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ON THE SYMMETRIES OF THE Sol_3 LIE GROUP

Lakehal Belarbi

Abstract. In this work we consider the Sol_3 Lie group, equipped with the left-invariant metric, Lorentzian or Riemannian. We determine Killing vector fields and affine vectors fields. Also we obtain a full classification of Ricci, curvature and matter collineations.

1. Introduction

Symmetries in general relativity have been extensively studied because of their interest both from the mathematical and physical viewpoint. The term Symmetry here refers to a one-parameter group of diffeomorphisms of the space-time preserving certain mathematical or physical quantity. Many works are studying the symmetries of Lorentzian manifolds. See for example [1–6, 8–11].

The aim of this paper is to investigate symmetries of the Sol_3 Lie group. Let $(\mathcal{M}, g)$ denote a Lorentzian manifold. A vector field X on $\mathcal{M}$ preserving its metric tensor g, the corresponding Levi-Civita connection ∇, its curvature tensor R or its Ricci tensor Ric, is respectively known as a Killing vector field, an affine vector field, a curvature collineation or a Ricci collineation. If T is a tensor on $(\mathcal{M}, g)$, codifying some mathematical or physical quantity, a symmetry of a tensor field T is a one-parameter group of diffeomorphisms of $(\mathcal{M}, g)$, which leaves T invariant (i.e., If the C^1 local diffeomorphisms associated with X are denoted by ϕ_t, then $\phi_t^* T = T$). By this definition, each symmetry corresponds to a vector field X which satisfies $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. Famous symmetries are: symmetries of the metric tensor g which correspond to the Killing vector fields, symmetries of the Levi-Civita connection ∇ which correspond to the affine vector fields, and symmetries of the Ricci tensor Ric which correspond to the Ricci collineations, and symmetries of the curvature tensor R which correspond to the curvature collineations.

A matter collineation of a Lorentzian or Riemannian manifold $(\mathcal{M}, g)$ is a vector field X, corresponding to a symmetry of the energy-momentum tensor

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$T = Ric - \frac{1}{2}\tau g$, where τ denotes the scalar curvature. Matter collineations are more relevant from a physical point of view [7] and [8].

In this paper, we shall investigate symmetries of the three-dimensional solvable Lie group Sol_3, equipped with a left invariant metric g, Lorentzian or Riemannian. We determine Killing vector fields and affine vector fields of the Sol_3 Lie group. Also we obtain a full classification of Ricci, curvature and matter collineations, and we remark that all solutions of matter collineation coincide with curvature collineation of the Sol_3 Lie group.

2. Connection and curvature of the Sol_3 group

Let us consider the three-dimensional solvable Lie group Sol_3 which is diffeomorphic to the cartesian space $\mathbb{R}^3(x, y, z)$.

The group structure of three-dimensional Lie group Sol_3 is given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z').$$

The isometries under the Lorentzian metric (1) are

$$(x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c),$$

where a, b and c are any real numbers.

The isometries under the Riemannian metric (7) are

$$(x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c)$$

and

$$(x, y, z) \mapsto (\pm e^{-c}y + a, \pm e^{c}x + b, -z + c),$$

where a, b and c are any real numbers.

Throughout the paper, we shall endow the three-dimensional solvable Lie group Sol_3 with left-invariant Lorentzian and Riemannian metric g.

We will denote by ∇ the Levi-Civita connection of (Sol_3, g), by R its curvature tensor, taken with the sign convention:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and by Ric the Ricci tensor of (Sol_3, g), which is defined by

$$Ric\,(X,Y) = \sum_{k=1}^{3} g\,(E_k, E_k)\, g(R(E_k, X)Y, E_k),$$

where $\{E_k\}_{k=1,\ldots,3}$ is an orthonormal basis.

2.1. Lorentzian setting

The Lorentzian Sol_3 Lie group is a Lie group $\mathbb{R}^3$ endowed with a left-invariant Lorentzian metric

$$g = e^{2z}dx^2 - e^{-2z}dy^2 + dz^2, \tag{1}$$

where (x, y, z) the usual coordinates of $\mathbb{R}^3$.

