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THE FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH INFINITELY MANY SOLUTIONS

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Abstract. In this paper, we study the existence of infinitely many large energy solutions for the supercubic fractional Schrödinger-Poisson systems. We consider different superlinear growth assumptions on the nonlinearity, starting from the well-know Ambrosetti-Rabinowitz type condition. We obtain three different existence results in this setting by using the Fountain Theorem, all these results extend some results for semelinear Schrödinger-Poisson systems to the nonlocal fractional setting.

1. Introduction

In this paper, we consider the following system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\frac{3}{4} < s < 1$ and $(-\Delta)^s$ is the fractional Laplace operator which can be defined as

$$(-\Delta)^s u = C(s) P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy = C(s) \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy.$$

Here $P.V.$ stands for the Cauchy principal value, and $C(s)$ is a positive constant depending only on s, f is a continuous nonlinear function which satisfies different supercubic conditions and V is the real valued external potential function.

In recent years, this kind of systems were studied in some papers, due to the fact that solutions $(u(x), \phi(x))$ of (1.1) correspond to standing wave solutions

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$(e^{-iEt}u(x), \phi(x))$ of the time-dependent system

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = (-\Delta)^s\Psi + \widetilde{V}(x)\Psi + \phi\Psi - \widetilde{f}(x, |\Psi|)\Psi & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ (-\Delta)^s\phi = |\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where i is the imaginary unit, $\widetilde{V}(x) = V(x) + E$ and $\widetilde{f}(x, |u|)u = f(x, u)$.

The first equation in (1.2) was introduced by Laskin [15, 16], which is the so-called fractional Schrödinger equation, describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. An interesting Schrödinger equation class is when the potential $\phi(x)$ is determined by the charge of wave function itself, that is, when the second equation in (1.2) (Poisson equation) holds. For this reason, (1.2) is refereed to as a fractional nonlinear Schrödinger-Poisson system (also called Schrödinger-Maxwell system).

In the local case that $s = 1$, we have the semilinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

which was proposed by Benci and Fortunato [6] in 1998 on a bounded domain, and is related to the Hartree equation ([18]). In the past several years, the existence and multiplicity of solutions to the systems similar to (1.3) has been studied extensively by means of variational tools, we refer the interested readers to see [1, 3, 34, 35] and the references therein.

When $s \in (0, 1)$, as pointed out in [24], the fractional Laplacian operator is a nonlocal one. This makes the system (1.1) different with the local one (1.3) with $s = 1$. Therefore, there are only few references on the existence of solutions to the fractional Schrödinger-Poisson systems, see [21, 23, 27, 30–33], maybe because the standard techniques that were developed for the local Laplacian do not work immediately.

Motivated by an evident and increasing interest in the current literature on fractional elliptic problems, the aim of our paper is finding infinitely many large energy solution (i.e., high energy solutions) under different supercubic growth assumptions on $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and

$$F(x, t) = \int_0^t f(x, \tau)d\tau$$

for every $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. Such a type of problems is classical and one of the main difficulties is the lack of compactness for Sobolev embedding theorem for the whole space $\mathbb{R}^3$ case. To overcome this difficulty, motivated by the approach used in [26], we will establish the existence results for (1.1), under the following assumptions for potential V:

(V_0) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $0 < V_0 = \inf_{x\in\mathbb{R}^3} V(x)$.

(V_1) For any $M > 0$, there exists $r > 0$ such that

$$\lim_{|y|\to\infty} meas\{x \in \mathbb{R}^3 : |x - y| \le r, V(x) \le M\} = 0,$$

where *meas* denotes the Lebesgue measure.

Moreover, the nonlinearity f satisfies the following assumptions:

(f_1) there exist $c_1 > 0$ and $p \in (2, 2_s^*)$, where $2_s^* = \frac{6}{3-2s}$, such that

$$|f(x,t)| \le c_1(1 + |t|^{p-1}).$$

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly for any $x \in \mathbb{R}^3$.

(f_3) $f(x,-t) = -f(x,t)$ for any $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

(f_4) there exists $\mu > 4$ such that

$$0 < \mu F(x,t) \le tf(x,t).$$

We note that (f_4) is a variant Ambrosetti-Rabinowitz type condition (AR for short) which was originally introduced by Ambrosetti and Rabinowitz in [2], where, as an application of the famous Mountain Pass Theorem, they obtained the existence of nontrivial solutions of problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under superlinear and subcritical growth conditions on the righthand side.

A lot of works concerning superlinear elliptic problem have been written by using this usual (AR) condition (see, for instance, [10,25,28] and the references therein), whose role consists in ensuring the boundedness of the Palais-Smale sequences of the energy functional associated with the problem under consideration. However, there are many functions which are supercubic at infinity, but for which condition (f_4) fails. Indeed, from (f_4) it follows that for some $c_2, c_3 > 0$

$$F(x,t) \ge c_2|t|^\mu - c_3. \tag{1.4}$$

Obviously, it is easy to see that the function

$$f(x,t) = t^3 \log(1 + |t|) \tag{1.5}$$

does not satisfies the growth condition (1.4). At this purpose, we would note that from (1.4) and the fact that $\mu > 4$ mean that

(f_5) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^4} = +\infty$ uniformly for any $x \in \mathbb{R}^3$.

