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NONLIFT WEIGHT TWO PARAMODULAR EIGENFORM CONSTRUCTIONS

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Abstract. We complete the construction of the nonlift weight two cusp paramodular Hecke eigenforms for prime levels $N < 600$, which arise in conformance with the paramodular conjecture of Brumer and Kramer.

1. Introduction

We construct all the paramodular nonlift newforms whose existence is suggested in [16], thus completing that article. That article and this one provide evidence for the paramodular conjecture of A. Brumer and K. Kramer. A *suitable* paramodular form f of level N is a cuspidal, nonlift Siegel paramodular newform of degree 2, weight 2, and level N with rational Hecke eigenvalues. The paramodular cusp form space is denoted $\mathcal{S}_2(\mathrm{K}(N))$—the subscript 2 indicates the weight, $\mathrm{K}(N)$ denotes the paramodular group of degree 2 and level N, and the degree is omitted from the notation because all paramodular forms in this article have degree 2. Newforms on $\mathrm{K}(N)$ are by definition Hecke eigenforms orthogonal to the images of level-raising operators from paramodular forms of lower levels [18, 19]. The lift space in $\mathcal{S}_2(\mathrm{K}(N))$ is $\mathrm{Grit}(\mathrm{J}_{2,N}^{\mathrm{cusp}})$, the Gritsenko lift of the Jacobi cusp form space of weight 2 and index N. A partial statement of the paramodular conjecture, sufficient for the purposes of this article, is:

> *Let N be a squarefree positive integer. Let $\mathcal{A}_N$ be the set of isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}}A = \mathbb{Z}$, and let $\mathcal{P}_N$ be the set of suitable paramodular forms f of level N, up to nonzero scaling. There is a bijection $\mathcal{A}_N \longleftrightarrow \mathcal{P}_N$ such that*
>
> $$L(A, s, \text{Hasse–Weil}) = L(f, s, \text{spin}).$$

Initially stated in [2], the paramodular conjecture is now modified in Section 8 of [3] to capture phenomena that can arise for some N divisible by a square, after F. Calegari pointed out an oversight in the earlier statement.

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We summarize previous work and state the main theorem of this article. In [16], the first and third authors of this article studied $\mathcal{S}_2(\mathrm{K}(N))$ for prime levels $N < 600$, proving by algorithm that $\mathcal{S}_2(\mathrm{K}(N)) = \mathrm{Grit}(\mathrm{J}_{2,N}^{\mathrm{cusp}})$ for all such N except the cases $N = 277, 349, 353, 389, 461, 523, 587$, precisely the primes $N < 600$ for which relevant abelian surfaces exist [2]. Also, [16] showed that in these cases there is at most one nonlift dimension, lying in the Fricke plus space, except that for $N = 587$ there is at most one Fricke plus space nonlift dimension and at most one Fricke minus space nonlift dimension. We refer to these last two settings as the cases $N = 587^{\pm}$. For each case N, conditionally on the existence of the nonlift, the newform f_N is known (see the website [17] for the article [16]), as are Euler factors of $L(f_N, s, \mathrm{spin})$ for small primes [16]. Further, [16] showed that $\mathcal{S}_2(\mathrm{K}(277))$ contains a nonlift dimension $\mathbb{C}f_{277}$, by constructing it. In [9], V. Gritsenko and the first and third authors of this article constructed a nonlift in $\mathcal{S}_2(\mathrm{K}(587))^{-}$, using a Borcherds product. In [4], A. Brumer and A. Pacetti and G. Tornaría and J. Voight and the first and third authors of this article showed the equality of $L(f_N, s, \mathrm{spin})$ and $L(A_N, s, \text{Hasse–Weil})$, for $N = 277, 353, 587^{-}$, citing this article for the existence of the $N = 353$ nonlift. We will describe the $N = 353$ nonlift in detail in Section 4. In this article we call $N = 277, 349, 353, 389, 461, 523, 587^{\pm}$ the *outstanding levels*, even though nonlifts are already constructed for $N = 277, 587^{-}$. This article describes at least one nonlift construction for each outstanding level. All but two outstanding levels have nonlift Borcherds products; constructing nonlifts at the two that do not, $N = 461, 587^{+}$, requires tracing methods. In [13], the authors of this article studied squarefree composite levels $N < 300$, showing that there is one Fricke plus space nonlift dimension for $N = 249, 295$ and otherwise no nonlifts. In this article, the adjective *outstanding* doesn't include these two composite levels, only the prime levels from above, but our main theorem does include them. So altogether our main theorem is as follows.

Theorem 1.1. *The following dimensions are established.*

N	$\dim \mathrm{J}_{2,N}^{\mathrm{cusp}}$	$\dim \mathcal{S}_2(\mathrm{K}(N))^{+}$	$\dim \mathcal{S}_2(\mathrm{K}(N))^{-}$
249	5	6	0
277	10	11	0
295	6	7	0
349	11	12	0
353	11	12	0
389	11	12	0
461	12	13	0
523	17	18	0
587	18	19	1

With the asserted dimensions already established as dimension upper bounds in [16], the proof of Theorem 1.1 is a matter of constructing a nonlift at each level, including both of $587^{\pm}$. As just discussed, constructions already exist for the outstanding levels $N = 277, 587^-$ and for the composite levels $N = 249, 295$, but the results in Theorem 1.1 that a nonlift Fricke plus space dimension exists for $N = 349, 353, 389, 461, 523, 587$ are new. This article proves Theorem 1.1 by giving in Section 4 at least one construction for each relevant level $N \neq 461, 587^+$, and by giving in Section 5 constructions for levels $N = 461, 587^+$. More details of the nonlift computations are given at the website [15] for this article. In particular the website gives an expression for each outstanding nonlift eigenform f_N as a linear combination of a nonlift Borcherds product and a Gritsenko lift, excepting levels $N = 461, 587^+$. These formulas for $N = 277, 349, 353, 389, 523$ are new to this article.

This article further completes [16] as follows.

Theorem 1.2. *The conjectured nonlift eigenform formulas $f_N = Q_N/L_N$ of* [16] *and its website* [17] *are correct for $N = 349, 353, 389, 461, 523, 587^{\pm}$. Here Q_N lies in $\mathcal{S}_4(\mathrm{K}(N))$ and L_N in $\mathrm{Grit}(\mathrm{J}_{2,N}^{\mathrm{cusp}})$.*

We will prove Theorem 1.2 in Section 6.

Having a nonlift eigenform f expressed either as a linear combination of a nonlift Borcherds product and a Gritsenko lift or as a quotient Q/L of a quadratic form in Gritsenko lifts by a linear form in Gritsenko lifts allows the computation of comparatively many f-eigenvalues. In particular, [4] used this idea to establish the equality of L-functions in the paramodular conjecture for $N = 277, 353, 587^-$.

