j=1}^{m-2} |a_{m+1,j}|^2 + \sum_{j=m-1}^{m+1} |a_{m+1,j}|^2 =$$
$$(B_{m+1}^{[*]}B_{m+1})_{m+1,m+1} = -\sum_{j=1}^{m-2} |a_{j,m+1}|^2 + \sum_{j=m-1}^{m+1} |a_{j,m+1}|^2.$$

But $a_{m+1,1} = ... = a_{m+1,m+1} = 0$ and $a_{1,m+1} = ... = a_{m-1,m+1} = 0$, because of Hessenberg structure of A and Eq. (2.1), respectively. Thus the following equality is earned:

$$\left(B_{m+1}.B^{[*]}{}_{m+1} \right)_{m+1,m+1} = \left(B^{[*]}{}_{m+1}.B_{m+1} \right)_{m+1,m+1}. \tag{2.4}$$

For each of the remaining positions (i, j), with attention to J_A-normality of A, we earn the following equality:

$$\left(B_{m+1}.B^{[*]}{}_{m+1} \right)_{ij} = \left(AA^{[*]} \right)_{ij} = \left(A^{[*]}A \right)_{ij} = \left(B^{[*]}{}_{m+1}.B_{m+1} \right)_{ij}. \tag{2.5}$$

Indeed:

$$(B_{m+1}B_{m+1}^{[*]})_{i,j} = \sum_{t=1}^{m-2} a_{i,t} \times \overline{a_{j,t}} - \sum_{t=m-1}^{m+1} a_{i,t} \times \overline{a_{j,t}} \tag{2.6}$$

and

$$(B_{m+1}^{[*]}B_{m+1})_{i,j} = \sum_{t=1}^{m-2} \overline{a_{t,i}} \times a_{t,j} - \sum_{t=m-1}^{m+1} \overline{a_{t,i}} \times a_{t,j}. \tag{2.7}$$

On the other side

$$(AA^{[*]})_{i,j} = \sum_{t=1}^{m-2} a_{i,t} \times \overline{a_{j,t}} - \sum_{t=m-1}^{n} a_{i,t} \times \overline{a_{j,t}} =$$

$$(A^{[*]}A)_{i,j} = \sum_{t=1}^{m-2} \overline{a_{t,i}} \times a_{t,j} - \sum_{t=m-1}^{n} \overline{a_{t,i}} \times a_{t,j}.$$

But according to the Hessenberg structure of A

$$a_{j+2,j} = ... = a_{n,j} = a_{i,1} = ... = a_{i,i-2} = a_{i,1} = ... = a_{i,i-2} = 0$$

and because of Eq. (2.1), $a_{2,m+1} = 0$. By considering this zero clauses and by comparing the sentences of the two sides of the recent equality with the remaining sentences of Eqs. (2.6)-(2.7), the relation Eq. (2.5) is earned. Thus, in this case the J_B-normal $m \times m$ matrix B is also embedded into the bordering matrix B_{m+1} of order $m+1$.

Now we show that with the proper definition of J_{m-1}, as described in Section Introduction, the principal submatrix B_{m-1}, for which B is bordering matrix, is also J_{m-1}-normal matrix. In fact,

$$\left(B^{[*]}_{m+1}.B_{m+1} \right)_{m+1,j} = 0, \qquad j = 1, ..., m-2.$$

Therefore,

$$0 = \left(B_{m+1}.B^{[*]}{}_{m+1} \right)_{m+1,j} = a_{j,m}\overline{a_{m+1,m}}$$

which implies

$$a_{j,m} = 0, \qquad j = 1, ..., m-2.$$

From J_B-normality of B and with contrast entry (m, m) in two sides of the equality $B^{[*]}B = BB^{[*]}$, we earn,

$$|a_{m,m-1}| = |a_{m-1,m}|.$$

All details which mentioned in the above about B_{m+1}, with similar calculations to the details of the Eq. (2.3) and before of that, can be analyzed. Now we can prove the J_{m-1}-normality of B_{m-1}. From the equality (2.8) and with contrast the $(m-1)$st row and $(m-1)$st column of B and with attention to zero-nonzero patterns in entries, the following relation is concluded,

$$\left(B_{m-1}.B^{[*]}{}_{m-1} \right)_{m-1,m-1} = \left(B^{[*]}{}_{m-1}.B_{m-1} \right)_{m-1,m-1}. \tag{2.8}$$

For each of the remaining index pairs, as above, we have

$$\left(B_{m-1}.B^{[*]}{}_{m-1} \right)_{ij} = \left(BB^{[*]} \right)_{ij} = \left(B^{[*]}B \right)_{ij} = \left(B^{[*]}{}_{m-1}.B_{m-1} \right)_{ij}. \tag{2.9}$$

The details of these conclusions, are similar to the same results a bout B_{m+1} which were observed in Eqs. (2.4)-(2.5). Equations (2.8)-(2.9) prove the J_{m-1}-normality of B_{m-1}.