It is easily seen that the function (1.5) satisfies the condition (f_5) but does not verify (f_4). In recent years, such a condition was often applied to the existence of nontrivial solutions for superlinear problem without (f_4) condition. See, for instance, [7,19,20] and references therein. Jeanjean introduced in [13] the following assumption on f:

(f_6) there exists $\theta \geq 1$ such that

$$\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,\zeta t)$$

for every $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and $\zeta \in [0,1]$, where we set

$$\mathcal{F}(x,t) := f(x,t)t - 4F(x,t). \tag{1.6}$$

Note that (f_6) is a global condition and the function (1.5) also satisfies (f_6). Another interesting condition which is stronger than (f_6) is the following one (see e.g. [20]).

(f_7) The function $t \to \frac{f(x,t)}{t^3}$ is increasing on $(0,+\infty)$ and decreasing on $(-\infty,0)$, a.e. $x \in \mathbb{R}^3$.

After this overview on the assumptions on nonlinearity f, we can state our main result as follows.

Theorem 1.1. *Assume that* (V_0)-(V_1) *holds and let* $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *be a function verifying* (f_1)-(f_3) *and one of the following assumptions*

(a) (f_4),
(b) (f_5) *and* (f_6),
(c) (f_5) *and* (f_7).

Then the system (1.1) *has infinitely many solutions* $\{(u_k,\phi_k)\}$ *in* $H^s(\mathbb{R}^3) \times \mathcal{D}^{s,2}(\mathbb{R}^3)$ *satisfying*

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u_k|^2 + V(x)|u_k|^2)dx - \frac{1}{4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_k|dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx - \int_{\mathbb{R}^3}F(x,u_k)dx \to +\infty.
\end{aligned}$$

From the viewpoint of variational method, the system (1.1) is a strongly indefinite variational problem (see Section 2.2 below for more details). The proof of Theorem 1.1 rely on the similar arguments for a single fractional equation as used in [7, 22]. More precisely, the strategy of our proofs consists in looking for infinitely many critical points for the energy functional associated with problem (1.1), namely here we apply the Fountain Theorem proved by Bartsch in [5]. For this purpose, we have to analyze the compactness properties of the functional and its geometric features. As for the compactness, when the nonlinearity satisfies the (AR) assumption (f_4), we shall prove that the Palais-Smale condition is satisfied; when f is assumed to satisfy conditions (f_5) and (f_6) or (f_7), the Cerami condition will be considered. In both cases the main difficulty is related to the proof of the boundedness of the Palais-Smale (or Cerami) sequence. Finally, we would note that Theorem 1.1 represents the nonlocal counterpart of [11, 17, 29], where the limit case as $s \to 1$ (that is, the Laplace case) was considered.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2, including the functional space setting and some useful Lemmas. In Section 3, we are devoted to a completion the proof of Theorem 1.1.

2. Variational settings and preliminary results

Throughout this paper, we denote $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, $1 \le p < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, c or c_i $(i = 1, 2, \dots)$ denote some positive constants may change from line to line. $a_n \rightharpoonup a$ and $a_n \to a$ mean the weak and strong convergence, respectively, as $n \to \infty$.

2.1. The functional space setting

Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketchs of the fractional order Sobolev spaces and the complete introduction can be found in [12]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} := (\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^3)}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous and for any $s \in (0, 1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{2_s^*(\mathbb{R}^3)}}.$$

The fractional Laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3,$$

that is

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi\cdot x}\phi(x)dx$$

for functions ϕ in the Schwartz class. Also $(-\Delta)^s u$ can be equivalently represented [12] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s) \int_{\mathbb{R}^3} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{3+2s}} dy, \ \forall x \in \mathbb{R}^3,$$

where

$$C(s) = (\int_{\mathbb{R}^3} \frac{(1-\cos\xi_1)}{|\xi|^{3+2s}} d\xi)^{-1},\ \xi = (\xi_1, \xi_2, \xi_3).$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)} \|(-\Delta)^{\frac{s}{2}} u\|_2^2.$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined above

$$u \longmapsto (\int_{\mathbb{R}^3} |u|^2 dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy)^{\frac{1}{2}};$$

$$u \longmapsto (\int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi)^{\frac{1}{2}};$$

$$u \longmapsto (\int_{\mathbb{R}^3} |u|^2 dx + \|(-\Delta)^{\frac{s}{2}} u\|_2^2)^{\frac{1}{2}}$$

are equivalent.

In view of the presence of potential $V(x)$, we introduce the subspace

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 dx < +\infty \right\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_E = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x) uv dx,$$

and the norm

$$\|u\|_E^2 = (u,u) = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx.$$

We denote $\|\cdot\|_E$ by $\|\cdot\|$ in the sequel for convenience.

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces:

Lemma 2.1 ([12])**.** *Let $0 < s < 1$, then there exists a constant $C = C(s) > 0$ such that*

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^3)} \le C[u]^2_{H^s(\mathbb{R}^3)}$$

for every $u \in H^s(\mathbb{R}^3)$, where $2_s^* = \frac{6}{3-2s}$ is the fractional critical exponent. Moreover, the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2_s^*]$ and is locally compact whenever $r \in [2, 2_s^*)$.

We recall the following embedding properties of E.