Beyond proving Theorem 1.1 and Theorem 1.2, this article aims to make known our various nonlift construction methods. Computing a space of paramodular forms first requires good estimation methods to get a dimension upper bound that we believe is tight, and then, separately, it requires constructions of nonlifts to achieve a matching dimension lower bound. Although nonlift Borcherds product paramodular cusp forms are finitely determined, the search-spaces for their constructions are enormous, making naïve searches infeasible, and so we hope that the methods described in this article will be useful. Some of them give remarkably simple constructions in light of the complexity of the situation, and they showcase the versatility of theta blocks, particular Jacobi forms due to V. Gritsenko, N.-P. Skoruppa, and D. Zagier [10] to be discussed below.

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2. Background: Paramodular forms, Atkin–Lehner eigenforms, theta blocks

The background section of [13] introduces terminology and notation for paramodular forms and for Fricke and Atkin–Lehner involutions. Section 4 of [14] introduces terminology and notation for theta blocks. Here we repeat some of this background more briefly.

The degree 2 symplectic group Sp(2) of 4×4 matrices is defined by the condition $g'Jg = J$, where the prime denotes matrix transpose and J is the skew form $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ with each block 2×2. The Siegel upper half space $\mathcal{H}_2$ consists of the 2×2 symmetric complex matrices that have positive definite imaginary part. Elements of $\mathcal{H}_2$ are notated

$$\Omega = \begin{bmatrix} \tau & z \\ z & \omega \end{bmatrix},$$

and also, letting $\mathrm{e}(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$, we use throughout this article the notation

$$q = \mathrm{e}(\tau), \quad \zeta = \mathrm{e}(z), \quad \xi = \mathrm{e}(\omega).$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathcal{H}_2$ via fractional linear transformations, $g(\Omega) = (a\Omega + b)(c\Omega + d)^{-1}$ for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, and the Siegel factor of automorphy is $j(g, \Omega) = \det(c\Omega + d)$.

For any positive integer N, the paramodular group $\mathrm{K}(N)$ of degree 2 and level N is the group of rational symplectic matrices that stabilize the column vector lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$. In coordinates,

$$\mathrm{K}(N) = \{ \left[\begin{array}{cc|cc} * & *N & * & * \\ * & * & * & */N \\ \hline * & *N & * & * \\ *N & *N & *N & * \end{array}\right] \in \mathrm{Sp}_2(\mathbb{Q}) : \text{all } * \text{ entries integral}\}.$$

The upper right entries of the four subblocks are "more integral by a factor of N" than implied immediately by the definition of the paramodular group as a lattice stabilizer, but the extra conditions hold because the matrices are symplectic.

Fix an integer k. Any function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ and any real symplectic matrix $g \in \mathrm{Sp}_2(\mathbb{R})$ combine to form another such function through the weight k operator, $f[g]_k(\Omega) = j(g, \Omega)^{-k} f(g(\Omega))$. A Siegel paramodular form of weight k and level N is a holomorphic function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ that is $[\mathrm{K}(N)]_k$-invariant; the Köcher Principle says that the function $f[g]_k$ is bounded on $\{\mathrm{Im}(\Omega) > Y_o\}$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$ and all positive 2×2 real matrices Y_o. The space of weight k, level N Siegel paramodular forms is denoted $\mathcal{M}_k(\mathrm{K}(N))$. Siegel's Φ map takes any holomorphic function that has a Fourier series of the form $f(\Omega) = \sum_t a(t; f)\, \mathrm{e}(\langle t, \Omega \rangle)$, summing over matrices $t = \left[\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right]$ with $n, m \in \frac{1}{M}\mathbb{Z}_{\geq 0}$ and $r \in \frac{1}{M}\mathbb{Z}$ for some positive integer M and with $nm - r^2 \geq 0$, to the function $(\Phi f)(\tau) = \lim_{\omega \to i\infty} f(\left[\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right])$. A Siegel paramodular form f is a cusp form if $\Phi(f[g]_k) = 0$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$. The space of weight k, level N Siegel paramodular cusp forms is denoted $\mathcal{S}_k(\mathrm{K}(N))$. The dimension of $\mathcal{S}_k(\mathrm{K}(N))$ is known for squarefree N and $k \geq 3$ [11], but no dimension formula is known for $k = 2$.

Every paramodular cusp form of weight k and level N has a Fourier expansion

$$f(\Omega) = \sum_{t \in \mathcal{X}_2(N)} a(t; f)\, \mathrm{e}(\langle t, \Omega \rangle),$$

summing over the index set

$$\mathcal{X}_2(N) = \{ \left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right] : n, m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}, 4nmN - r^2 > 0 \},$$

and with $\langle t, \Omega \rangle = \mathrm{tr}(t\Omega)$. The Fourier–Jacobi expansion of a paramodular cusp form is

$$f(\Omega) = \sum_{m \geq 1} \phi_m(f)(\tau, z) \xi^{mN},$$

with Jacobi coefficients

$$\phi_m(f)(\tau, z) = \sum_{t = \left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right] \in \mathcal{X}_2(N)} a(t; f) q^n \zeta^r. \tag{1}$$

Each Jacobi coefficient $\phi_m(f)$ lies in the space $\mathrm{J}_{k,mN}^{\mathrm{cusp}}$ of weight k, index mN Jacobi cusp forms, whose dimension is known [5, 21]. For the theory of Jacobi forms, see [5, 7, 21].

We review Atkin–Lehner involutions, including the Fricke involution. Let N be a positive integer, and let c be a positive divisor of N such that $\gcd(c, N/c) = 1$. In this article N is always squarefree, so c can be any positive divisor of N. For any integers $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta c - \beta\gamma N/c = 1$, an elliptic c-Atkin–Lehner matrix is

$$\alpha_c = \frac{1}{\sqrt{c}} \begin{bmatrix} \alpha c & \beta \\ \gamma N & \delta c \end{bmatrix}.$$

Especially, for $c = 1$ we may take $\alpha, \delta = 1$ and $\beta, \gamma = 0$ to get the identity matrix, and for $c = N$ we may take $\alpha, \delta = 0$ and $\beta, \gamma = \mp 1$ to get the usual Fricke involution matrix $\frac{1}{\sqrt{N}} \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$. Let $\Gamma_0(N)$ denote the level N Hecke subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By quick calculations, the inverse of any α_c is another $\tilde{\alpha}_c$, any product $\tilde{\alpha}_c \alpha_c$ lies in $\Gamma_0(N)$ and so the set of all $\tilde{\alpha}_c$ lies in the coset $\Gamma_0(N)\alpha_c$, and this coset also lies in the set of all $\tilde{\alpha}_c$, making them equal. Consequently, α_c squares into $\Gamma_0(N)$ and normalizes $\Gamma_0(N)$. Now, with an asterisk and a box-plus denoting the matrix transpose-inverse and direct sum operators, a paramodular c-Atkin–Lehner matrix is

$$\mu_c = \alpha_c^* \boxplus \alpha_c = \begin{bmatrix} \alpha_c^* & 0 \\ 0 & \alpha_c \end{bmatrix}.$$

