J. Korean Math. Soc. **57** (2020), No. 2, pp. 539–552
https://doi.org/10.4134/JKMS.j190201
pISSN: 0304-9914 / eISSN: 2234-3008

GRADIENT ALMOST RICCI SOLITONS WITH VANISHING CONDITIONS ON WEYL TENSOR AND BACH TENSOR

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Abstract. In this paper we consider gradient almost Ricci solitons with weak conditions on Weyl and Bach tensors. We show that a gradient almost Ricci soliton has harmonic Weyl curvature if it has fourth order divergence-free Weyl tensor, or it has divergence-free Bach tensor. Furthermore, if its Weyl tensor is radially flat, we prove such a gradient almost Ricci soliton is locally a warped product with Einstein fibers. Finally, we prove a rigidity result on compact gradient almost Ricci solitons satisfying an integral condition.

1. Introduction

The concept of almost Ricci solitons was introduced by Pigola, Rigoli, Rimoldi, and Setti as a generalization of Ricci solitons in [17]. An n-dimensional Riemannian manifold (M, g) is an almost Ricci soliton if there exit a vector field X and a smooth function $\lambda : M \to \mathbb{R}$ such that

$$r_g + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where r_g is the Ricci tensor of the metric g and $\mathcal{L}$ is the Lie derivative. If the vector field X is given by the gradient of a smooth function $f : M \to \mathbb{R}$, the manifold is called a gradient almost Ricci soliton. In this case, we have

$$r_g + D_g df = \lambda g, \tag{1.1}$$

where $D_g df$ is the Hessian of f. When λ is constant, this is the usual gradient Ricci soliton. And we say the Ricci soliton (M, g, λ) is *shrinking, steady, expanding* if $\lambda > 0$, $= 0$, < 0, respectively. A gradient almost Ricci soliton has been studied in [2], where Barros and Ribeiro derived several useful identities on structure for almost Ricci solitons which generalize corresponding equivalent for Ricci solitons. In particular, using these identities, they show an integral

Received March 8, 2019; Revised April 19, 2019; Accepted April 24, 2019.
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 58E11.
Key words and phrases. Gradient almost Ricci soliton, harmonic Weyl curvature, conformally flat.
The second author was supported by the National Research Foundation of Korea (NRF-2018R1D1A1B05042186).

formula for compact gradient almost Ricci solitons which enables to obtain a rigidity result. In other words, they proved that a compact nontrivial gradient almost Ricci soliton is isometric to a sphere provided either it has constant scalar curvature or it satisfies an integral condition given by the potential functions.

On the other hand, since classifying complete gradient Ricci solitons in higher dimension is more difficult than 3-dimension, some recent work has focused on complete gradient Ricci solitons with vanishing Weyl tensor. For example, in [19], Z.-H. Zhang proved that any complete gradient shrinking soliton with vanishing Weyl tensor must be a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, or $\mathbb{S}^n$. This is a generalization of 3-dimensional case due to Cao et al. in [5] or Chen in [11]. Note that a 3-dimensional manifold automatically has vanishing Weyl tensor, and for $n \geq 4$, a metric is locally conformally flat if the Weyl tensor $\mathcal{W}$ vanishes.

In case of gradient steady Ricci solitons vanishing Weyl tensor, H.-D. Cao and Q. Chen proved that any n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton ([6]). For $n \geq 3$, R. Bryant proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on $\mathbb{R}^n$. For details, see, e.g., Chow et al. in [12].

There has been various vanishing conditions on the Weyl tensor. A Riemannian manifold (M, g) is said to have *harmonic Weyl curvature* if $\delta\mathcal{W} = 0$. In [13] and [16], M. Fernández-López and E. García-Río, and O. Munteanu and N. Sesum proved that any n-dimensional complete gradient shrinking Ricci soliton with harmonic Weyl tensor is a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, or $\mathbb{S}^n$. Related to gradient almost Ricci solitons with harmonic Weyl tensor, G. Catino proved the following result.