A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in the Lorentzian Sol_3 Lie group is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}. \tag{2}$$

The Levi-Civita connection ∇ of the Lorentzian Sol_3 Lie group with respect to this frame is

$$\begin{cases} \nabla_{E_1}E_1 = -E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = E_1, \\ \nabla_{E_2}E_1 = 0, \nabla_{E_2}E_2 = -E_3, \nabla_{E_2}E_3 = -E_2, \\ \nabla_{E_3}E_1 = 0, \nabla_{E_3}E_2 = 0, \nabla_{E_3}E_3 = 0. \end{cases} \tag{3}$$

The non-vanishing curvature tensor R components are computed as

$$\begin{cases} R(E_1,E_2)E_1 = -E_2, R(E_1,E_2)E_2 = -E_1, \\ R(E_1,E_3)E_1 = E_3, R(E_1,E_3)E_3 = -E_1, \\ R(E_2,E_3)E_2 = -E_3, R(E_2,E_3)E_3 = -E_2. \end{cases} \tag{4}$$

The Ricci curvature components $\{Ric_{ij}\}$ are computed as

$$Ric_{11} = Ric_{12} = Ric_{13} = Ric_{23} = Ric_{22} = 0,\ \ Ric_{33} = -2. \tag{5}$$

The scalar curvature τ of the Lorentzian Sol_3 Lie group is constant and we have

$$\tau = trRic = \sum_{i=1}^{3} g(E_i,E_i)Ric(E_i,E_i) = -2. \tag{6}$$

2.2. Riemannian setting

The Riemannian Sol_3 Lie group is a Lie group $\mathbb{R}^3$ endowed with a left-invariant Riemannian metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{7}$$

where (x, y, z) the usual coordinates of $\mathbb{R}^3$.

A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in the Riemannian Sol_3 Lie group is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}. \tag{8}$$

The Levi-Civita connection ∇ of the Riemannian Sol_3 Lie group with respect to this frame is

$$\begin{cases} \nabla_{E_1}E_1 = -E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = E_1, \\ \nabla_{E_2}E_1 = 0, \nabla_{E_2}E_2 = E_3, \nabla_{E_2}E_3 = -E_2, \\ \nabla_{E_3}E_1 = 0, \nabla_{E_3}E_2 = 0, \nabla_{E_3}E_3 = 0. \end{cases} \tag{9}$$

The non-vanishing curvature tensor R components are computed as

$$\begin{cases} R(E_1,E_2)E_1 = -E_2, R(E_1,E_2)E_2 = E_1, \\ R(E_1,E_3)E_1 = E_3, R(E_1,E_3)E_3 = -E_1, \\ R(E_2,E_3)E_2 = E_3, R(E_2,E_3)E_3 = -E_2. \end{cases} \tag{10}$$

The Ricci curvature components $\{Ric_{ij}\}$ are computed as

$$Ric_{11} = Ric_{12} = Ric_{13} = Ric_{23} = Ric_{22} = 0, \quad Ric_{33} = -2. \tag{11}$$

The scalar curvature τ of the Riemannian Sol_3 Lie group is constant and we have

$$\tau = trRic = \sum_{i=1}^{3} g(E_i, E_i)Ric(E_i, E_i) = -2. \tag{12}$$

3. Killing and affine vector fields of the Sol_3 Lie group

In this section we completely classify Killing vector fields and affine vector fields of three-dimensional Lorentzian and Riemannian solvable Lie group (Sol_3, g). We will denote the coordinates basis$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ by $\{\partial_x, \partial_y, \partial_z\}$.

3.1. Lorentzian metric

We first classify Killing and affine vector fields of the Lorentzian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 3.1. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Lorentzian Sol_3 Lie group.*

- *X is a Killing vector field if and only if*

$$\begin{cases} f_1 = -(\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \quad \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

- *X is an affine vector field if and only if*

$$\begin{cases} f_1 = -\beta x e^z, \\ f_2 = \beta y e^{-z}, \\ f_3 = \beta, \quad \beta \in \mathbb{R}. \end{cases}$$

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Lorentzian Sol_3 Lie group, for some arbitrary smooth functions f_1, f_2, f_3 on Lorentzian Sol_3 group. Starting from (1), a direct calculation yields the following description of the Lie derivative of the metric tensor g:

$$\begin{cases} (\mathcal{L}_X g)(E_1, E_1) = 2[f_3 + e^{-z}\partial_x f_1], \\ (\mathcal{L}_X g)(E_1, E_2) = -e^{-z}\partial_x f_2 + e^z\partial_y f_1, \\ (\mathcal{L}_X g)(E_1, E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1, \\ (\mathcal{L}_X g)(E_2, E_2) = 2[f_3 - e^z\partial_y f_2], \\ (\mathcal{L}_X g)(E_2, E_3) = e^z\partial_y f_3 - f_2 - \partial_z f_2, \\ (\mathcal{L}_X g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{13}$$

In order to determine the Killing vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative are equal to zero.