Lemma 2.2 ([26])**.** *The embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $r \in [2, 2_s^*]$ and is compact whenever $r \in [2, 2_s^*)$.*

2.2. A reduced problem

It is clear that the system (1.1) is the Euler-Lagrange equations of the functional $J : E \times \mathcal{D}^{s,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.1}$$

Evidently, the energy functional $J \in C^1(E \times \mathcal{D}^{s,2}(\mathbb{R}^3), \mathbb{R})$ and its critical points are the solutions of (1.1). It is easy to show that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [6]. First of all, for a fixed $u \in E$, there exists a unique $\phi_u^s \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^s \phi = u^2 \quad \text{in } \mathbb{R}^3.$$

We can write an integral expression for ϕ_u^s in the form

$$\phi_u^s(x) = C_s \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2s}} dy, \quad \forall\, x \in \mathbb{R}^3,$$

which is called s-Riesz potential (see [14]), where

$$C_s = \frac{1}{\pi^{\frac{3}{2}}} \frac{\Gamma(3-2s)}{2^{2s}\Gamma(s)}.$$

Then the system (1.1) can be reduced to the first equation with ϕ represented by the solution of the fractional Poisson equation. This is the basic strategy of solving (1.1). To be more precise about the solution ϕ of the fractional Poisson equation, we have the following Lemma.

Lemma 2.3 ([27])**.** *For any $u \in E$, we have*
(i) $\phi_u^s \geq 0$;
(ii) $\|\phi_u^s\| \leq c\|u\|^2_{\frac{12}{3+2s}}$, *where $c > 0$ does not depend on u. As a consequence there exists $M > 0$ such that*

$$\int_{\mathbb{R}^3} \phi_u^s u^2 \leq M\|u\|^4.$$

Putting $\phi = \phi_u^s$ into the first equation of (1.1), we obtain a semilinear elliptic equation

$$(-\Delta)^s u + V(x)u + \phi_u^s u = f(x,u) \quad \text{in } \mathbb{R}^3$$

with a nonlocal term. The corresponding functional $\mathcal{I} : E \to \mathbb{R}$ is defined by
(2.2)

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

Note that if $s > \frac{3}{4}$, there holds $2 \leq \frac{12}{3+2s} \leq 2_s^*$ and thus $E \hookrightarrow L^{\frac{12}{3+2s}}(\mathbb{R}^3)$, then by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_u^s u^2 dx &\leq \Big(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\Big)^{\frac{3+2s}{6}} \Big(\int_{\mathbb{R}^3} |\phi_u^s|^{2_s^*} dx\Big)^{\frac{1}{2_s^*}} \\ &\leq \mathcal{S}_s^{-\frac{1}{2}} \Big(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\Big)^{\frac{3+2s}{6}} \|\phi_u^s\|_{\mathcal{D}^{s,2}} \\ &\leq C\|u\|^2 \|\phi_u^s\|_{\mathcal{D}^{s,2}} < \infty.\end{aligned}$$

Therefore, the functional $\mathcal{I}$ is well-defined for every $u \in E$ and belongs to $\mathcal{C}^1(E, \mathbb{R})$. Moreover, for any $u, v \in E$, we have

$$\begin{aligned}\langle \mathcal{I}'(u), v\rangle = &\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x) uv dx \\ &+ \int_{\mathbb{R}^3} \phi_u^s uv dx - \int_{\mathbb{R}^3} f(x, u) v dx.\end{aligned} \tag{2.3}$$

It is standard to verify that a critical point u of the functional $\mathcal{I}$ corresponds to a weak solution (u, ϕ) of (1.1) with $\phi = \phi_u^s$. Hence in the following, we consider critical points of $\mathcal{I}$ using variational method.

3. Proof of the main result

In order to prove our main results, we shall apply the Fountain Theorem due to Bartsch (see [5]), which, under suitable compactness and geometric assumptions on a functional, provides the existence of an unbounded sequence of critical value for it. The compactness condition assumed in the Fountain Theorem is the well-known Palais-Smale condition (see, for instance, [28]), which in our framework reads as follows:

Definition. Assume functional $I \in C^1$ and $c \in R$, if any sequence u_n satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ has a convergence subsequence, we say I satisfies Palais-Smale condition ((PS) in short) at the level c.

In [8,9] Cerami introduced the so-called Cerami condition, as a weak version of the Palais-Smale assumption. With our notation, it can be written as follows:

Definition. Assume functional $I \in C^1$ and $c \in R$, if any sequence u_n satisfying $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ has a convergence subsequence, we say I satisfies Cerami condition ((C) in short) at the level c.

We would remark that Cerami condition is weaker than the Palais-Smale condition. However, it was shown in [4] that from Cerami condition a deformation lemma follows and, as a consequence, we can also get minimax theorems. Hence, the Fountain Theorem holds true also under this different compactness assumption.

Theorem 3.1. *Let X be a Banach space with the norm $\|\cdot\|$ and let X_j be a sequence of subspace of X with* $\dim X_j < +\infty$ *for each* $j \in \mathbb{N}$. *Further* $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, *the closure of the direct sum of all* X_j. *Set*

$$W_k = \bigoplus_{j=0}^{k} X_j, \;\; Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}\,, n \in \mathbb{N}.$$

Consider an even functional $I \in C^1(X, \mathbb{R})$ *(i.e.,* $I(-u) = I(u)$ *for all* $u \in X$*). If for ever* $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

(A_1) $a_k := \max\limits_{\substack{u\in W_k\\ \|u\|=\rho_k}} I(u) \le 0,$

(A_2) $b_k := \inf\limits_{\substack{u\in Z_k\\ \|u\|=r_k}} I(u) \to \infty, k \to \infty,$

(A_3) *I satisfies the* $(PS)_c$ *or* $(C)_c$ *condition for every* $c > 0$,

then I has an unbounded sequence of critical values.