Theorem 1.1 ([9]). *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton with harmonic Weyl tensor, $n \geq 3$, and $\mathcal{W}(\cdot, \cdot, \cdot, \nabla f) = 0$. Then, around any regular point of f, the manifold (M, g) is locally a warped product with $(n-1)$-dimensional Einstein fibers.*

In [14], the second author and G. Yun proved that if (M, g, f) is a compact gradient shrinking Ricci soliton satisfying $\delta\mathcal{W}(\cdot, \cdot, \nabla f) = 0$, then (M, g) is Einstein. In noncompact case, they also showed that if (M, g) is complete and satisfies this weakly harmonic Weyl curvature condition, then (M, g) is rigid in the sense that M is given by a quotient of product of an Einstein manifold with Euclidean space.

In order to find a weaker vanishing condition on the Weyl tensor, Catino, Mastrolia, and Monticelli ([10]) introduced a fourth order vanishing condition on the Weyl tensor as follows:

$$\operatorname{div}^4 \mathcal{W} = \nabla_k \nabla_l \nabla_j \nabla_i \mathcal{W}_{ikjl}.$$

Under the condition that $\mathrm{div}^4\,\mathcal{W} = 0$, they classified gradient shrinking Ricci solitons for $n \geq 4$. Their result clearly generalized previous results concerning gradient shrinking solitons with harmonic Weyl curvature.

In the case of steady and expanding solitons, under natural Ricci curvature assumptions, they showed that the solitons has harmonic Weyl curvature. Namely, they proved the followings.

Theorem 1.2 ([10])**.** *Let (M, g) be an n-dimensional complete gradient steady Ricci soliton, $n \geq 4$, with positive Ricci curvature and such that the scalar curvature attains its maximum at some point. If $\mathrm{div}^4\mathcal{W} = 0$, then (M, g) has harmonic Weyl curvature.*

Theorem 1.3 ([10])**.** *Let (M, g) be an n-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature, $n \geq 4$. If $\mathrm{div}^4\mathcal{W} = 0$, then (M, g) has harmonic Weyl curvature.*

In this paper, we consider gradient almost Ricci solitons with a fourth order vanishing condition on the Weyl tensor, and prove some structural properties which generalize previous results mentioned above. Throughout the paper, we will assume that $n \geq 4$.

First, we prove that a gradient almost Ricci soliton has harmonic Weyl curvature if $\mathrm{div}^4\,\mathcal{W} = 0$ for $n \geq 4$.

Theorem 1.4. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton with $\mathrm{div}^4\,\mathcal{W} = 0$ such that f is bounded below. Assume either each level set of f is compact, or f has polynomial growth at infinity. Then M has harmonic Weyl curvature.*

Note that if (M, g, f) is a complete shrinking Ricci soliton with positive constant λ, then f is bounded below. This easily follows from the facts that

$$s_g + |\nabla f|^2 - 2\lambda f = C_0,$$

where C_0 is some constant and the scalar curvature s_g is nonnegative (cf. see Lemma 2.2 of [8]). However, for almost gradient Ricci solitons, there is neither a lower bound for f nor λ and s_g are necessarily nonnegative. Therefore, we need some bounded condition on f.

Also it should be noted that a potential function was characterized under the nonnegativity of the Ricci curvature in [4] and [10]. However, such conditions were not required in Theorem 1.4. We only required a potential function, which has polynomial growth at infinity and is bounded below.

As a consequence, we have the following result (cf. [18]).

Corollary 1.5. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 4$, be an n-dimensional compact gradient almost Ricci soliton with $\mathrm{div}^4\,\mathcal{W} = 0$. Then M has harmonic Weyl curvature.*

As applications of our main results, we have the following structural results which generalize a result in [9].