A straightforward calculation leads to prove that

$$\begin{cases} f_1 = -(\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \quad \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

To determine the affine vector fields, we need to calculate the Lie derivative of the Levi-Civita connection ∇. Staring from (3), we find the following possibly non-vanishing components:

$$\begin{cases} (\mathcal{L}_X\nabla)(E_1, E_1) = [-f_1 + 2e^{-z}\partial_x f_3 + \partial_z f_1 + e^{-2z}\partial_x^2 f_1]E_1 \\ \qquad + [f_2 + \partial_z f_2 + e^{-2z}\partial_x^2 f_2]E_2 \\ \qquad + [\partial_z f_3 - 2f_3 - 2e^{-z}\partial_x f_1 + e^{-2z}\partial_x^2 f_3]E_3, \\ (\mathcal{L}_X\nabla)(E_1, E_2) = [e^z\partial_y f_3 + \partial_x\partial_y f_1]E_1 + [-e^{-z}\partial_x f_3 + \partial_x\partial_y f_2]E_2 \\ \qquad + [-e^{-z}\partial_x f_2 - e^z\partial_y f_1 + \partial_x\partial_y f_3]E_3, \\ (\mathcal{L}_X\nabla)(E_1, E_3) = [-e^{-z}\partial_x f_1 + e^{-z}\partial_x\partial_z f_1 + \partial_z f_3]E_1 \\ \qquad + [-e^{-z}\partial_x f_2 + e^{-z}\partial_x\partial_z f_2]E_2 \\ \qquad + [-e^{-z}\partial_x f_3 + e^{-z}\partial_x\partial_z f_3 + f_1 - \partial_z f_1]E_3, \\ (\mathcal{L}_X\nabla)(E_2, E_2) = [-f_1 + \partial_z f_1 + e^{2z}\partial_y^2 f_1]E_1 \\ \qquad + [f_2 + \partial_z f_2 - 2e^z\partial_y f_3 + e^{2z}\partial_y^2 f_2]E_2 \\ \qquad + [\partial_z f_3 + 2f_3 - 2e^z\partial_y f_2 + e^{2z}\partial_y^2 f_3]E_3, \\ (\mathcal{L}_X\nabla)(E_2, E_3) = [e^z\partial_y f_1 + e^z\partial_y\partial_z f_1]E_1 \\ \qquad + [e^z\partial_y f_2 + e^z\partial_y\partial_z f_2 - \partial_z f_3]E_2 \\ \qquad + [e^z\partial_y f_3 + e^z\partial_y\partial_z f_3 - f_2 - \partial_z f_2]E_3, \\ (\mathcal{L}_X\nabla)(E_3, E_3) = [-f_1 + \partial_z^2 f_1]E_1 + [-f_2 + \partial_z^2 f_2]E_2 + [\partial_z^2 f_3]E_3. \end{cases} \tag{14}$$

In order to determine the affine vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative are equal to zero.

From $(\mathcal{L}_X\nabla)^3(E_3, E_3) = 0$ it follows that $f_3 = H(x, y)z + \overline{H}(x, y)$, where H and $\overline{H}$ are real-valuable smooth functions on $\mathbb{R}^2$.

We then replace f_3 in the equation given by $(\mathcal{L}_X\nabla)^3(E_2, E_2) = 0$ we get

$$\partial_y f_2 = \frac{1}{2}e^{-z}[H + 2Hz + 2\overline{H} + e^{2z}\partial_y^2 Hz + e^{2z}\partial_y^2\overline{H}]$$

and

$$\partial_y\partial_z f_2 = \frac{1}{2}e^{-z}[H - 2Hz - 2\overline{H} + e^{2z}\partial_y^2 Hz + e^{2z}\partial_y^2 H + e^{2z}\partial_y^2\overline{H}].$$

And we replace f_3, $\partial_y f_2$ and $\partial_y\partial_z f_2$ in the equation given by $(\mathcal{L}_X\nabla)^2(E_2,E_3) = 0$ we get

$$\begin{cases} \partial_y^2 H = 0, \\ \partial_y^2 \overline{H} = 0. \end{cases} \tag{15}$$

From the equation given by $(\mathcal{L}_X\nabla)^3(E_1,E_1) = 0$ and we replace f_3 we get

$$\partial_x f_1 = \frac{1}{2}e^z[H - 2Hz - 2\overline{H} + e^{-2z}\partial_x^2 Hz + e^{-2z}\partial_x^2\overline{H}]$$

and

$$\partial_x\partial_z f_1 = \frac{1}{2}e^z[-H - 2Hz - 2\overline{H} - e^{-2z}\partial_x^2 Hz + e^{-2z}\partial_x^2 H - e^{-2z}\partial_x^2\overline{H}].$$

We replace f_3, $\partial_x f_1$ and $\partial_x\partial_z f_1$ in the equation given by $(\mathcal{L}_X\nabla)^1(E_1,E_3) = 0$ and we get

$$\begin{cases} \partial_x^2 H = 0, \\ \partial_x^2 \overline{H} = 0. \end{cases} \tag{16}$$