We choose an orthogonal basis $\{e_j\}$ of $X := E$ and define $W_k := span\{e_1, \ldots, e_k\}$, $Z_k := W_{k-1}^{\perp}$. In order to perform the proof of Theorem 1.1, we first need the following result:

Lemma 3.2. *For any* $2 \le p < 2_s^*$, *we have that*

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_p \to 0 \;\; as \;\; k \to \infty.$$

Proof. Since the embedding from E into $L^p(\mathbb{R}^3)$ is compact, then Lemma 3.1 can be proved by a similar way as Lemma 3.8 in [28]. □

3.1. The case (f_4) holds

This section is devoted to the problem (1.1) in presence of a nonlinear term satisfying condition (f_4). In this framework we shall prove the following result about the compactness of the functional $\mathcal{I}$:

Lemma 3.3. *Let* $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *be a function verifying conditions* (f_1)-(f_4). *Then $\mathcal{I}$ satisfies the (PS) condition at any level* $c \in \mathbb{R}$.

Proof. We split the proof into two steps. First, we show that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E and then that it admits a strongly convergent subsequence in E.

Step 1. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. From (f_4), for n large enough we have

$$\begin{aligned} 1 + c + \|u_n\| &\ge \mathcal{I}(u_n) - \frac{1}{\mu}\langle \mathcal{I}'(u_n), u_n\rangle \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_n\|^2 + (\frac{1}{4} - \frac{1}{\mu}) \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx \end{aligned}$$

$$+\int_{\mathbb{R}^3}\frac{1}{\mu}f(x,u_n)u_n-F(x,u_n)dx$$
$$\geq(\frac{1}{2}-\frac{1}{\mu})\|u_n\|^2+(\frac{1}{4}-\frac{1}{\mu})\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx.$$

Since $\mu>4$ and $\phi_u^s\geq 0$, it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E.

Step 2. Up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ strongly converges in E. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E, we may assume that there exists $u\in E$ such that

$$u_n\rightharpoonup u \text{ in } E,$$
$$u_n\to u \text{ in } L^r(\mathbb{R}^3),\ 2\leq r<2_s^*,$$
$$u_n(x)\to u(x) \text{ a.e. in } \mathbb{R}^3$$

as $n\to+\infty$. Observe that

$$\|u_n-u\|^2=\langle\mathcal{I}'(u_n)-\mathcal{I}'(u),u_n-u\rangle+\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)dx$$
$$-\int_{\mathbb{R}^3}(\phi_{u_n}^s u_n-\phi_u^s u)(u_n-u)dx.$$

It is clear that

$$\langle\mathcal{I}'(u_n)-\mathcal{I}'(u),u_n-u\rangle\to 0 \text{ as } n\to+\infty.$$

According to assumptions (f_1)-(f_2) and the Hölder inequality, we obtain

$$\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)dx$$
$$\leq\int_{\mathbb{R}^3}c_1(|u_n|+|u|+|u_n|^{p-1}+|u|^{p-1})|u_n-u|dx$$
$$\leq c_2(\|u_n\|_2+\|u\|_2)\|u_n-u\|_2+c_2(\|u_n\|_p^{p-1}+\|u\|_p^{p-1})\|u_n-u\|_p.$$

Since $u_n\to u$ in $L^r(\mathbb{R}^3)$ for all $r\in[2,2_s^*)$, we have that

$$\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)dx\to 0 \text{ as } n\to+\infty.$$

By Hölder inequality, the Sobolev inequality and Lemma 2.3 we have

$$|\int_{\mathbb{R}^3}\phi_{u_n}^s u_n(u_n-u)dx|\leq\|\phi_{u_n}^s\|_{2_s^*}\|u_n\|_{\frac{3}{s}}\|u_n-u\|_2$$
$$\leq c_3\|\phi_{u_n}^s\|_{\mathcal{D}^{s,2}}\|u_n\|_{\frac{3}{s}}\|u_n-u\|_2$$
$$\leq c_3c_4\|u_n\|_{\frac{12}{3+2s}}^2\|u_n\|_{\frac{3}{s}}\|u_n-u\|_2,$$

where $c_i>0$ is a constant. Again using $u_n\to u$ in $L^r(\mathbb{R}^3)$ for any $r\in[2,2_s^*)$, we have

$$\int_{\mathbb{R}^3}\phi_{u_n}^s u_n(u_n-u)dx\to 0 \text{ as } n\to\infty,$$

where we need $s > \frac{3}{4}$. Similarly, we obtain

$$\int_{\mathbb{R}^3} \phi_u^s u(u_n - u)dx \to 0 \text{ as } n \to \infty.$$

Thus

$$\int_{\mathbb{R}^3} (\phi_{u_n}^s u_n - \phi_u^s u)(u_n - u)dx \to 0 \text{ as } n \to \infty,$$

so that $\|u_n - u\| \to 0$. Thus, we get $\mathcal{I}(u)$ satisfies Palais-Smale condition. □

Proof of Theorem 1.1 under assumption (a). By Lemma 3.2 we have that $\mathcal{I}$ satisfies the Palais-Smale condition, while by (f_3), we get that $\mathcal{I}(-u) = \mathcal{I}(u)$ for any $u \in E$. In order to apply the Fountain Theorem 3.1, it remains to study the geometry of the functional $\mathcal{I}$. For this purpose, we will verify $\mathcal{I}$ satisfies the remaining conditions of Theorem 3.1.