Theorem 1.6. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton with* $\operatorname{div}^4 \mathcal{W} = 0$ *such that f is bounded below. Assume either each level set of f is compact, or f has polynomial growth at infinity. If $\mathcal{W}(\cdot, \cdot, \cdot, \nabla f) = 0$, then (M, g) is locally a warped product with $(n-1)$-dimensional Einstein fibers.*

Corollary 1.7. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional compact gradient almost Ricci soliton with* $\operatorname{div}^4 \mathcal{W} = 0$. *If $\mathcal{W}(\cdot, \cdot, \cdot, \nabla f) = 0$, then (M, g) is locally a warped product with $(n-1)$-dimensional Einstein fibers.*

Next, we consider gradient almost Ricci solitons with divergence-free Bach tensor. The Bach tensor discussed first by Bach in [1] is deeply related to general relativity and conformal geometry (cf. [15]). In dimension $n = 4$, it is well-known ([3]) that the Bach tensor is conformally invariant, and arises as a gradient of the total Weyl curvature functional which is given by the integral of the square norm of Weyl tensor. The Bach tensor B of an n-dimensional Riemannian manifold (M^n, g), $n \geq 4$, is defined by

$$B = -\frac{1}{n-3}\, \delta^D \operatorname{div} \mathcal{W} + \frac{1}{n-2}\, \mathring{\mathcal{W}} r_g, \tag{1.2}$$

where δ^D is L^2 adjoint of d^D, and $\mathring{\mathcal{W}} r_g$ is defined by

$$\mathring{\mathcal{W}} r_g(X, Y) = \sum_{i=1}^{n} r_g(\mathcal{W}(X, E_i)Y, E_i)$$

for some orthonormal basis $\{E_i\}_{i=1}^n$. Recall that r_g is the Ricci tensor of g. It is easy to observe that if (M, g) is either locally conformally flat or Einstein, then it is Bach-flat, i.e., $B = 0$. As classifications for gradient Ricci solitons with vanishing Weyl tensor or harmonic Weyl tensor are known, many results on gradient Ricci solitons with Bach-flat are proved (cf. [4, 7]).

A Riemannian manifold (M, g) is said to have divergence-free Bach tensor if $\operatorname{div} B = 0$. This condition is clearly a generalization of Bach flatness condition. Applying [18], we can obtain the following.

Theorem 1.8. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional complete gradient almost Ricci soliton with divergence-free Bach tensor satisfying $\mathcal{W}(\cdot, \cdot, \cdot, \nabla f) = 0$. Assume that each level set of f is compact. Then either (M, g) is Einstein, or is locally a warped product with $(n-1)$-dimensional Einstein fibers. In particular, (M, g) has harmonic Weyl curvature.*

Finally, we prove the following rigidity result for compact gradient almost Ricci solitons.

Theorem 1.9. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional compact gradient almost Ricci soliton satisfying $T = 0$. If $\int_M s_g^2\, dv_g \leq n \int_M \lambda s_g dv_g$, then (M, g) is isometric to an Euclidean sphere $\mathbb{S}^n(r)$.*

2. Preliminaries

In this section, we introduce various notations and derive several identities on an n-dimensional Riemannian manifold M for $n \geq 3$. From now on, we will denote r_g and s_g by r and s, respectively, for simplicity and convenience if there are no ambiguities.

From Riemannian curvature decomposition, the Weyl curvature tensor is given by

$$\mathcal{W}_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}r_{jl} - g_{il}r_{jk} - g_{jk}r_{il} + g_{jl}r_{ik}) + \frac{s}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{2.1}$$

where $r_{ij} = r(E_i, E_j)$.

The differential operator d^D from $C^\infty(S^2M)$ to $C^\infty\left(\Lambda^2 M \otimes T^*M\right)$ is defined by

$$d^D\eta(X,Y,Z) = (D_X\eta)(Y,Z) - (D_Y\eta)(X,Z)$$

for $\eta \in C^\infty(S^2M)$ and vectors X, Y, and Z. For a function $\varphi \in C^\infty(M)$ and $\eta \in C^\infty(S^2M)$, $d\varphi \wedge \eta$ is defined by

$$(d\varphi \wedge \eta)(X,Y,Z) = d\varphi(X)\eta(Y,Z) - d\varphi(Y)\eta(X,Z),$$

where $d\varphi$ denotes the total differential of φ. Then, Cotton tensor $C \in \Gamma(\Lambda^2 M \otimes T^*M)$ is defined by

$$C = d^D r - \frac{1}{2(n-1)} ds \wedge g, \tag{2.2}$$

where s is the scalar curvature of g. It is skew-symmetric for the first two indices and trace free for all other indices, and it satisfies

$$C_{ijk} + C_{jki} + C_{kij} = 0. \tag{2.3}$$

When $n = 3$, it is well known that a metric g is conformally flat if and only if $C = 0$. The Cotton and Weyl tensors are related by

$$C = \frac{n-2}{n-3}\operatorname{div}\mathcal{W}. \tag{2.4}$$

Thus we have

$$\operatorname{div}^3 C = \frac{n-2}{n-3}\operatorname{div}^4\mathcal{W}. \tag{2.5}$$