From (15) and (16) we conclude that

$$\begin{cases} H(x,y) = \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4, \alpha_i \in \mathbb{R}, \\ \overline{H}(x,y) = \beta_1 x + \beta_2 y + \beta_3 xy + \beta_4, \beta_i \in \mathbb{R}, \end{cases} \tag{17}$$

which together with equations $(\mathcal{L}_X\nabla)^1(E_1,E_2) = 0$ and $(\mathcal{L}_X\nabla)^2(E_1,E_2) = 0$ gives

$$\begin{cases} f_1 = \dfrac{1}{2}e^z[\alpha_4 x - 2\alpha_4 xz - \beta_1 x^2 - 2\beta_2 xy - \beta_3 x^2 y - 2\beta_4 x] + \alpha_5, \\ f_2 = \dfrac{1}{2}e^{-z}[\alpha_4 y + 2\alpha_4 yz + 2\beta_1 xy + \beta_2 y^2 + \beta_3 xy^2 + 2\beta_4 y] + \beta_5, \\ f_3 = \alpha_4 z + \overline{H}, \quad \alpha_i, \beta_i \in \mathbb{R}. \end{cases} \tag{18}$$

Replacing f_1 into equations $(\mathcal{L}_X\nabla)^1(E_2,E_2) = 0$ and $(\mathcal{L}_X\nabla)^1(E_2,E_3) = 0$ we get

$$\alpha_4 = \alpha_5 = \beta_2 = \beta_3 = 0.$$

And similarly replacing f_2 into equations

$$(\mathcal{L}_X\nabla)^2(E_1,E_1) = 0 \text{ and } (\mathcal{L}_X\nabla)^2(E_1,E_3) = 0$$

we get

$$\alpha_4 = \alpha_5 = \beta_1 = \beta_3 = 0.$$

Thus the final solution of PDEs system obtained requiring that all the coefficients in the above Lie derivative are equal to zero are given by

$$\begin{cases} f_1 = -\beta_4 x e^z, \\ f_2 = \beta_4 y e^{-z}, \\ f_3 = \beta_4, \quad \beta_4 \in \mathbb{R}. \end{cases} \tag{19}$$

□

3.2. Riemannian metric

We first classify Killing and affine vector fields of the Riemannian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 3.2. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Riemannian Sol_3 Lie group.*

- *X is a Killing vector field if and only if*

$$\begin{cases} f_1 = -(\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \quad \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

- *X is an affine vector field if and only if*

$$\begin{cases} f_1 = -\beta x e^z, \\ f_2 = \beta y e^{-z}, \\ f_3 = \beta, \quad \beta \in \mathbb{R}. \end{cases}$$

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Riemannian Sol_3 Lie group, for some arbitrary smooth functions f_1, f_2, f_3 on Riemannian Sol_3 Lie group. Starting from (7), a direct calculation yields the following description of the Lie derivative of the metric tensor g:

$$\begin{cases} (\mathcal{L}_X g)(E_1, E_1) = 2[f_3 + e^{-z}\partial_x f_1], \\ (\mathcal{L}_X g)(E_1, E_2) = e^{-z}\partial_x f_2 + e^z \partial_y f_1, \\ (\mathcal{L}_X g)(E_1, E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1, \\ (\mathcal{L}_X g)(E_2, E_2) = -2[f_3 - e^z \partial_y f_2], \\ (\mathcal{L}_X g)(E_2, E_3) = e^z \partial_y f_3 + f_2 + \partial_z f_2, \\ (\mathcal{L}_X g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{20}$$

In order to determine the Killing vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative are equal to zero.

A straightforward calculation leads to prove that

$$\begin{cases} f_1 = -(\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \quad \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

To determine the affine vector fields, we need to calculate the Lie derivative of the Levi-Civita connection ∇. Staring from (9), we find the following possibly non-vanishing components:

$$
(21) \quad \begin{cases}
(\mathcal{L}_X\nabla)(E_1,E_1) = [-f_1 + 2e^{-z}\partial_x f_3 + \partial_z f_1 + e^{-2z}\partial_x^2 f_1]E_1 \\
\qquad + [f_2 + \partial_z f_2 + e^{-2z}\partial_x^2 f_2]E_2 \\
\qquad + [\partial_z f_3 - 2f_3 - 2e^{-z}\partial_x f_1 + e^{-2z}\partial_x^2 f_3]E_3, \\
(\mathcal{L}_X\nabla)(E_1,E_2) = [e^{z}\partial_y f_3 + \partial_x\partial_y f_1]E_1 \\
\qquad + [-e^{-z}\partial_x f_3 + \partial_x\partial_y f_2]E_2 \\
\qquad + [e^{-z}\partial_x f_2 - e^{z}\partial_y f_1 + \partial_x\partial_y f_3]E_3, \\
(\mathcal{L}_X\nabla)(E_1,E_3) = [-e^{-z}\partial_x f_1 + e^{-z}\partial_x\partial_z f_1 + \partial_z f_3]E_1 \\
\qquad + [-e^{-z}\partial_x f_2 + e^{-z}\partial_x\partial_z f_2]E_2 \\
\qquad + [-e^{-z}\partial_x f_3 + e^{-z}\partial_x\partial_z f_3 + f_1 - \partial_z f_1]E_3, \\
(\mathcal{L}_X\nabla)(E_2,E_2) = [f_1 - \partial_z f_1 + e^{2z}\partial_y^2 f_1]E_1 \\
\qquad + [-f_2 - \partial_z f_2 - 2e^{z}\partial_y f_3 + e^{2z}\partial_y^2 f_2]E_2 \\
\qquad + [-\partial_z f_3 - 2f_3 + 2e^{z}\partial_y f_2 + e^{2z}\partial_y^2 f_3]E_3, \\
(\mathcal{L}_X\nabla)(E_2,E_3) = [e^{z}\partial_y f_1 + e^{z}\partial_y\partial_z f_1]E_1 \\
\qquad + [e^{z}\partial_y f_2 + e^{z}\partial_y\partial_z f_2 - \partial_z f_3]E_2 \\
\qquad + [e^{z}\partial_y f_3 + e^{z}\partial_y\partial_z f_3 + f_2 + \partial_z f_2]E_3, \\
(\mathcal{L}_X\nabla)(E_3,E_3) = [-f_1 + \partial_z^2 f_1]E_1 + [-f_2 + \partial_z^2 f_2]E_2 + [\partial_z^2 f_3]E_3.
\end{cases}
$$

In order to determine the affine vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative are equal to zero.

A straightforward calculation leads to prove that

$$
\begin{cases}
f_1 = -\beta x e^{z}, \\
f_2 = \beta y e^{-z}, \\
f_3 = \beta, \quad \beta \in \mathbb{R}.
\end{cases}
$$

□

4. Ricci and curvature collineations vector fields of the *Sol*$_3$ Lie group

In this section we completely classify Ricci and curvature collineations vector fields and affine vector fields of three-dimensional Lorentzian and Riemannian solvable Lie group (Sol_3, g).

4.1. Lorentzian metric

In this subsection we give a full classification of Ricci and curvature collineations vector fields of the Lorentzian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 4.1. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Lorentzian Sol_3 Lie group.*

- *X is a Ricci collineation if and only if*

$$\begin{cases} f_1 = H(x, y, z), \\ f_2 = F(x, y, z), \\ f_3 = \alpha, \ \ \alpha \in \mathbb{R}, \end{cases}$$

where H and F are any smooth functions on Sol_3 Lie group.

- *X is a curvature collineation vector field if and only if*

$$\begin{cases} f_1 = (-\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \ \ \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Lorentzian Sol_3 Lie group, for some arbitrary smooth functions f_1, f_2, f_3 on Lorentzian Sol_3 Lie group. Starting from (5), a direct calculation yields the following description of the Lie derivative of the Ricci tensor Ric in the direction of X given by:

$$\begin{cases} (\mathcal{L}_X Ric)(E_1, E_1) = 0, \\ (\mathcal{L}_X Ric)(E_1, E_2) = 0, \\ (\mathcal{L}_X Ric)(E_1, E_3) = -2e^{-z}\partial_x f_3, \\ (\mathcal{L}_X Ric)(E_2, E_2) = 0, \\ (\mathcal{L}_X Ric)(E_2, E_3) = -2e^{z}\partial_y f_3, \\ (\mathcal{L}_X Ric)(E_3, E_3) = -4\partial_z f_3. \end{cases} \tag{22}$$

Ricci collineations are then calculated by solving the system of PDEs obtained by requiring that all the above coefficients of $\mathcal{L}_X Ric$ vanish. From equations given by $(\mathcal{L}_X Ric)(E_1, E_3) = 0$, $(\mathcal{L}_X Ric)(E_2, E_3) = 0$, and $(\mathcal{L}_X Ric)(E_3, E_3) = 0$ we get

$$f_3 = \alpha,$$

where $\alpha \in \mathbb{R}$. And f_1, f_2 are any smooth functions on Sol_3 Lie group.