Step 1. We verify that $\mathcal{I}$ satisfies (A_1). Observe that assumption (f_1)-(f_4) implies there exist $c_1, c_2 > 0$ such that

$$F(x, u) \geq c_1|u|^\mu - c_2|u|^2$$

for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$. Hence we have

$$\mathcal{I}(u) \leq \frac{1}{2}\|u\|^2 + \frac{M}{4}\|u\|^4 - c_1\|u\|^\mu + c_2\|u\|_2^2.$$

Since, on the finitely dimensional space W_k all norms are equivalent, $\mu > 4$ implies that

$$a_k := \max_{u \in W_k, \|u\| = \rho_k} \mathcal{I}(u) \leq 0$$

for some $\rho_k > 0$ large enough.

Step 2. We prove $\mathcal{I}$ satisfies (A_2). It follows form (f_1) and (f_2) that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + c_\varepsilon|u|^{p-1}$$

for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$. By the equality $F(x, t) = \int_0^t f(x, \tau)d\tau$ we obtain

$$F(x, u) \leq \varepsilon|u|^2 + c_\varepsilon|u|^p$$

for all $x \in \mathbb{R}^3$ and $z \in \mathbb{R}$. Hence, we have

$$\begin{aligned}\mathcal{I}(u) &\geq \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_2^2 - c_\varepsilon\|u\|_p^p \\ &\geq (\frac{1}{2} - \frac{\varepsilon}{V_0})\|u\|^2 - c_3\beta_k^p\|u\|^p,\end{aligned}$$

where V_0 is a lower bound of $V(x)$ from (V_1) and β_k are defined in Lemma 3.1. Choosing $r_k := (c_3 p \beta_k^p)^{\frac{1}{2-p}}$, we obtain

$$\begin{aligned}b_k :&= \inf_{u \in Z_k, \|u\| = r_k} \mathcal{I}(u) \\ &\geq \inf_{u \in Z_k, \|u\| = r_k} [(\frac{1}{2} - \frac{\varepsilon}{V_0})\|u\|^2 - c_3\beta_k^p\|u\|^p]\end{aligned}$$

$$\geq (\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{p})(c_3 p \beta_k^p)^{\frac{2}{2-p}}.$$

Because $\beta_k \to 0$ as $k \to 0$ and $p > 2$, we have

$$b_k \geq (\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{p})(c_3 p \beta_k^p)^{\frac{2}{2-p}} \to +\infty$$

for enough small ε. This prove (A_2).

The proof of theorem under assumption (a) is completed and we would like to emphasize that in the verification of the geometric structure of the functional $\mathcal{I}$ the Ambrosetti-Rabinowitz condition (namely, (f_4)) was used only for proving Step 1. □

3.2. The case (f_5) and (f_6) hold

In this section we shall deal with the problem (1.1) when superlinear conditions on the term f different from the Ambrosetti-Rabinowitz are satisfied. Now, we show that the functional $\mathcal{I}$ satisfies the Cerami condition.

Lemma 3.4. *Under the assumptions* (f_1)-(f_2) *and* (f_5)-(f_6)*, the functional* $\mathcal{I}(u)$ *satisfies the Cerami condition an any positive level.*

Proof. We suppose that $\{u_n\}_{n\in\mathbb{N}}$ is the Cerami sequence, that is for some $c > 0$

$$\begin{aligned}\mathcal{I}(u_n) = &\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^s |u_n|^2 dx \\ &- \int_{\mathbb{R}^3} F(x, u_n) dx \to c \text{ as } n \to \infty,\end{aligned} \tag{3.1}$$

and

$$(1 + \|u_n\|)\mathcal{I}'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.2}$$

From (3.1) and (3.2), for n large enough, we have

$$\begin{aligned}1 + c &\geq \mathcal{I}(u_n) - \frac{1}{4}\langle \mathcal{I}'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} f(x, u_n)u_n dx - \int_{\mathbb{R}^3} F(x, u_n) dx.\end{aligned} \tag{3.3}$$

We claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Otherwise there should exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ satisfying $\|u_n\| \to \infty$ as $n \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, then v_n is bounded. Passing to a subsequence, for some $v \in E$, we obtain

$$\begin{aligned}&v_n \rightharpoonup v \text{ in } E, \\ &v_n \to v \text{ in } L^r(\mathbb{R}^3),\ 2 \leq r < 2_s^*, \\ &v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^3\end{aligned}$$

as $n \to +\infty$ and there exists $h \in L^r(\mathbb{R}^3)$ such that

$$|v_n(x)| \leq h(x) \text{ a.e. in } \mathbb{R}^3 \text{ for any } n \in \mathbb{N} \tag{3.4}$$

(see [28]). In the sequel we will consider separately the cases when $v \equiv 0$ and $v \not\equiv 0$ and we will prove that in both these situations a contradiction occurs.