Finally, we define the covariant 3-tensor T introduced in [7, 3.2] by

$$(n-2)\,T = df \wedge r + \frac{1}{n-1} i_{\nabla f} r \wedge g - \frac{s}{n-1} df \wedge g. \tag{2.6}$$

Then T is skew-symmetric for first two indices and trace free for any other indices.

The following is a well-known equation for gradient almost Ricci soliton [18, Proposition 2.2] or [7, Lemma 3.1]. We briefly summarize its proof. We denote the interior product $\tilde{i}$ to the final factor by

$$\tilde{i}_\xi \omega(X, Y, Z) = \omega(X, Y, Z, \xi)$$

for a 4-tensor ω and a vector field ξ.

Lemma 2.1. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional almost gradient Ricci soliton. Then we have*

$$C + \tilde{i}_{\nabla f}\mathcal{W} = T. \tag{2.7}$$

Proof. By taking d^D to (1.1), we have

$$d^D r + \tilde{i}_{\nabla f} R = d\lambda \wedge g,$$

where R is the Riemann full tensor. Thus, we get

$$C + \tilde{i}_{\nabla f} R = d\lambda \wedge g - \frac{1}{2(n-1)} ds \wedge g. \tag{2.8}$$

From the decomposition of the Riemann tensor given by (2.1), we have

$$d^D r + \tilde{i}_{\nabla f}\mathcal{W} - \frac{1}{n-2} i_{\nabla f} r \wedge g + \frac{s}{(n-1)(n-2)} df \wedge g - \frac{1}{n-2} df \wedge r = d\lambda \wedge g.$$

By the definition of the Cotton tensor and T,

$$C + \tilde{i}_{\nabla f}\mathcal{W} = T + \frac{1}{n-1} i_{\nabla f} r \wedge g - \frac{1}{2(n-1)} ds \wedge g + d\lambda \wedge g. \tag{2.9}$$

On the other hand, by taking the trace of (1.1) we get

$$s + \Delta f = n\lambda,$$

implying that

$$ds + d\Delta f = n d\lambda.$$

By taking the divergence of (1.1) we have

$$-\frac{1}{2} ds - d\Delta f - i_{\nabla f} r = -d\lambda.$$

By adding the last two equations, we have

$$\frac{1}{2} ds - i_{\nabla f} r = (n-1) d\lambda. \tag{2.10}$$

Our Lemma follows by substituting this equation into (2.9). □

We also have the following result (cf. [7, Lemma 5.1]).

Proposition 2.2.

$$\operatorname{div}^2 C(X) = \frac{1}{2}\langle \tilde{i}_X R, C\rangle.$$

Proof. Recall that Schouten tensor A is defined by

$$A = r_g - \frac{s}{2(n-1)} g.$$

Note that

$$\begin{aligned} \delta\delta C(X) &= \delta\delta d^D A(X) \\ &= D_{E_i} D_{E_k} (D_{E_k} A(E_i, X) - D_{E_i} A(E_k, X)) \\ &= (D_{E_i} D_{E_k} - D_{E_k} D_{E_i}) D_{E_k} A(E_i, X). \end{aligned}$$

Since

$$D^3_{X,Y,Z} h - D^3_{Y,X,Z} h = -R(X, Y) D_Z h + D_{R(X,Y)Z} h$$

for any tensor h (see Corollary 1.22 of [3]), we have

$$\begin{aligned} \operatorname{div}^2 C(X) &= R(E_k, E_i) D_{E_k} A(E_i, X) - D_{R(E_k,E_i)E_k} A(E_i, X) \\ &= -D_{E_k} A(R(E_k, E_i)E_i, X) - D_{E_k} A(E_i, R(E_k, E_i)X) \\ &\quad - r_{is} D_{E_s} A(E_i, X) \\ &= r_{kj} D_{E_k} A(E_j, X) + \langle R(E_k, E_i)E_s, X\rangle D_{E_k} A(E_i, E_s) \\ &\quad - r_{ik} D_{E_k} A(E_i, X) \\ &= \frac{1}{2} \langle R(E_k, E_i)E_s, X\rangle C(E_k, E_i, E_s). \end{aligned}$$