The inverse of any μ_c is another $\tilde{\mu}_c$, and any product $\tilde{\mu}_c \mu_c$ lies in $\mathrm{K}(N)$, so that the set of all $\tilde{\mu}_c$ lies in the coset $\mathrm{K}(N)\mu_c$ and they all give the same action on paramodular forms, although now the containment is proper. Again μ_c squares into $\mathrm{K}(N)$, and a blockwise check shows that μ_c normalizes $\mathrm{K}(N)$. For $c = 1$ we take $\mu_1 = 1_4$. For $c = N$, the paramodular Fricke involution is $\mu_N = \frac{1}{\sqrt{N}}(\left[\begin{smallmatrix} 0 & -N \\ 1 & 0 \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]) : \left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right] \longmapsto \left[\begin{smallmatrix} \omega N & -z \\ -z & \tau/N \end{smallmatrix}\right]$. The space $\mathcal{S}_k(\mathrm{K}(N))$

decomposes as the direct sum of the Fricke eigenspaces for the two eigenvalues ± 1, $\mathcal{S}_k(\mathrm{K}(N)) = \mathcal{S}_k(\mathrm{K}(N))^+ \oplus \mathcal{S}_k(\mathrm{K}(N))^-$. More generally the paramodular Atkin–Lehner involutions satisfy $[\mu_c]_k[\mu_{\tilde{c}}]_k = [\mu_{c\tilde{c}}]_k$ for coprime c and $\tilde{c}$, and so they commute. Thus $\mathcal{S}_k(\mathrm{K}(N))$ decomposes as a direct sum of spaces $\mathcal{S}_k(\mathrm{K}(N))^v$ where v is a vector of $\pm$ entries indexed by the prime divisors of the level N. Such a vector is called an Atkin–Lehner signature.

We quickly review some basic terminology of theta blocks. Let τ be a variable from the complex upper half plane and let z be a complex variable. With q and ζ as before, the Dedekind eta function η and the odd Jacobi theta function ϑ are

$$\begin{aligned}\eta(\tau) &= q^{1/24}\prod_{n\geq 1}(1-q^n),\\ \vartheta(\tau,z) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2/2}\zeta^{n+1/2}\\ &= q^{1/8}(\zeta^{1/2}-\zeta^{-1/2})\prod_{n\geq 1}(1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n).\end{aligned}$$

Here η is a Jacobi cusp form of weight $1/2$ and index 0 and a multiplier denoted ϵ, and ϑ is a Jacobi cusp form of weight $1/2$ and index $1/2$ and a multiplier $\epsilon^3 v_H$ where v_H is a character of the Heisenberg group [7]. For any $r \in \mathbb{Z}_{\geq 1}$, define $\vartheta_r \in \mathrm{J}^{\mathrm{cusp}}_{1/2,r^2/2}(\epsilon^3 v_H^r)$ to be $\vartheta_r(\tau,z) = \vartheta(\tau, rz)$, so that

$$\vartheta_r(\tau,z)/\eta(\tau) = q^{1/12}(\zeta^{r/2}-\zeta^{-r/2})\prod_{n\geq 1}(1-q^n\zeta^r)(1-q^n\zeta^{-r}).$$

A theta block is a meromorphic function of the form

$$\mathrm{TB}(\tau,z) = \mathrm{TB}(\varphi)(\tau,z) = \eta(\tau)^{\varphi(0)}\prod_{r\geq 1}(\vartheta_r(\tau,z)/\eta(\tau))^{\varphi(r)},$$

where $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is even and finitely supported. The theta block of φ has the product form

$$\mathrm{TB}(\tau,z) = q^A b(\zeta)\prod_{n\geq 1,\, r\in\mathbb{Z}}(1-q^n\zeta^r)^{\varphi(r)}, \qquad A = \frac{1}{24}\sum_{r\in\mathbb{Z}}\varphi(r),$$

in which $b(\zeta) = \prod_{r\geq 1}(\zeta^{r/2}-\zeta^{-r/2})^{\varphi(r)}$ is the baby theta block, or

$$b(\zeta) = \zeta^{-B}\prod_{r\geq 1}(\zeta^r - 1)^{\varphi(r)}, \qquad B = \frac{1}{2}\sum_{r\geq 1} r\varphi(r).$$

The weight of the theta block is $k = \frac{1}{2}\varphi(0)$ and the index is $N = \frac{1}{2}\sum_{r\geq 1} r^2\varphi(r)$, so that $\mathrm{TB}(\varphi) \in \mathrm{J}^{\mathrm{mero}}_{k,N}(\epsilon^{24A}v_H^{2B})$.

3. Overview of computing a space of paramodular forms

Let $\mathcal{S}$ denote any Fricke eigenspace or Atkin–Lehner eigenspace of $\mathcal{S}_k(\mathrm{K}(N))$. To describe our computational method for studying $\mathcal{S}$ in weight $k = 2$, introduce for any positive integer d the notation

$$\mathcal{S}(d) = \{\mathcal{S}\text{-elements with vanishing first } d-1 \text{ Jacobi coefficients}\},$$
$$\mathcal{S}[d] = \{\mathcal{S}\text{-elements truncated to the first } d \text{ Jacobi coefficients}\}.$$

Thus there is an exact sequence

$$0 \longrightarrow \mathcal{S}(d+1) \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}[d] \longrightarrow 0.$$

In broad strokes, our methodology to study $\mathcal{S}$ in weight 2 is as follows.

(1) Show for some nonnegative integer d that $\mathcal{S}(d+1) = 0$, i.e., every element of $\mathcal{S}$ is determined by its first d Jacobi coefficients.
(2) Compute a small superspace $\mathcal{J}$ of $\mathcal{S}[d]$ (specifics will be given below). We can identify elements of $\mathcal{S}$ uniquely as linear combinations of our basis of $\mathcal{J}$.
(3) Show that $\dim \mathcal{S} = \dim \mathcal{J}$, and span $\mathcal{S}$ in the process, by constructing that many linearly independent elements in $\mathcal{S}$.

This section amplifies the outline just given, explaining why the work can be carried out in a prime characteristic p, and saying more about parts (1) and (2). Part (3) is carried out for the outstanding levels of this article in Sections 4 and 5 to follow.