Firstly, let us suppose that $v \not\equiv 0$ in E. Dividing by $\|u_n\|^4$ in both side of (3.1) and Lemma 2.3(i) we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{4F(x,u_n)}{\|u_n\|^4}dx = \lim_{n\to\infty}\frac{2\|u_n\|^2 + \int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx - 4\mathcal{I}(u_n)}{\|u_n\|^4} \le C. \tag{3.5}$$

Let $\Omega := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, then $\operatorname{meas}(\Omega) > 0$ and $\lim_{n\to\infty}|u_n(x)| \to \infty$ for a.e. $x \in \Omega$. By (f_5) and Fatou's Lemma, for large n we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{4F(x,u_n)}{\|u_n\|^4}dx &\ge \lim_{n\to\infty}\int_{\Omega}\frac{4F(x,u_n)v_n^4}{|u_n|^4}dx\\ &\ge \int_{\Omega}\liminf_{n\to\infty}\frac{4F(x,u_n)v_n^4}{|u_n|^4}dx \to \infty\end{aligned}$$

as $n \to \infty$. This contradicts (3.5).

Now, suppose that $v \equiv 0$. As in [13], we can say that for any $n \in \mathbb{N}$ there exists $t_n \in [0,1]$ such that

$$\mathcal{I}(t_n u_n) = \max_{t\in[0,1]}\mathcal{I}(tu_n). \tag{3.6}$$

Since $\|u_n\| \to \infty$, for any $m \in \mathbb{N}$, we can choose $r_m = 2\sqrt{m}$ such that

$$r_m\|u_n\|^{-1} \in (0,1), \tag{3.7}$$

provided n is large enough.

By the continuity of the function F, we get that

$$F(x, r_m v_n) \to F(x, r_m v) \ \text{ a.e. } x \in \mathbb{R}^3$$

as $n \to +\infty$ for any $m \in \mathbb{N}$. Moreover, by (f_1) and (f_2), we have

$$\begin{aligned}|F(x,r_m v_n)| &\le \varepsilon|r_m v_n|^2 + c_\varepsilon|r_m v_n|^p\\ &\le \varepsilon(r_m h(x))^2 + c_\varepsilon(r_m h(x))^p \in L^1(\mathbb{R}^3),\end{aligned} \tag{3.8}$$

a.e. $x \in \mathbb{R}^3$ and for any $m, n \in \mathbb{N}$. Hence, (3.7), (3.8) and the Dominated Convergence Theorem yield that

$$F(x, r_m v_n) \to F(x, r_m v) \text{ in } L^1(\mathbb{R}^3) \tag{3.9}$$

as $n \to +\infty$ for any $m \in \mathbb{N}$. Since $F(x,0) = 0$ for any $x \in \mathbb{R}^3$ and $v \equiv 0$ holds true, (3.9) gives that

$$\int_{\mathbb{R}^3}F(x, r_m v_n)dx \to 0 \tag{3.10}$$

as $n \to +\infty$ for any $m \in \mathbb{N}$. Thus, (3.6), (3.7) and (3.10) yield

$$\begin{aligned} \mathcal{I}(t_n u_n) &\geq \mathcal{I}(r_m \|u_n\|^{-1} u_n) \\ &= \mathcal{I}(r_m v_n) \\ &= \frac{1}{2}\|r_m v_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi^s_{r_m v_n} |r_m v_n|^2 dx - \int_{\mathbb{R}^3} F(x, r_m v_n) dx \\ &\geq 2m - \int_{\mathbb{R}^3} F(x, r_m v_n) dx \geq m, \end{aligned} \tag{3.11}$$

provided n is large enough and for any $m \in \mathbb{N}$. From this we deduce that

$$\mathcal{I}(t_n u_n) \to +\infty \tag{3.12}$$

as $n \to +\infty$. Since $\mathcal{I}(0) = 0$ and $\mathcal{I}(u_n) \to c$, then $0 < t_n < 1$ if n large enough, we have

$$\begin{aligned} &\|t_n u_n\| + \int_{\mathbb{R}^3} \phi^s_{t_n u_n} |t_n u_n|^2 dx - \int_{\mathbb{R}^3} f(x, t_n u_n) t_n u_n dx \\ &= \langle \mathcal{I}'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \mathcal{I}(t u_n) = 0. \end{aligned}$$

Thus, by (f_6) we obtain

$$\begin{aligned} &\mathcal{I}(u_n) - \frac{1}{4}\langle \mathcal{I}'(u_n), u_n \rangle \\ &= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} [\frac{1}{4} f(x, u_n) u_n - F(x, u_n)] dx \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} H(x, u_n) dx \\ &\geq \frac{1}{4\theta}\|t_n u_n\|^2 + \frac{1}{4\theta}\int_{\mathbb{R}^3} H(x, t_n u_n) dx \\ &= \frac{1}{4\theta}\|t_n u_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^3} [\frac{1}{4} f(x, t_n u_n) t_n u_n - F(x, t_n u_n)] dx \\ &= \frac{1}{\theta}\mathcal{I}(t_n u_n) - \frac{1}{4\theta}\langle \mathcal{I}'(t_n u_n), t_n u_n \rangle \to +\infty. \end{aligned} \tag{3.13}$$

This contradicts (3.3). So $\{u_n\}_{n \in \mathbb{N}}$ is bounded. In order to prove the Lemma from now on we can argue as in Step 2 of the proof of Lemma 3.2. □

Proof of Theorem 1.1 under assumption (b). Due to Lemma 3.3, $\mathcal{I}(u)$ satisfies Cerami condition. Next, we verify that $\mathcal{I}(u)$ satisfies the rest conditions of Theorem 3.1. The verification of geometric assumption (A_2) of the Fountain Theorem follows as in Step 2 in Section 3.1. It remains to verify that $\mathcal{I}(u)$ satisfies (A_1).