□

Consequently, using (2.8) and Proposition 2.2 we have

$$\begin{aligned} \operatorname{div}^2 C(\nabla f) &= \frac{1}{2} \langle \tilde{i}_{\nabla f} R, C\rangle \\ &= \frac{1}{2} \langle d\lambda \wedge g - \frac{1}{2(n-1)} ds \wedge g - C, C\rangle = -\frac{1}{2} |C|^2, \end{aligned} \tag{2.11}$$

since C is trace free in any two indices. Equation (2.11) is known to be true when $n = 3$ (cf. [10]). Here, we proved (2.11) for $n \geq 4$.

Now we briefly discuss the boundness of f. Let $\Lambda = s + |\nabla f|^2 - 2\lambda f$. It is well known that Λ is constant when (M, g) is a gradient Ricci soliton. However, this is not true in the case of an almost Ricci soliton. Suppose that $\lambda > 0$ and s is bounded below on M. From the equations derived above, we have

$$\nabla(s + |\nabla f|^2 - 2\lambda f) = ((n-1) - f)\nabla\lambda.$$

Thus, if there exists a maximum point of $s + |\nabla f|^2 - 2\lambda f$, then f is bounded below. However, once λ takes both positive and negative values on M, the lower boundness of f depends deeply on s, λ, and Λ.

Finally, we have the following equation for the divergence of the Bach tensor B.

Lemma 2.3 ([7, Lemma 5.1])**.** *For any vector field X we have*

$$(n-2) \operatorname{div} B(X) = \frac{n-4}{n-2} \langle i_X C, z\rangle.$$

3. Proof of Theorems 1.4 and 1.6

First, to prove Theorem 1.4, it suffices to prove that C vanishes identically on all of M.

First, assume that each level set of f is compact. For a specific value of t, let $M_t = \{x \in M \mid f(x) < t\}$. From the assumption on f,

$$\int_{M_t} \operatorname{div}\big((f-t)\operatorname{div}^2 C\big)$$

is finite. When t is a regular value of f, $N = \nabla f / |\nabla f|$ is well-defined, hence,

$$\int_{M_t} \operatorname{div}\big((f-t)\operatorname{div}^2 C\big) = \int_{\partial M_t} (f-t)\operatorname{div}^2 C(N) = 0.$$

Since

$$\operatorname{div}\big((f-t)\operatorname{div}^2 C\big) = \operatorname{div}^2 C(\nabla f) + (f-t)\operatorname{div}^3 C,$$

from the assumption that $\operatorname{div}^3 C = 0$ and (2.11) we have

$$\int_{M_t} \operatorname{div}\big((f-t)\operatorname{div}^2 C\big) = \int_{M_t} \operatorname{div}^2 C(\nabla f) = -\frac{1}{2}\int_{M_t} |C|^2.$$

Hence, $C = 0$ on M_t for any regular value t of f. Since t is arbitrary, we may conclude that $C = 0$ on all of M by continuity.

Next, assume that f has polynomial growth at infinity. In this case, we prove Theorem 1.4 by using the standard cut off function method. Choose $\psi(f) = e^{-f}\phi(f)$, where for any fixed $s > 0$, $\phi : \mathbb{R} \to \mathbb{R}$ is a nonnegative C^3 function such that $\phi \equiv 1$ on $[0, s]$, $\phi \equiv 0$ on $[2s, \infty)$ and $\phi' \le 0$ on $[s, 2s]$. From the assumption that f has polynomial growth at infinity, for every $s > 0$, the cutoff function $\psi(f)$ has compact support in M, and by (2.11) and integrating by parts we obtain

$$\begin{aligned}
\frac{1}{2}\int_M |C|^2 e^{-f}\phi(f) &= -\int_M \operatorname{div}^2 C(E_k) e^{-f}\phi(f)\nabla_k f \\
&= \int_M \operatorname{div}^2 C(E_k)(e^{-f})_k \phi(f) \\
&= -\int_M \big(\operatorname{div}^3 C\, e^{-f}\phi(f) + \operatorname{div}^2 C(\nabla f) e^{-f}\phi'(f)\big) \\
&= \frac{1}{2}\int_M |C|^2 e^{-f}\phi'(f).
\end{aligned}$$