The methods described in the previous paragraph are facilitated by working in some prime characteristic p, rather than in characteristic 0; as will be explained below, this does not lose any dimensions. We describe the transition from characteristic 0 to characteristic p. Continue to let $\mathcal{S}$ denote any Fricke eigenspace or Atkin–Lehner eigenspace of $\mathcal{S}_k(\mathrm{K}(N))$, and let $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ denote the space of weight k, index m Jacobi cusp forms for any m. Using the Fourier–Jacobi expansions of paramodular cusp forms, we view two maps of complex vector spaces as containments for simplicity, with $\mathbb{C}^\infty$ indexed by $\mathcal{X}_2(N)$ in the next display and with the Fourier series of elements of $\mathrm{J}_{k,jN}^{\mathrm{cusp}}$ indexed by the matrices $\left[\begin{smallmatrix} n & r/2 \\ r/2 & jN \end{smallmatrix}\right]$ of $\mathcal{X}_2(N)$ (cf. (1) in Section 2),

$$\mathcal{S} \subset \bigoplus_{j=1}^{\infty} \mathrm{J}_{k,jN}^{\mathrm{cusp}} \subset \mathbb{C}^\infty.$$

For any vector subspace V of $\mathbb{C}^\infty$, let $V(\mathbb{Z}) = V \cap \mathbb{Z}^\infty$. For any prime p, let $R_p : \mathbb{Z}^\infty \longrightarrow \mathbb{F}_p^\infty$ be the reduction modulo p map. Define a map $V \mapsto V_p$ from the set of vector subspaces of $\mathbb{C}^\infty$ to the set of vector subspaces of $\mathbb{F}_p^\infty$ by reducing the integral elements of the input vector space,

$$V_p = R_p(V(\mathbb{Z})), \quad V \text{ a vector subspace of } \mathbb{C}^\infty.$$

From the previous two displays,

$$\mathcal{S}_p \subset \bigoplus_{j=1}^{\infty} (\mathrm{J}_{k,jN}^{\mathrm{cusp}})_p \subset \mathbb{F}_p^{\infty}.$$

Extending the notation from above, again for any positive integer d,

$$\begin{aligned}\mathcal{S}_p(d) &= \{\text{elements of } \mathcal{S}_p \text{ with vanishing first } d-1 \text{ Jacobi coefficients in } \mathbb{F}_p\},\\ \mathcal{S}[d]_p &= \{\text{reductions modulo } p \text{ of the integral elements of } \mathcal{S}[d]\},\end{aligned}$$

and we will show at the end of this section that because $\mathcal{S}$ has an integral basis,

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{S}(d) &\le \dim_{\mathbb{F}_p} \mathcal{S}_p(d),\\ \dim_{\mathbb{C}} \mathcal{S}[d] &= \dim_{\mathbb{F}_p} \mathcal{S}[d]_p.\end{aligned}$$

Thus, establishing that $\mathcal{S}_p(d) = 0$ establishes that $\mathcal{S}(d) = 0$, and bounding $\dim \mathcal{S}[d]_p$ gives the same bound of $\dim \mathcal{S}[d]$. Here the spaces $\mathcal{S}_p(d)$ and $\mathcal{S}[d]_p$, which we use, have different definitions from the spaces $\mathcal{S}(d)_p$ and $\mathcal{S}_p[d]$, which we do not.

For the first part of our three-part method in weight 2, the methods of [1,13] let us attempt to show for a given d that $\mathcal{S}(d) = 0$ or $\mathcal{S}_p(d) = 0$, where $\mathcal{S}$ is either Fricke eigenspace of $\mathcal{S}_2(\mathrm{K}(N))$, or even all of $\mathcal{S}_2(\mathrm{K}(N))$. From [16] we already know that $\mathcal{S}_2(\mathrm{K}(N))^+$ is determined by the first two Fourier–Jacobi coefficients for each outstanding level, and so we may take $d \ge 2$ in this article.

For the second part of our method, the Jacobi restriction method [1, 12, 13] gives a superspace $\mathcal{J}$ of $\mathcal{S}[d]$ or $\mathcal{S}[d]_p$ that provides a good upper bound of $\dim \mathcal{S}[d]$ or $\dim \mathcal{S}[d]_p$. The space $\mathcal{J}$ is a space of finite sequences of Jacobi forms whose Fourier coefficients altogether satisfy the linear relations generically required of the Fourier coefficients of a paramodular Atkin–Lehner eigenform. Specifically, let v denote either a Fricke eigenvalue or a vector of Atkin–Lehner eigenvalues, and let $\mathcal{S}^v$ denote the corresponding Fricke or Atkin–Lehner eigenspace of $\mathcal{S}_k(\mathrm{K}(N))$. Jacobi restriction runs with d as a parameter, returning a basis of a finite-dimensional complex vector space $\mathcal{J}_d^v$ of truncated formal Fourier–Jacobi expansions such that

$$\mathcal{S}^v[d] \subset \mathcal{J}_d^v \subset \bigoplus_{j=1}^{d} \mathrm{J}_{k,jN}^{\mathrm{cusp}}.$$

Jacobi restriction is experimentally remarkable in that the equality $\mathcal{S}^v[d] = \mathcal{J}_d^v$ tends to hold for small values of d. An additional parameter, $\det_{\max}$, bounds the Fourier coefficient index determinants in the calculation, thus making the method an algorithm; this parameter must be chosen so that each space $\mathrm{J}_{k,jN}^{\mathrm{cusp}}$ with $j \le d$ is determined by the Fourier coefficients whose indices satisfy the determinant bound. For simplicity we suppress $\det_{\max}$ from the notation $\mathcal{J}_d^v$, along with the weight k and the level N. In the basis of $\mathcal{J}_d^v$ given by Jacobi restriction, each basis element is represented as a finite collection of data $a(n,r,m)$ where $1 \le m \le d$ and $0 < nmN - r^2/4 \le \det_{\max}$; the truncation of any paramodular cusp form $f \in \mathcal{S}_k(\mathrm{K}(N))$ to its first d Jacobi coefficients is determined by its Fourier coefficients $a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right]; f)$ for such n, r, m. Jacobi restriction also can be run modulo p, in which case it returns a basis of a finite dimensional vector space $\mathcal{J}^v_{d,p}$ over $\mathbb{F}_p$ such that

$$\mathcal{S}^v[d]_p \subset \mathcal{J}^v_{d,p} \subset \bigoplus_{j=1}^{d} (\mathrm{J}^{\mathrm{cusp}}_{k,jN})_p.$$

Here $\mathcal{J}^v_{d,p}$ is different from $(\mathcal{J}^v_d)_p$ but contains it. For this article, we already have a tight dimension bound $\dim \mathcal{S}_2(\mathrm{K}(N))^+ \le \dim \mathrm{J}^{\mathrm{cusp}}_{2,N} + 1$ from [16]; however, Jacobi restriction is still required in Section 5 to construct a small superspace $\mathcal{J}^{-,+}_{5,p}$ of $\mathcal{S}_2(\mathrm{K}(2N))^{-,+}[5]_p$ for $N = 461, 587^+$ in order to extend our Fourier coefficient computations out to the fifth Fourier–Jacobi coefficient.