Indeed, it following from (f_5) that for any $c_1 > 0$, there exists $\delta > 0$ such that

$$F(x, u) \geq c_1 |u|^4$$

for all $x \in \mathbb{R}^3$, and all $|u| > \delta$. From (f_1)-(f_2), there exists $c_2 > 0$ such that for all $x \in \mathbb{R}^3$, $0 < |u| < \delta$, we have

$$\frac{|f(x,u)u|}{|u|^2} \le c_2.$$

Denote $c_3 = c_1|\delta|^2 + \frac{c_2}{2}$ we have

$$F(x,u) \ge c_1|u|^4 - c_3|u|^2.$$

Hence, we have

$$\mathcal{I}(u) \le \frac{1}{2}\|u\|^2 + \frac{M}{4}\|u\|^4 - \frac{1}{4}c_1\|u\|_4^4 + c_3\|u\|_2^2.$$

Since, on the finitely dimensional space W_k all norms are equivalent, we have that

$$\mathcal{I}(u) \le \frac{1}{2}\|u\|^2 + \frac{C}{4}\|u\|^4 - \frac{1}{4}c_1c_4\|u\|^4 + c_3c_5\|u\|^2,$$

where c_i is a constant. Now take c_1 sufficiently large such that

$$\frac{C}{4} - \frac{1}{4}c_1c_4 < 0,$$

it follows that

$$a_k := \max_{u\in W_k, \|u\|=\rho_k} \mathcal{I}(u) \le 0$$

for some $\rho_k > 0$ large enough. This proves that $\mathcal{I}$ satisfies condition (A_1) of Theorem 3.1 and the proof of Theorem 1.1 is completed. □

3.3. The case (f_5) and (f_7) hold

In this setting we need the following Lemma, which will be crucial in the proof of this subsection.

Lemma 3.5 ([20]). *If (f_7) holds, then for any $x \in \mathbb{R}^3$, the function $\mathcal{F}(x,t) := f(x,t)t - 4F(x,t)$ is increasing when $t > 0$ and decreasing when $t < 0$. That is*

$$\mathcal{F}(x,s) \le \mathcal{F}(x,t), \quad \forall\ 0 \le s < t \text{ or } t < s \le 0 \text{ for } x \in \mathbb{R}^3.$$

Lemma 3.6. *Under the assumptions (f_1), (f_2), (f_5) and (f_7), the functional $\mathcal{I}(u)$ satisfies the Cerami condition an any positive level.*

Proof. We can argue exactly as in the proof of Lemma 3.3. We only have to modify the proof of inequality (3.13). Here we will show the validity of (3.13) by making use of assumption (f_7) and Lemma 3.4. We point out that our notation is the one used in the proof of Lemma 3.3. In view of Lemma 3.4 we have that

$$\begin{aligned}
&\mathcal{I}(u_n) - \frac{1}{4}\langle \mathcal{I}'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3}[\frac{1}{4}f(x,u_n)u_n - F(x,u_n)]dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} H(x,u_n)dx \\
&\geq \frac{1}{4}\|t_n u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} H(x,t_n u_n)dx \\
&= \frac{1}{4}\|t_n u_n\|^2 + \int_{\mathbb{R}^3} [\frac{1}{4} f(x,t_n u_n)t_n u_n - F(x,t_n u_n)]dx \\
&= \mathcal{I}(t_n u_n) - \frac{1}{4}\langle \mathcal{I}'(t_n u_n), t_n u_n\rangle \to +\infty
\end{aligned}
$$

as $n \to \infty$. Thus, we get a contradiction. Combining the arguments in Lemma 3.3, the proof of this Lemma is completed. □

Proof of Theorem 1.1 under assumption (c). The functional $\mathcal{I}$ satisfies the Cerami condition by Lemma 3.5, as for the geometric features of $\mathcal{I}$, condition (A_2) of the Fountain Theorem follows as in Step 2 of the proof of Section 3.1, whereas condition (A_1) can be proved as in the proof of Section 3.2. Hence, the assertion of Theorem 1.1 under assumption (c) is obtained. □

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References

[1] A. Ambrosetti, *On Schrödinger-Poisson systems*, Milan J. Math. **76** (2008), 257–274. https://doi.org/10.1007/s00032-008-0094-z

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7

[3] A. Azzollini, *Concentration and compactness in nonlinear Schrödinger-Poisson system with a general nonlinearity*, J. Differential Equations **249** (2010), no. 7, 1746–1763. https://doi.org/10.1016/j.jde.2010.07.007

[4] P. Bartolo, V. Benci, and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. **7** (1983), no. 9, 981–1012. https://doi.org/10.1016/0362-546X(83)90115-3

[5] T. Bartsch, *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal. **20** (1993), no. 10, 1205–1216. https://doi.org/10.1016/0362-546X(93)90151-H

[6] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), no. 2, 283–293. https://doi.org/10.12775/TMNA.1998.019

[7] Z. Binlin, G. Molica Bisci, and R. Servadei, *Superlinear nonlocal fractional problems with infinitely many solutions*, Nonlinearity **28** (2015), no. 7, 2247–2264. https://doi.org/10.1088/0951-7715/28/7/2247

[8] G. Cerami, *An existence criterion for the critical points on unbounded manifolds*, Istit. Lombardo Accad. Sci. Lett. Rend. A **112** (1978), no. 2, 332–336 (1979).