Since $\phi'(f) \le 0$, we may conclude that $C = 0$ on M. This completes the proof of Theorem 1.4.

For the proof of Theorem 1.6, first note that the manifold M has harmonic Weyl curvature by Theorem 1.4. Then, by Theorem 1.1 or following the proof of Theorem 4.7 in [18], we may conclude that (M, g) is locally a warped product with $(n-1)$-dimensional Einstein fibers.

4. Proof of Theorem 1.8

In this section we prove Theorem 1.8. Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton. Note that, by the definition T is trace free in any two indices, and so

$$|T|^2 = \frac{1}{n-2}\langle T, df \wedge r\rangle = \frac{2}{(n-2)}\langle i_{\nabla f}T, r\rangle, \tag{4.1}$$

where i_X is the usual interior product with respect to X. In particular, if $T = 0$, we have the following result.

Lemma 4.1 ([18, Lemma 3.3 and Lemma 4.2])**.** *Assume that $T = 0$. Then, for X orthogonal to ∇f,*

$$r(\nabla f, X) = 0. \tag{4.2}$$

Let $\alpha = r(N, N)$ with $N = \nabla f/|\nabla f|$. Moreover, we have

$$\begin{aligned} 0 = & \frac{ns-(n-1)^2\lambda-\alpha}{n-1} r_{-}\nabla_{\nabla f} r - r\circ r \\ & + \frac{n-3}{2(n-1)} df \otimes ds + \frac{1}{n-1} df \otimes d\alpha \\ & + \frac{1}{n-1}(ds(f) - df(\alpha) + s + (n-1)\lambda(\alpha - s))g. \end{aligned} \tag{4.3}$$

Here, for arbitrary vector fields X and Y, $r \circ r$ is defined by

$$r\circ r = \sum_{i=1}^{n} r(X, E_i) r(E_i, Y),$$

where $\{E_i\}_{i=1}^n$ is an orthonormal frame.

Proof. From the definition of T, for X orthogonal to ∇f we have

$$(n-2)T(\nabla f, X, \nabla f) = \frac{n-2}{n-1} r(\nabla f, X)|\nabla f|^2.$$

Thus, (4.2) follows from the assumption that $T = 0$. In particular, $i_{\nabla f} r = \alpha\, df$. Therefore,

$$\operatorname{div}(i_{\nabla f} r \wedge g) = df(\alpha)g + ((n-1)\lambda + s)\alpha g - df \otimes d\alpha + \alpha\, r.$$

Moreover,

$$\operatorname{div}(df \wedge r) = ((n-1)\lambda - s)r + D_{\nabla f} r + r\circ r - \frac{1}{2} df \otimes ds,$$

and

$$\operatorname{div}(s\, df \wedge g) = df(s)g + s((n-1)\lambda - s)g - df \otimes ds + s\, r.$$

Hence,

$$(n-2)\operatorname{div} T = \frac{(n-1)^2\lambda + \alpha - ns}{n-1} r + D_{\nabla f} r_g + r \circ r$$

$$- \frac{n-3}{2(n-1)} df \otimes ds - \frac{1}{n-1} df \otimes d\alpha$$
$$+ \frac{1}{n-1}(df(\alpha) - df(s) - s + (n-1)\lambda(\alpha - s))g.$$

Thus, (4.3) follows from $\operatorname{div} T = 0$. □

Consequently, we have

$$r \circ r(\nabla f, \nabla f) = \alpha^2 |\nabla f|^2, \tag{4.4}$$

where $\alpha = r(N, N)$ with $N = \nabla f / |\nabla f|$.