The third part of our method is the main substance of this article. We must construct a weight 2 nonlift at each outstanding level N except $N = 277, 587^-$, where nonlifts are already constructed. In Section 4 we construct nonlifts by constructing Borcherds products at level N. In Section 5 we address the levels where no Borcherds product exists by tracing level $2N$ Borcherds products down to level N.

Let p be prime. To end this section we prove the inequality $\dim_{\mathbb{C}} \mathcal{S}(d) \le \dim_{\mathbb{F}_p} \mathcal{S}_p(d)$ and the equality $\dim_{\mathbb{C}} \mathcal{S}[d] = \dim_{\mathbb{F}_p} \mathcal{S}[d]_p$ for any $d \ge 1$, as stated above. The key is that reducing a saturated lattice modulo p does not decrease its dimension, as follows.

Lemma 3.1. *Let $M \subset \mathbb{Z}^\infty$ be a $\mathbb{Z}$-module of finite rank. Suppose that M is saturated, meaning that the general containment* $\mathrm{span}_{\mathbb{Q}}(M) \cap \mathbb{Z}^\infty \supset M$ *is equality,*

$$\mathrm{span}_{\mathbb{Q}}(M) \cap \mathbb{Z}^\infty = M.$$

Let p be prime, and let $M_p \subset \mathbb{F}_p^\infty$ denote the image of M under reduction modulo p, a vector space over $\mathbb{F}_p$. Then $\mathrm{rank}_{\mathbb{Z}} M = \dim_{\mathbb{F}_p} M_p$.

Proof. Consider a basis $\{v_1, \dots, v_r\}$ of M. The corresponding set of reductions, $\{v_{1,p}, \dots, v_{r,p}\}$, spans M_p. Consider any linear relation over $\mathbb{F}_p$ among the reductions, $\sum_i c_{i,p} v_{i,p} = 0$. The relation gives a congruence $\sum_i c_i v_i = 0 \bmod p$ in $\mathbb{Z}^\infty$, and so the vector $\sum_i (c_i/p) v_i$ lies in $\mathrm{span}_{\mathbb{Q}}(M) \cap \mathbb{Z}^\infty$, which by hypothesis is M, which is $\bigoplus_i \mathbb{Z} v_i$. So each c_i lies in $p\mathbb{Z}$, and the linear combination over $\mathbb{F}_p$ is trivial. Thus $\{v_{1,p}, \dots, v_{r,p}\}$ is a basis of M_p. □

Now, let $M = \mathcal{S}(\mathbb{Z}) = \mathcal{S} \cap \mathbb{Z}^\infty$. Because $\mathcal{S}$ has an integral basis by [20], M is a saturated module with $\dim_{\mathbb{C}} \mathcal{S} = \mathrm{rank}_{\mathbb{Z}} M$. We have $\mathrm{rank}_{\mathbb{Z}} M = \dim_{\mathbb{F}_p} M_p$ by the lemma, and $M_p = \mathcal{S}_p$. The previous three equalities give $\dim_{\mathbb{C}} \mathcal{S} = \dim_{\mathbb{F}_p} \mathcal{S}_p$, proving that for $d = 1$ the inequality $\dim_{\mathbb{C}} \mathcal{S}(d) \le \dim_{\mathbb{F}_p} \mathcal{S}_p(d)$ holds and is in fact equality. We complete the argument by proving the inequality for $d \ge 2$ and the equality $\dim_{\mathbb{C}} \mathcal{S}[d] = \dim_{\mathbb{F}_p} \mathcal{S}[d]_p$ for $d \ge 1$. Consider any such d. Because $\mathcal{S}$ has an integral basis, the projection map from $\mathcal{S}$ to $\mathcal{S}[d]$ is defined over $\mathbb{Z}$, and hence so are its kernel and image. That is, $\mathcal{S}(d+1)$ and $\mathcal{S}[d]$ have integral bases. Consequently the lattices $M(d+1) = \mathcal{S}(d+1) \cap \mathbb{Z}^\infty$ and $M[d] = \mathcal{S}[d] \cap \mathbb{Z}^\infty$ have respective ranks $\dim_{\mathbb{C}} \mathcal{S}(d+1)$ and $\dim_{\mathbb{C}} \mathcal{S}[d]$, and they are saturated. By the lemma, their ranks equal the dimensions of their reductions $M(d+1)_p$ and $M[d]_p$. Because $M(d+1)_p$ lies in $\mathcal{S}_p(d+1)$, this gives the desired inequality, $\dim_{\mathbb{C}} \mathcal{S}(d) \le \dim_{\mathbb{F}_p} \mathcal{S}_p(d)$, with $d+1$ in place of d, i.e., for all $d \ge 2$. And because $M[d]_p = \mathcal{S}[d]_p$, we have the desired equality as well, $\dim_{\mathbb{C}} \mathcal{S}[d] = \dim_{\mathbb{F}_p} \mathcal{S}[d]_p$.

4. Jacobi form constructions for Borcherds products

Borcherds products are a source of nonlift paramodular forms, although some Borcherds products are lifts. The theory of Borcherds products for paramodular forms is given by V. Gritsenko and V. Nikulin in [7]. A Borcherds product takes the form $f = \mathrm{Borch}(\psi)$ where $\psi \in \mathrm{J}_{0,N}^{\mathrm{w.h.}}$ is a weight 0 weakly holomorphic Jacobi form. Theorem 3.3 of [8], which in turn is quoted from [6, 7] and relies on the work of R. Borcherds, gives sufficient conditions for a Borcherds product to be a paramodular Fricke eigenform in $\mathcal{M}_k(\mathrm{K}(N))$; see also Section 7 of [13].