[9] ———, *On the existence of eigenvalues for a nonlinear boundary value problem*, Ann. Mat. Pura Appl. (4) **124** (1980), 161–179. https://doi.org/10.1007/BF01795391

[10] X. Chang and Z.-Q. Wang, *Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian*, J. Differential Equations **256** (2014), no. 8, 2965–2992. https://doi.org/10.1016/j.jde.2014.01.027

[11] S.-J. Chen and C.-L. Tang, *High energy solutions for the superlinear Schrödinger-Maxwell equations*, Nonlinear Anal. **71** (2009), no. 10, 4927–4934. https://doi.org/10.1016/j.na.2009.03.050

[12] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004

[13] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 4, 787–809. https://doi.org/10.1017/S0308210500013147

[14] N. S. Landkof, *Foundations of Modern Potential Theory*, translated from the Russian by A. P. Doohovskoy, Springer-Verlag, New York, 1972.

[15] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), no. 4-6, 298–305. https://doi.org/10.1016/S0375-9601(00)00201-2

[16] ———, *Fractional Schrödinger equation*, Phys. Rev. E (3) **66** (2002), no. 5, 056108, 7 pp. https://doi.org/10.1103/PhysRevE.66.056108

[17] L. Li and S.-J. Chen, *Infinitely many large energy solutions of superlinear Schrödinger-Maxwell equations*, Electron. J. Differential Equations **2012** (2012), no. 224, 9 pp.

[18] P.-L. Lions, *Solutions of Hartree-Fock equations for Coulomb systems*, Comm. Math. Phys. **109** (1987), no. 1, 33–97. http://projecteuclid.org/euclid.cmp/1104116712

[19] S. Liu, *On ground states of superlinear p-Laplacian equations in* $\mathbf{R}^N$, J. Math. Anal. Appl. **361** (2010), no. 1, 48–58. https://doi.org/10.1016/j.jmaa.2009.09.016

[20] ———, *On superlinear problems without the Ambrosetti and Rabinowitz condition*, Nonlinear Anal. **73** (2010), no. 3, 788–795. https://doi.org/10.1016/j.na.2010.04.016

[21] W. Liu, *Existence of multi-bump solutions for the fractional Schrödinger-Poisson system*, J. Math. Phys. **57** (2016), no. 9, 091502, 17 pp. https://doi.org/10.1063/1.4963172

[22] G. Molica Bisci, D. Repovš, and R. Servadei, *Nontrivial solutions of superlinear nonlocal problems*, Forum Math. **28** (2016), no. 6, 1095–1110. https://doi.org/10.1515/forum-2015-0204

[23] E. G. Murcia and G. Siciliano, *Positive semiclassical states for a fractional Schrödinger-Poisson system*, Differential Integral Equations **30** (2017), no. 3-4, 231–258. http://projecteuclid.org/euclid.die/1487386824

[24] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), no. 3, 031501, 17 pp. https://doi.org/10.1063/1.4793990

[25] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, 1990. https://doi.org/10.1007/978-3-662-02624-3

[26] K. Teng, *Multiple solutions for a class of fractional Schrödinger equations in* $\mathbb{R}^N$, Nonlinear Anal. Real World Appl. **21** (2015), 76–86. https://doi.org/10.1016/j.nonrwa.2014.06.008

[27] ———, *Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent*, J. Differential Equations **261** (2016), no. 6, 3061–3106. https://doi.org/10.1016/j.jde.2016.05.022

[28] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, **24**, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1

[29] M.-H. Yang and Z.-Q. Han, *Existence and multiplicity results for the nonlinear Schrödinger-Poisson systems*, Nonlinear Anal. Real World Appl. **13** (2012), no. 3, 1093–1101. https://doi.org/10.1016/j.nonrwa.2011.07.008

[30] Z. Yang, Y. Yu, and F. Zhao, *The concentration behavior of ground state solutions for a critical fractional Schrödinger-Poisson system*, Math. Nachr. **292** (2019), 1837–1868. https://doi.org/10.1002/mana.201700398

[31] ———, *Concentration behavior of ground state solutions for a fractional Schrödinger-Poisson system involving critical exponent*, Commun. Contemp. Math. (2019), 1850027. https://doi.org/10.1142/S021919971850027X

[32] Y. Yu, F. Zhao and L. Zhao, *The concentration behavior of ground state solutions for a fractional Schrödinger-Poisson system*, Calc. Var. Partial Differential Equations **56** (2017), no. 4, Art. 116, 25 pp. https://doi.org/10.1007/s00526-017-1199-4

[33] J. Zhang, J. M. do Ó, and M. Squassina, *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. **16** (2016), no. 1, 15–30. https://doi.org/10.1515/ans-2015-5024

[34] L. Zhao, H. Liu, and F. Zhao, *Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential*, J. Differential Equations **255** (2013), no. 1, 1–23. https://doi.org/10.1016/j.jde.2013.03.005

[35] L. Zhao and F. Zhao, *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl. **346** (2008), no. 1, 155–169. https://doi.org/10.1016/j.jmaa.2008.04.053

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