To determine the curvature collineations, we need to calculate the Lie derivative of the curvature tensor R in the direction of X. Staring from (4), we find the following possibly non-vanishing components:

$$
(23)\quad \begin{cases}
(\mathcal{L}_X R)(E_1, E_2, E_1) = [e^z \partial_y f_1 - e^{-z}\partial_x f_2]E_1 - 2[f_3 + e^{-z}\partial_x f_1]E_2 \\
\qquad\qquad + 2[e^z \partial_y f_3]E_3, \\
(\mathcal{L}_X R)(E_1, E_2, E_2) = 2[f_3 - e^z \partial_y f_2]E_1 + [-e^z \partial_y f_1 + e^{-z}\partial_x f_2]E_2 \\
\qquad\qquad + 2[e^{-z}\partial_x f_3]E_3, \\
(\mathcal{L}_X R)(E_1, E_2, E_3) = [-e^z \partial_y f_3 - f_2 - \partial_z f_2]E_1 \\
\qquad\qquad + [f_1 + e^{-z}\partial_x f_3 - \partial_z f_1]E_2, \\
(\mathcal{L}_X R)(E_1, E_3, E_1) = [f_1 - \partial_z f_1 - e^{-z}\partial_x f_3]E_1 - 2[f_2 + \partial_z f_2]E_2 \\
\qquad\qquad + 2[f_3 + e^{-z}\partial_x f_1]E_3, \\
(\mathcal{L}_X R)(E_1, E_3, E_2) = -[f_2 + \partial_z f_2 + e^z \partial_y f_3]E_1 \\
\qquad\qquad + [e^z \partial_y f_1 - e^{-z}\partial_x f_2]E_3, \\
(\mathcal{L}_X R)(E_1, E_3, E_3) = -2[\partial_z f_3]E_1 + [e^{-z}\partial_x f_3 - f_1 + \partial_z f_1]E_3, \\
(\mathcal{L}_X R)(E_2, E_3, E_1) = -[e^{-z}\partial_x f_3 + f_1 - \partial_z f_1]E_2 \\
\qquad\qquad + [e^z \partial_y f_1 - e^{-z}\partial_x f_2]E_3, \\
(\mathcal{L}_X R)(E_2, E_3, E_2) = -2[f_1 - \partial_z f_1]E_1 + [f_2 + \partial_z f_2 - e^z \partial_y f_3]E_2 \\
\qquad\qquad + 2[f_3 - e^z \partial_y f_2]E_3, \\
(\mathcal{L}_X R)(E_2, E_3, E_3) = -2[\partial_z f_3]E_2 + [e^z \partial_y f_3 - f_2 - \partial_z f_2]E_3.
\end{cases}
$$

In order to determine the curvature collineation vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero, which together with equations $(\mathcal{L}_X R)^3(E_1, E_2, E_1) = 0$, $(\mathcal{L}_X R)^3(E_1, E_2, E_2) = 0$ and $(\mathcal{L}_X R)^1(E_1, E_3, E_3) = 0$ gives

$$f_3 = \alpha_1, \ \ \alpha_1 \in \mathbb{R}.$$

From the equations given by $(\mathcal{L}_X R)^1(E_1, E_2, E_2) = 0$ and $(\mathcal{L}_X R)^2(E_1, E_3, E_1) = 0$ we get

$$f_2 = \alpha_1 y e^{-z} + \overline{H}(x)e^{-z} + \alpha_2$$

for some arbitrary smooth function $\overline{H} = \overline{H}(x)$ on $\mathbb{R}$ and $\alpha_2 \in \mathbb{R}$. Replacing f_2 in equation $(\mathcal{L}_X R)^2(E_1, E_3, E_1) = 0$, we get $\alpha_2 = 0$. Together with equations $(\mathcal{L}_X R)^2(E_1, E_2, E_1) = 0$ and $(\mathcal{L}_X R)^1(E_2, E_3, E_2) = 0$ we get

$$f_1 = -\alpha_1 x e^z + \overline{F}(y)e^z + \alpha_3$$

for some arbitrary smooth function $\overline{F} = \overline{F}(y)$ on $\mathbb{R}$ and $\alpha_3 \in \mathbb{R}$. Replacing f_1 in equation $(\mathcal{L}_X R)^1(E_2, E_3, E_2) = 0$, we get $\alpha_3 = 0$. Then we replace f_1 and f_2 in the equation given by $(\mathcal{L}_X R)^1(E_1, E_2, E_1) = 0$ and we get

$$\begin{cases} \overline{F} = \alpha_4, \\ \overline{H} = \alpha_5, \ \ \alpha_4, \alpha_5 \in \mathbb{R}. \end{cases}$$

The final solution of the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero are given by

$$\begin{cases} f_1 = (-\alpha_1 x + \alpha_4)e^z, \\ f_2 = (\alpha_1 y + \alpha_5)e^{-z}, \\ f_3 = \alpha_1, \ \ \alpha_i \in \mathbb{R}. \end{cases}$$

□

4.2. Riemannian metric

In this subsection we give a full classification of Ricci and curvature collineations vector fields of the Riemannian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 4.2. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Riemannian Sol_3 Lie group.*

- *X is a Ricci collineation if and only if*

$$\begin{cases} f_1 = \varphi(x, y, z), \\ f_2 = \psi(x, y, z), \\ f_3 = \alpha, \ \alpha \in \mathbb{R}, \end{cases}$$

where φ and ψ are any smooth functions on Sol_3 Lie group.