One source of weight 0 weakly holomorphic Jacobi forms ψ is the containment $\mathrm{J}_{12r,N}^{\mathrm{cusp}}/\Delta^r \subset \mathrm{J}_{0,N}^{\mathrm{w.h.}}$ for $r \in \mathbb{Z}_{\ge 1}$, with Δ the discriminant function from elliptic modular forms, a weight 12 cusp form. In particular, for all outstanding levels except $N = 461, 587^{\pm}$, there exist Jacobi forms $\psi \in \mathrm{J}_{12,N}^{\mathrm{cusp}}/\Delta$ having nonlift Borcherds products in $\mathcal{S}_2(\mathrm{K}(N))$ (this statement makes no reference to the squarefree composite levels $N = 249, 295$). For $N = 587^-$, a Jacobi form $\psi = \phi|V_2/\phi + \psi_o$ with ϕ a theta block in $\mathrm{J}_{2,587}^{\mathrm{cusp}}$ and $\psi_o \in \mathrm{J}_{12,N}^{\mathrm{cusp}}/\Delta$ has a nonlift Borcherds product in $\mathcal{S}_2(\mathrm{K}(N))$; here V_2 is the index-raising operator of [5], page 41, which takes weakly holomorphic Jacobi forms to weakly holomorphic Jacobi forms, as noted tacitly in [7]. These nonlifts are shown at [15], and they prove Theorem 1.1 for $N \ne 249, 295, 461, 587^+$. A Jacobi form $\psi \in \mathrm{J}_{24,N}^{\mathrm{cusp}}/\Delta^2$ also has a nonlift Borcherds product in $\mathcal{S}_2(\mathrm{K}(587))^-$, but this is not shown at the website. A drawback of creating Jacobi forms by the method of this paragraph is that they tend to be long linear combinations of a basis of $\mathrm{J}_{12,N}^{\mathrm{cusp}}/\Delta$, and their coefficients tend to be rational numbers with large numerators and denominators. Although they establish nonlift dimensions, we sought to create nonlifts from more simply expressed Jacobi forms in order to make further computations more efficient.

A second source of weight 0 weakly holomorphic Jacobi forms ψ is the *inflation method*. This method builds ψ from a combination of two sums, one that involves the V_2 raising operator from the previous paragraph and a second that involves inflations (to be explained below),

$$\psi = \sum_{i=1}^{m} \alpha_i (\phi_{1,i}|V_2)/\phi_{1,i} + \sum_{j=1}^{n} \beta_j \Theta_j/\phi_{2,j}, \quad \text{all } \alpha_i, \beta_j \in \mathbb{Z}.$$

Here the $\phi_{1,i}$ and the $\phi_{2,j}$ and the Θ_j are *basic* theta blocks, by which we mean theta blocks that are weakly holomorphic Jacobi forms of integral weight and level. Each theta block $\phi_{1,i}$ lies in $\mathrm{J}^{\text{w.h.}}_{k,N}$, has q-vanishing order $\nu_{1,i}$, and has baby theta block $b_{1,i}(\zeta)$ such that $b_{1,i}(\zeta) \mid b_{1,i}(\zeta^2)$; thus the first sum in the previous display lies in $\mathrm{J}^{\text{w.h.}}_{0,N}$ by Theorem 4.2 of [14]. Each theta block $\phi_{2,j}$ lies in $\mathrm{J}^{\text{w.h.}}_{k,r_jN}$ for some r_j and has q-vanishing order $\nu_{2,j}$ and baby theta block $b_{2,j}(\zeta)$, and each theta block Θ_j lies in $\mathrm{J}^{\text{w.h.}}_{k,(r_j+1)N}$ and has q-vanishing order $\tilde{\nu}_j$ and baby theta block $\tilde{b}_j(\zeta)$, and $b_{2,j}(\zeta) \mid \tilde{b}_j(\zeta)$; thus the second sum in the previous display lies in $\mathrm{J}^{\text{w.h.}}_{0,N}$ by Lemma 4.6 of [14]. The property that the baby theta block of Θ_j is a multiple of the baby theta block of $\phi_{2,j}$ makes Θ_j what we call an *inflation* of $\phi_{2,j}$ [9]. Some special cases of the inflation method are as follows.

- *Case 1.* $n = 0$, so that $\psi = \sum_{i=1}^{m} \alpha_i(\phi_i|V_2)/\phi_i$, and furthermore all ν_i are 1.
- *Case 2.* $(m, n) = (1, 1)$ with one ϕ, having q-vanishing order $\nu \in \{1, 2\}$, and $\psi = (-1)^\nu(\phi|V_2)/\phi + \beta\Theta/\phi$. Here β is usually ±1. The first term of ψ determines $r = 1$ in the second.
- *Case 3.* $(m, n) = (0, 1)$, so that $\psi = \Theta/\phi$. Usually $(\nu, \tilde{\nu}) = (1, 2)$ or $(\nu, \tilde{\nu}) = (2, 2)$.

The inflation method provides a second proof of Theorem 1.1 for each level $N \neq 461, 587^+$, including the squarefree composite levels $N = 249, 295$. The table in Figure 1 shows the method of constructing a relevant Jacobi form ψ that gives rise to a nonlift Borcherds product. At level $N = 277$, the Case 1 and Case 2 methods produce the same nonlift Borcherds product, its leading Jacobi coefficient in $\mathrm{J}^{\text{cusp}}_{2,277}$, while the Case 3 method produces a different nonlift Borcherds product, its leading Jacobi coefficient in $\mathrm{J}^{\text{cusp}}_{2,554}$.

N	Case 1	Case 2	Case 3
249			$(r, \nu, \tilde{\nu}) = (2, 2, 2)$
277	$m = 3$	$(\nu, \tilde{\nu}) = (1, 2),\ \beta = -1$	$(r, \nu, \tilde{\nu}) = (2, 2, 2)$
295			$(r, \nu, \tilde{\nu}) = (2, 2, 2)$
349	$m = 22$		
353	$m = 3$	$(\nu, \tilde{\nu}) = (1, 2),\ \beta = -1$	
389		$(\nu, \tilde{\nu}) = (1, 2),\ \beta = 1$	
523	$m = 52$		
587^-		$(\nu, \tilde{\nu}) = (2, 2),\ \beta = -1$	

FIGURE 1. Inflation method Jacobi form constructions

To convey a tangible sense of inflation method nonlift eigenform constructions, we illustrate the Cases 1 and 2 constructions of the level 353 suitable paramodular form f_{353}. The two constructions use theta blocks to create the same weakly holomorphic Jacobi form ψ of weight 0 and index 353; this Jacobi form has a nonlift weight 2, level 353 paramodular form Borcherds product $\mathrm{Borch}(\psi)$, and the Borcherds product linearly combines with Gritsenko lifts of further theta blocks to make the sought nonlift eigenform f_{353}. For theta blocks $\phi_1, \phi_2, \phi_3, \phi, \Theta$ to be described below, the Cases 1 and 2 constructions of the weakly holomorphic Jacobi form ψ are

$$\psi = \frac{\phi_1|V_2}{\phi_1} - \frac{\phi_2|V_2}{\phi_2} - \frac{\phi_3|V_2}{\phi_3} = -\frac{\phi|V_2}{\phi} - \frac{\Theta}{\phi}.$$