Corollary 4.2. *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton. Assume that $T = 0$. Then, $|\nabla f|^2$, $(n-3)s + 2\alpha$, $s + 2(1-n)\alpha$, and $s + 2(1-n)\lambda$ are constant on each level set of f.*

Proof. From (4.2), for X orthogonal to ∇f we have

$$\frac{1}{2} X(|\nabla f|^2) = \langle D_X df, \nabla f \rangle = \lambda\, df(X) - r(X, \nabla f) = 0.$$

Moreover, by putting $(X, \nabla f)$ in (4.3) with $\operatorname{div} T = 0$

$$D_{\nabla f} r(X, \nabla f) = 0.$$

Now, by putting $(\nabla f, X)$ in (4.3) with $\operatorname{div} T = 0$ again, we get

$$0 = -\frac{n-3}{2(n-1)} |\nabla f|^2 ds(X) - \frac{1}{n-1} |\nabla f|^2 d\alpha(X),$$

since $r(X, \nabla f) = 0$ and

$$D_{\nabla f} r(\nabla f, X) = D_{\nabla f} r(X, \nabla f).$$

On the other hand, from (2.7) we have

$$C(X, \nabla f, \nabla f) = -\mathcal{W}(X, \nabla f, \nabla f, \nabla f) + T(X, \nabla f, \nabla f) = 0$$
$$= D_X r(\nabla f, \nabla f) - \frac{1}{2(n-1)}\, ds(X) |\nabla f|^2.$$

Thus, from

$$r(D_X df, \nabla f) = \lambda\, r(X, \nabla f) - r \circ r(X, \nabla f) = 0,$$

it follows that

$$X(\alpha) = \frac{1}{|\nabla f|^2}\, X(r(\nabla f, \nabla f))$$
$$= \frac{1}{|\nabla f|^2}\, (D_X r(\nabla f, \nabla f) + 2r(D_X df, \nabla f)) = \frac{1}{2(n-1)} ds(X).$$

This implies that $s + 2(1-n)\alpha$ is constant on each level sets of f.

Finally, by taking the trace of (1.1)

$$s + \Delta f = n\lambda$$

and hence,

$$ds + d\Delta f = nd\lambda.$$

Moreover, by putting a vector X orthogonal to ∇f into (2.10) we get

$$\frac{1}{2}ds(X) = (n-1)d\lambda(X).$$

□

By Corollary 4.2, we may conclude that s, α, and λ are constant on each regular level set of f.

Corollary 4.3 ([18, Lemma 4.6]). *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional gradient almost Ricci soliton. Assume that $T = 0$. Then the Ricci tensor has two eigenvalues.*

Proof. Let $\{E_i\}$, where $1 \le i \le n$, be an orthonormal frame with $E_n = N = \nabla f/|\nabla f|$ and $\alpha = r(N, N)$. Considering the second fundamental form of each level set of f, we have

$$II_{ij} = \langle \nabla_{E_i} N, E_j\rangle = \frac{1}{|\nabla f|} Ddf_{ij} = \frac{1}{|\nabla f|}(\lambda g_{ij} - r_{ij}) \tag{4.5}$$

and

$$\operatorname{tr} II = m = \frac{n-1}{|\nabla f|}\left(\lambda + \frac{\alpha - s}{n-1}\right).$$

Thus, for each level set of f, the mean curvature m is constant, and

$$\begin{aligned}
\left|II - \frac{m}{n-1}g\right|^2 &= |II|^2 - \frac{m^2}{n-1} \\
&= \frac{1}{|\nabla f|^2}\langle \lambda g_{ij} - r_{ij}, \lambda g_{ij} - r_{ij}\rangle - \frac{n-1}{|\nabla f|^2}\left(\lambda + \frac{\alpha - s}{n-1}\right)^2 \\
&= \frac{1}{|\nabla f|^2}\{(n-1)\lambda^2 - 2\lambda(s-\alpha) + |r|^2 - \alpha^2 - \lambda^2(n-1) \\
&\quad - \frac{(\alpha - s)^2}{n-1} - 2\lambda(\alpha - s)\} \\
&= \frac{1}{|\nabla f|^2}\{|r|^2 - \frac{n}{n-1}\alpha^2 + \frac{2s\alpha}{n-1} - \frac{s^2}{n-1}\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{n-2}{2}|T|^2 &= \langle i_{\nabla f} T, r\rangle \\
&= |r|^2|\nabla f|^2 - \frac{n}{n-1} r \circ r(\nabla f, \nabla f) + \frac{2s}{n-1} r(\nabla f, \nabla f) - \frac{s^2}{n-1}|\nabla f|^2. \qquad (4.6)
\end{aligned}$$