So from J_B-normality of B, we conclude J_{m+1}-normality of B_{m+1} and as well J_{m-1}-normality of B_{m-1}. Note that from triple (B_{m-1}, B, B_{m+1}), we earned $a_{j,m} = 0$, for $j = 1, ..., m-2$. Applying the same argument successively to the triples (B_{m-2}, B_{m-1}, B), $(B_{m-3}, B_{m-2}, B_{m-1})$, and so on, we finally conclude that

$$|a_{j,m-1}| = 0, \qquad for \;\; j = 1, ..., m-3,$$

$$|a_{j,m-2}| = 0, \qquad for \;\; j = 1, ..., m-4,$$

$\ddots$

As the final result, B is tridiagonal. Note that for each of this triples, for B, B_{m-1} and B_{m+1}, we consider J as mentioned before. But for B_{m-2}, B_{m-3},..., we let $J_{m-2} = I_{m-2}$, $J_{m-3} = I_{m-3}, \ldots$. For example, in (B_{m-2}, B_{m-1}, B), B_{m-1} is J_{m-1}-normal and it is also embedded into the bordering J_m-normal matrix B of order $m \times m$. If we let $J_{m-2} = I_{m-2}$, it is easy to see normality of B_{m-2}. In fact all details of the calculations to check normality of B_{m-2}, is exactly similar to operations for the triple (B_{m-1}, B, B_{m+1}) with regard to the J_A-normality of A and same results are earned and so on for other triples.

According to the above discussions, it suffices to show the tridiagonality of E. From Eq. (1.2),

$$-D^*J_BD + EE^* = -C^*J_BC + E^*E, \tag{2.10}$$

as mentioned, both C^*C and D^*D are $l \times l$ matrices with the only nonzero entry s^2 in the position $(1, 1)$. Thus D^*J_BD and C^*J_BC, have the only nonzero entry $-s^2$ in the position $(1, 1)$. So according to Eq. (2.10), the matrix E must be normal. As well we know that E has irreduceble Hessenberg structure. So, according to Theorem (2) of [4], E has the tridiagonal structure.

As the final point, with attention to structure of submatrices C and D which have the only one nonzero entry,(in position (m,1)) and with attention to tridiagonal structure of B and E, we conclude that A is actually tridiagonal. □

3. TRIDIAGONALITY OF J-CONJUGATE NORMAL HESSENBERG MATRICES

As close talk to being normal, is conjugate normal, beside J-normality talk, J-conjugate normality appears. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers and let $J = I_r \bigoplus (-I_{n-r})$, in which $0 < r \leq n$. A matrix $A \in M_n(\mathbb{C})$ is said to be J-conjugate normal if $AA^{[*]} = \overline{A^{[*]}A}$, where $A^{[*]} = JA^*J$ denotes the J-adjoint of A and let J and its different representations such as $J_A, J_B, \ldots$ be the same as they were discussed in the introduction section. With these assumptions, we have:

Theorem 3.1. *Let in (1.1), $A = \begin{pmatrix} a_{i,j} \end{pmatrix} \in M_n(\mathbb{C})$ be irreducible upper Hessenberg and B is its leading principal submatrix of order m, $(3 \leq m < n)$. If A is J_A-conjugate normal and B is J_B-conjugate normal, then A is actually tridiagonal.*

Proof. Overall, all details of the proof is analogous to the proof of theorem(2.1). Except that the bar sign over some entries $(\overline{a})$ be replaced or added during the computations, however this does not affect the basic final results. The only significant difference between the process of proving these two theorem is that the equality (2.10) will change as following:

$$-D^*J_BD + EE^* = \overline{-C^*J_BC + E^*E},$$

But according to the explanations, $-D^*J_BD = \overline{-C^*J_BC}$ and so $EE^* = \overline{E^*E}$. This leads to conjugate normality of E. This result along with Hessenberg structure of E, according to theorem (1) in [5], indicate the tridiagonality of E. Another important difference in other parts of these two proof does not exist and everything is similar. □

4. CONCLUTION

We know that an irreducible upper Hessenberg and normal matrix with a leading principal normal submatrix is tridiagonal[4]. Here, we have extended this theorem to a more general case and theorem 2 in [4] is a special case of this generalization in which $J_A = I$. Indeed, we show that if A is J_A-normal and B is its J_B-normal leading principal submatrix, then A is a tridiagonal matrix. Also, we prove a similar theorem for J-conjugate normal matrices.

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