As for the theta blocks being used here, recall from Section 2 that $\eta(\tau)$ and $\vartheta(\tau, z)$ denote the Dedekind eta function and the odd Jacobi theta function, and $\vartheta_r(\tau, z) = \vartheta(\tau, rz)$ for $r \geq 1$. Then, with 0^e and r^e abbreviating η^e and $(\vartheta_r/\eta)^e$,

$$\begin{aligned}
\phi_1 &= 0^4 2^2 3^1 4^1 5^1 6^1 7^1 9^1 11^1 19^1,\\
\phi_2 &= 0^4 1^1 2^2 4^1 6^1 7^1 9^1 11^1 13^1 15^1,\\
\phi_3 &= 0^4 1^1 2^1 3^2 4^1 5^1 6^1 7^1 14^1 19^1,\\
\phi &= 0^4 1^2 2^1 3^1 4^1 6^1 7^1 13^1 14^1 15^1,\\
\Theta &= 0^4 1^2 2^2 3^2 4^2 5^2 6^2 7^1 8^1 9^2 10^1 11^1 12^1 13^1 14^1 15^1.
\end{aligned}$$

For instance, $\phi_1(\tau, z) = \eta(\tau)^{-6} \prod_{r=2,2,3,4,5,6,7,9,11,19} \vartheta(\tau, rz)$. All five of these theta blocks have weight 2, and the first four have index 353 while the last has index 706. The sought nonlift eigenform f_{353} is a linear combination of $\mathrm{Borch}(\psi)$ and a Gritsenko lift,

$$f_{353} = -11\mathrm{Borch}(\psi) + \sum_{i=1}^{11} c_i \,\mathrm{Grit}(\tilde{\phi}_i),$$

where $(c_1, \ldots, c_{11}) = (2, 1, -2, 4, 2, 0, -6, -7, 0, -5, -1)$ and the $\tilde{\phi}_i$ are more theta blocks,

$$\begin{aligned}
\tilde{\phi}_1 &= 0^4 3^2 4^2 5^1 7^2 10^1 12^1 17^1, & \tilde{\phi}_2 &= 0^4 3^1 4^3 6^1 7^1 8^1 10^1 12^1 16^1,\\
\tilde{\phi}_3 &= 0^4 2^2 3^1 4^1 5^2 7^1 9^1 13^1 18^1, & \tilde{\phi}_4 &= 0^4 2^2 3^1 4^1 5^1 6^1 7^1 9^1 11^1 19^1,\\
\tilde{\phi}_5 &= 0^4 2^2 3^1 4^1 5^1 8^1 10^1 12^2 14^1, & \tilde{\phi}_6 &= 0^4 2^2 3^1 4^1 6^1 7^1 8^1 10^2 18^1,\\
\tilde{\phi}_7 &= 0^4 2^2 3^1 5^1 7^1 9^1 10^1 11^1 12^1 13^1, & \tilde{\phi}_8 &= 0^4 2^2 4^2 6^2 7^1 10^1 11^1 18^1,\\
\tilde{\phi}_9 &= 0^4 2^2 4^1 5^1 6^2 8^1 10^1 14^1 15^1, & \tilde{\phi}_{10} &= 0^4 1^2 2^1 3^2 4^2 5^1 7^1 24^1,\\
\tilde{\phi}_{11} &= 0^4 1^2 2^1 3^1 4^1 6^1 7^1 13^1 14^1 15^1. & &
\end{aligned}$$

The weight 0 Jacobi form formula $\psi = (\phi_1|V_2)/\phi_1 - (\phi_2|V_2)/\phi_2 - (\phi_3|V_2)/\phi_3$ gives the Borcherds product formula $\mathrm{Borch}(\psi) = \mathrm{Grit}(\phi_2)\,\mathrm{Grit}(\phi_3)/\,\mathrm{Grit}(\phi_1)$, and so the nonlift eigenform formula $f_{353} = -11\mathrm{Borch}(\psi) + \sum_{i=1}^{11} c_i \,\mathrm{Grit}(\tilde{\phi}_i)$ gives a construction $f_{353} = Q/L$ of the eigenform as a quotient of a quadratic form in Gritsenko lifts over a linear form in Gritsenko lifts. Such a quotient construction is crucial in that it lets us compute many eigenvalues (over a hundred), enough for the level 353 modularity proof in [4], though that proof used a more complicated construction $f_{353} = Q/L$ than the one that we have obtained here. The nonlift eigenform formula $f_{353} = -11\mathrm{Borch}(\psi) + \sum_{i=1}^{11} c_i\,\mathrm{Grit}(\tilde{\phi}_i)$ also shows that f_{353} is congruent modulo 11 to a Gritsenko lift.

5. Levels $N = 461, 567^+$

We complete the proof of Theorem 1.1 by constructing nonlifts at the two outstanding levels that lack nonlift Borcherds products.

For an elliptic Atkin–Lehner matrix α_c at level NM, understanding that $c \mid NM$ and $\gcd(c, NM/c) = 1$, if c divides N, then α_c is also an Atkin–Lehner matrix at level N, and so the same is true for the corresponding paramodular Atkin–Lehner matrix μ_c. This shows that for any prime divisor p of N such that $\gcd(p, NM/p) = 1$, any p-Atkin–Lehner eigenform f in $\mathcal{S}_k(\mathrm{K}(NM))$ traces down to a p-Atkin–Lehner eigenform in $\mathcal{S}_k(\mathrm{K}(N))$. Indeed, if $\mathrm{K}(NM)\mathrm{K}(N) = \bigsqcup_i \mathrm{K}(NM)h_i$, then the traced down image of f is $\mathrm{TD}\,f = \sum_i f[h_i]_k$, and taking a p-Atkin–Lehner matrix μ_p at level N that is also a p-Atkin–Lehner matrix at level NM gives, because $f[\mu_p]_k = \epsilon_p f$ and $\mathrm{K}(NM)\mathrm{K}(N) = \bigsqcup_i \mathrm{K}(NM)\mu_p^{-1}h_i\mu_p$,

$$(\mathrm{TD}\,f)[\mu_p]_k = \sum_i f[\mu_p\mu_p^{-1}h_i\mu_p]_k = \epsilon_p\,\mathrm{TD}\,f.$$

Especially we may take $N = 461, 587$ and $M = 2$, and trace down a level $2N$ Atkin–Lehner eigenform having 2-eigenvalue -1 and N-eigenvalue 1, obtaining a level N Fricke plus form. This is our route to constructing a nonlift at the outstanding levels that lack nonlift Borcherds products.