Thus, by (4.4)

$$|T|^2 = \frac{2}{(n-2)^2}|\nabla f|^4 \left|II - \frac{m}{n-1}g\right|^2.$$

It follows from $T=0$ that

$$II_{ij} = \frac{m}{n-1} g_{ij}. \tag{4.7}$$

Therefore, we may conclude that

$$r_{ij} = \frac{s-\alpha}{n-1} g_{ij} \tag{4.8}$$

for $1 \le i, j \le n-1$. □

To prove Theorem 1.9, we need the following.

Theorem 4.4 ([2, Corollary 1])**.** *Let $(M, g, \nabla f, \lambda)$ be an n-dimensional compact gradient almost Ricci soliton. If either s is constant, or the following inequality*

$$\int_M \{r(\nabla f, \nabla f) + (n-1)\langle \nabla\lambda, \nabla f\rangle\}\, dv_g \le 0$$

holds, then (M, g) is isometric to an Euclidean sphere $\mathbb{S}^n(r)$.

To prove Theorem 1.9, we have

$$\int_M \{r(\nabla f, \nabla f) + (n-1)\langle \nabla\lambda, \nabla f\rangle\}\, dv_g = \int_M \left\{\alpha|\nabla f|^2 + (n-1)\langle \nabla\lambda, \nabla f\rangle\right\} dv_g.$$

Then, by (2.10) we have

$$(n-1)\langle \nabla\lambda, \nabla f\rangle + \alpha|\nabla f|^2 = \frac{1}{2}\langle \nabla s, \nabla f\rangle.$$

Hence, from the assumption $T=0$ together with the fact $\Delta f = n\lambda - s$,

$$\begin{aligned}\int_M \{r(\nabla f, \nabla f) + (n-1)\langle \nabla\lambda, \nabla f\rangle\}\, dv_g &= \frac{1}{2}\int_M \langle \nabla s, \nabla f\rangle\, dv_g \\ &= -\frac{1}{2}\int_M s\Delta f dv_g \\ &= -\frac{1}{2}\int_M s(n\lambda - s) dv_g \le 0.\end{aligned}$$

The proof of Theorem 1.9 follows from Theorem 4.4.

Now, we are ready to prove Theorem 1.8. First, taking an inner product with a Ricci tensor on both sides of (2.7), we get

$$\langle i_X C, r\rangle + \langle i_X \tilde{i}_{\nabla f}\mathcal{W}, r\rangle = \langle i_X T, r\rangle. \tag{4.9}$$

Also, by Lemma 2.3

$$\frac{n-4}{(n-2)^2}\langle i_{\nabla f} C, r\rangle = \mathrm{div} B(\nabla f) = 0. \tag{4.10}$$

Thus, by substituting (4.9) with the assumption that $\mathcal{W}(\cdot,\cdot,\cdot,\nabla f) = 0$ into (4.10) we have

$$\langle i_{\nabla f} T, r\rangle = 0.$$

In particular, by (4.1) we have $T = 0$. Hence, from

$$C + \tilde{i}_{\nabla f}\mathcal{W} = T,$$

we have $C = 0$. In other words, M has harmonic Weyl curvature. Moreover, since

$$0 = \operatorname{div}(\tilde{i}_{\nabla f}\mathcal{W})(X,Y) = \frac{n-3}{n-2}C(Y,\nabla f,X) + \mathring{\mathcal{W}}r(X,Y)$$

(cf. [18, Section 3]), we have $\mathring{\mathcal{W}}r = 0$. Hence, (1.2) and (2.4) imply that g is Bach-flat. By taking $h = 1$ in Theorem 1.1 of [18], Theorem 1.8 immediately follows.

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