- *X is a curvature collineation vector field if and only if*

$$\begin{cases} f_1 = (-\alpha x + \beta)e^z, \\ f_2 = (\alpha y + \gamma)e^{-z}, \\ f_3 = \alpha, \ \ \alpha, \beta, \gamma \in \mathbb{R}. \end{cases}$$

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Riemannian Sol_3 Lie group, for some arbitrary smooth functions f_1, f_2, f_3 on Riemannian Sol_3 Lie group. Starting from (11), a direct calculation yields the following description of the Lie derivative of the Ricci tensor Ric in the direction of X given by:

$$\begin{cases} (\mathcal{L}_X Ric)(E_1, E_1) = 0, \\ (\mathcal{L}_X Ric)(E_1, E_2) = 0, \\ (\mathcal{L}_X Ric)(E_1, E_3) = -2e^{-z}\partial_x f_3, \\ (\mathcal{L}_X Ric)(E_2, E_2) = 0, \\ (\mathcal{L}_X Ric)(E_2, E_3) = -2e^{z}\partial_y f_3, \\ (\mathcal{L}_X Ric)(E_3, E_3) = -4\partial_z f_3. \end{cases} \tag{24}$$

Ricci collineations are then calculated by solving the system of PDEs obtained by requiring that all the above coefficients of $\mathcal{L}_X Ric$ vanish. From equations given by $(\mathcal{L}_X Ric)(E_1, E_3) = 0$, $(\mathcal{L}_X Ric)(E_2, E_3) = 0$, and $(\mathcal{L}_X Ric)(E_3, E_3) = 0$ we get

$$f_3 = \alpha,$$

where $\alpha \in \mathbb{R}$. And f_1, f_2 are any smooth functions on Sol_3 Lie group.

To determine the curvature collineations, we need to calculate the Lie derivative of the curvature tensor R in the direction of X. Staring from (10), we find the following possibly non-vanishing components:

$$
\begin{cases}
(\mathcal{L}_X R)(E_1,E_2,E_1) = [e^z\partial_y f_1 + e^{-z}\partial_x f_2]E_1 - 2[f_3 + e^{-z}\partial_x f_1]E_2 \\
\qquad\qquad + 2[e^z\partial_y f_3]E_3, \\
(\mathcal{L}_X R)(E_1,E_2,E_2) = -2[f_3 - e^z\partial_y f_2]E_1 - [e^z\partial_y f_1 + e^{-z}\partial_x f_2]E_2 \\
\qquad\qquad - 2[e^{-z}\partial_x f_3]E_3, \\
(\mathcal{L}_X R)(E_1,E_2,E_3) = [-e^z\partial_y f_3 + f_2 + \partial_z f_2]E_1 \\
\qquad\qquad + [f_1 + e^{-z}\partial_x f_3 - \partial_z f_1]E_2, \\
(\mathcal{L}_X R)(E_1,E_3,E_1) = [f_1 - \partial_z f_1 - e^{-z}\partial_x f_3]E_1 - 2[f_2 + \partial_z f_2]E_2 \\
\qquad\qquad + 2[f_3 + e^{-z}\partial_x f_1]E_3, \\
(\mathcal{L}_X R)(E_1,E_3,E_2) = [f_2 + \partial_z f_2 - e^z\partial_y f_3]E_1 \\
\qquad\qquad + [e^z\partial_y f_1 + e^{-z}\partial_x f_2]E_3, \\
(\mathcal{L}_X R)(E_1,E_3,E_3) = -2[\partial_z f_3]E_1 + [e^{-z}\partial_x f_3 - f_1 + \partial_z f_1]E_3, \\
(\mathcal{L}_X R)(E_2,E_3,E_1) = -[e^{-z}\partial_x f_3 + f_1 - \partial_z f_1]E_2 \\
\qquad\qquad + [e^z\partial_y f_1 + e^{-z}\partial_x f_2]E_3, \\
(\mathcal{L}_X R)(E_2,E_3,E_2) = 2[f_1 - \partial_z f_1]E_1 - [f_2 + \partial_z f_2 + e^z\partial_y f_3]E_2 \\
\qquad\qquad - 2[f_3 - e^z\partial_y f_2]E_3, \\
(\mathcal{L}_X R)(E_2,E_3,E_3) = -2[\partial_z f_3]E_2 + [e^z\partial_y f_3 + f_2 + \partial_z f_2]E_3.
\end{cases}
\tag{25}
$$

In order to determine the curvature collineation vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero. A straightforward calculation leads to prove that

$$
\begin{cases}
f_1 = (-\alpha_1 x + \alpha_2)e^z, \\
f_2 = (\alpha_1 y + \alpha_4)e^{-z}, \\
f_3 = \alpha_1, \ \ \alpha_i \in \mathbb{R}.
\end{cases}
$$

□

5. Matter collineations vector fields of the Sol_3 Lie group

In this section we classify matter collineation vector fields of the Lorentzian and Riemannian Sol_3 Lie group.