For levels $N = 461, 587$, the methods of [14] show that there is no nonlift Borcherds product in $\mathcal{S}_2(\mathrm{K}(N))^+$. To construct nonlifts in these two cases, we begin by constructing Borcherds products in $\mathcal{S}_2(\mathrm{K}(2N))^-$. Again by the methods of [14], there exist one Borcherds product in $\mathcal{S}_2(\mathrm{K}(922))^-$ and three in $\mathcal{S}_2(\mathrm{K}(1174))^-$, having integral Fourier coefficients, some of which we can compute (see [15]). Polarizing the Borcherds products creates Atkin–Lehner eigenforms $f_{2N}^{-,+}$ in $\mathcal{S}_2(\mathrm{K}(2N))^{-,+}$ (i.e., the polarization of a Borcherds product f is $f - f[\mu_2]_2 + f[\mu_N]_2 - f[\mu_2\mu_N]_2$, with μ_2, μ_N-eigenvalues $-1, 1$), again having integral Fourier coefficients, and the eigenforms trace down to $\mathcal{S}_2(\mathrm{K}(N))^+$, the tracing down also preserving the integrality of the Fourier coefficients; tracing down is described in [13]. However, we do not have enough Fourier coefficients of our eigenforms to compute many Fourier coefficients of their traced down images, so we need to produce more coefficients first. We do so in a finite characteristic p. Level $2N$ Jacobi restriction with $p = 12347$, with $d = 5$, and with $\det_{\max} = 2305$ for level 922 or $\det_{\max} = 3522$ for level 1174, gives a one-dimensional superspace $\mathcal{J}_{5,p}^{-,+}$ of $\mathcal{S}_2(\mathrm{K}(2N))^{-,+}[5]_p$ in each case. Thus any one nonzero integral Fourier coefficient modulo p of some $f_{2N}^{-,+}$ shows that

$\mathbb{F}_p f_{2N}^{-,+}[5]_p = \mathcal{J}_{5,p}^{-,+}$, and this gives us the Fourier coefficients of $f_{2N}^{-,+}[5]_p$ in $\mathbb{F}_p$ at the indices for which we have Fourier coefficients of the $\mathcal{J}_{5,p}^{-,+}$ basis element. We refer to this process as *prolonging* $(f_{2N}^{-,+})_p$. Because $f_{2N}^{-,+}$ is an Atkin–Lehner eigenform, we can obtain yet more of its Fourier coefficients modulo p by using the relations $a(\alpha t\alpha'; f_{2N}^{-,+}) = \epsilon a(t; f_{2N}^{-,+})$ where $\alpha \in \{\alpha_2, \alpha_N\}$ is a 2×2 Atkin–Lehner involution matrix and α' is its transpose, and ϵ is the corresponding eigenvalue. We refer to this process as *infilling* $(f_{2N}^{-,+})_p$. Prolonging and infilling our initial fragment of $(f_{2N}^{-,+})_p$ can give us enough information to demonstrate that the desired nonlift exists at level N. Now let TD denote the trace down operator both in characteristic 0 and in characteristic p. For $N = 461$, the available Fourier coefficients of $(f_{922}^{-,+})_p$ from prolonging and infilling lead to 255 coefficients of its traced down image $\mathrm{TD}((f_{922}^{-,+})_p)$. These are more than enough to show that the latter is linearly independent of the Gritsenko lifts modulo p. Because $\mathrm{TD}((f_{922}^{-,+})_p) = (\mathrm{TD}\, f_{922}^{-,+})_p$, it follows that $\mathrm{TD}\, f_{922}^{-,+}$ is a nonlift in $\mathcal{S}_2(\mathrm{K}(461))^+$. Similarly, for $N = 587$ we have 271 Fourier coefficients of the traced down image of the $(f_{1174}^{-,+})_p$ that arises from the second (or third) of the three nonlift Borcherds product in $\mathcal{S}_2(\mathrm{K}(1174))^-$, again plenty to determine that $\mathrm{TD}\, f_{1174}^{-,+}$ is a nonlift in $\mathcal{S}_2(\mathrm{K}(587))^+$. The ideas of this paragraph generalize beyond an odd prime level N and $M = 2$ and a one-dimensional space $\mathcal{J}_{d,p}^{-,+}$, but here we have described them only as needed for the situation at hand.

6. Proof of Theorem 1.2

Recall that Theorem 1.2 says that the conjectured nonlift eigenform formulas $f_N = Q_N/L_N$ of [16, 17] hold for $N = 349, 353, 389, 461, 523, 587^{\pm}$. Here Q_N lies in $\mathcal{S}_4(\mathrm{K}(N))$ and L_N in $\mathrm{Grit}(\mathrm{J}_{2,N}^{\mathrm{cusp}})$. We prove the theorem.

Proof. To keep the discussion and notation simple, we will prove only the cases $N = 349, 353, 389, 461, 523, 587^+$ here; the case $N = 587^-$ is similar. We may view $\mathcal{S}_2(\mathrm{K}(N))$ as a subset of $\mathbb{C}^\infty$, indexed by $\mathcal{X}_2(N)$, by considering the injective map that takes each element of $\mathcal{S}_2(\mathrm{K}(N))$ to its Fourier series. In Section 6 of [16], the "integral closure" method proves that for each of these $N = 349, 353, 389, 461, 523, 587$ there exists a space $Y_N \subseteq \mathbb{C}^\infty$ such that

$$\mathcal{S}_2(\mathrm{K}(N))^+ \subseteq Y_N \quad \text{and} \quad \dim Y_N = \dim \mathrm{J}_{2,N}^{\mathrm{cusp}} + 1.$$

By Theorem 1.1 we now know that $\dim \mathcal{S}_2(\mathrm{K}(N))^+ = \dim Y_N$, and consequently $\mathcal{S}_2(\mathrm{K}(N))^+ = Y_N$. Thus the computations that were carried out on Y_N in [16] to find an initial Fourier expansion of an eigenform f_N, conjectural in [16], are now proven rigorous.

Also in [16], for a chosen $L_N \in \mathrm{Grit}(\mathrm{J}_{2,N}^{\mathrm{cusp}})$, an initial expansion of the product $Q_N = f_N L_N$ was made to identify the form $Q_N \in S_4(K(N))^+$ as a linear combination of products of Gritsenko lifts or products of Gritsenko lifts with characters, possibly with Hecke operators applied to such products. We were able to do so because in [16] we spanned $\mathcal{S}_4(\mathrm{K}(N))^+$ by known forms, and we had determining Fourier coefficients. Therefore each formula $f_N = Q_N/L_N$ is now proven. □

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