5.1. Lorentzian metric

In this subsection we give a full classification of matter collineations vector fields of the Lorentzian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 5.1. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Lorentzian Sol_3 Lie group.*

- *X is a matter collineation vector field if and only if*

$$\begin{cases} f_1 = (-\alpha_1 x + \alpha_2)e^{z}, \\ f_2 = (\alpha_1 y + \alpha_3)e^{-z}, \\ f_3 = \alpha_1, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Lorentzian Sol_3 group, for some arbitrary smooth functions f_1, f_2, f_3 on Lorentzian Sol_3 Lie group. Starting from equations (1), (5) and (6), a direct calculation yields in the Lorentzian Sol_3 group, with respect to the orthonormal basis $\{E_i\}_{i\in\{1,2,3\}}$, the tensor $T = Ric - \frac{\tau}{2}g$ described by:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{26}$$

When we compute the Lie derivative of T with respect to X and we find:

$$\begin{cases} (\mathcal{L}_XT)(E_1,E_1) = 2[f_3 + e^{-z}\partial_x f_1], \\ (\mathcal{L}_XT)(E_1,E_2) = -e^{-z}\partial_x f_2 + e^{z}\partial_y f_1, \\ (\mathcal{L}_XT)(E_1,E_3) = -e^{-z}\partial_x f_3 - f_1 + \partial_z f_1, \\ (\mathcal{L}_XT)(E_2,E_2) = 2[f_3 - e^{z}\partial_y f_2], \\ (\mathcal{L}_XT)(E_2,E_3) = -e^{z}\partial_y f_3 - f_2 - \partial_z f_2, \\ (\mathcal{L}_XT)(E_3,E_3) = -2\partial_z f_3. \end{cases} \tag{27}$$

To determine matter collineation we solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the tensor field T in the direction of X are equal to zero (i.e., $\mathcal{L}_XT = 0$), we get that all solutions coincide with curvature collineation of Lorentzian Sol_3 Lie group. □

5.2. Riemannian metric

In this subsection we give a full classification of matter collineations vector fields of the Riemannian Sol_3 Lie group. The classifications we obtain are summarized in the following theorem.

Theorem 5.2. *Let $X = f_1E_1 + f_2E_2 + f_3E_3$ be an arbitrary vector field on the Riemannian Sol_3 Lie group.*

- *X is a matter collineation vector field if and only if*

$$\begin{cases} f_1 = (-\alpha_1 x + \alpha_2)e^{z}, \\ f_2 = (\alpha_1 y + \alpha_3)e^{-z}, \\ f_3 = \alpha_1, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Proof. Let $X = f_1E_1 + f_2E_2 + f_3E_3$ denote an arbitrary vector field on the Riemannian Sol_3 Lie group, for some arbitrary smooth functions f_1, f_2, f_3 on Lorentzian Sol_3 Lie group. Starting from equations (7), (11) and (12), a direct calculation yields in the Riemannian Sol_3 Lie group, with respect to the orthonormal basis $\{E_i\}_{i\in\{1,2,3\}}$, the tensor $T = Ric - \frac{\tau}{2}g$ described by:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{28}$$

When we compute the Lie derivative of T with respect to X and we find:

$$\begin{cases} (\mathcal{L}_XT)(E_1, E_1) = 2[f_3 + e^{-z}\partial_x f_1], \\ (\mathcal{L}_XT)(E_1, E_2) = e^{-z}\partial_x f_2 + e^{z}\partial_y f_1, \\ (\mathcal{L}_XT)(E_1, E_3) = -e^{-z}\partial_x f_3 - f_1 + \partial_z f_1, \\ (\mathcal{L}_XT)(E_2, E_2) = -2[f_3 - e^{z}\partial_y f_2], \\ (\mathcal{L}_XT)(E_2, E_3) = -e^{z}\partial_y f_3 + f_2 + \partial_z f_2, \\ (\mathcal{L}_XT)(E_3, E_3) = -2\partial_z f_3. \end{cases} \tag{29}$$

To determine matter collineation we solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the tensor field T in the direction of X are equal to zero (i.e., $\mathcal{L}_XT = 0$), we get that all solutions coincide with curvature collineation of Riemannian Sol_3 Lie group. □

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Lakehal Belarbi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (U.M.A.B.)
B.P.227, 27000, Mostaganem, Algeria
Email address: `lakehalbelarbi@gmail.com`