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CONGRUENCES MODULO POWERS OF 2 FOR OVERPARTITION PAIRS INTO ODD PARTS

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Abstract. We find congruences modulo 32, 64 and 128 for the partition function $\overline{pp}_o(n)$, the number of overpartition pairs of n into odd parts, with the aid of Ramamnujan's theta function identities and some known identities of $t_k(n)$, for $k = 6, 7$, where $t_k(n)$ denotes the number of representations of n as a sum of k triangular numbers. We also find two Ramanujan-like congruences for $\overline{pp}_o(n)$ modulo 128.

1. Introduction

Corteel and Lovejoy [7] introduced the notion of overpartitions. Several mathematicians studied arithmetic properties of overpartitions, for example, see Mahlburg [14], Hirschhorn and Sellers [10], and Kim [11]. An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined. For example, the eight overpartitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. Let $\overline{p}(n)$ denote the number of overpartitions of n. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2,q^2)_\infty}{(q,q)_\infty^2} = \frac{1}{\varphi(-q)},$$

where, as customary, for any complex number a and $|q| < 1$,

$$(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

An overpartition pair into odd parts is a pair of overpartitions (a, b) such that the parts of both overpartitions a and b are restricted to be odd integers. For

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example, the overpartition pairs of 3 into odd parts are

$$(3,\emptyset),(\overline{3},\emptyset),(1{+}1{+}1,\emptyset),(\overline{1}{+}1{+}1,\emptyset),(1{+}1,1),(\overline{1}{+}1,1),(1{+}1,\overline{1}),(\overline{1}{+}1,\overline{1}),$$
$$(\emptyset,3),(\emptyset,\overline{3}),(\emptyset,1{+}1{+}1),(\emptyset,\overline{1}{+}1{+}1),(1,1{+}1),(1,\overline{1}{+}1),(\overline{1},1{+}1),(\overline{1},\overline{1}{+}1).$$

If we denote by $\overline{pp}_o(n)$ the number of the overpartition pairs of n into odd parts, then $\overline{pp}_o(3)=16$. The generating function for $\overline{pp}_o(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_o(n)q^n &= \frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2}=\frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty^4}\\ &=\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^2}\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^4}=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4(q^4;q^4)_\infty^2}\\ &=\frac{\varphi(q)}{\varphi(-q)}. \qquad (1)\end{aligned}$$

This function has arisen in a number of recent papers, but in contexts which are very different from overpartitions. For example, see Bessenrodt [3], Chen and Lin [6], Bringmann and Lovejoy [4] and Kim [12].

In 2012, Lin [13] established several congruences for $\overline{pp}_o(n)$. Let p be a prime such that $p\equiv 1 \pmod 4$ and r an integer with $1\le r<p$. Then for all $n\ge 0$, Lin [13] proved that

$$\overline{pp}_o(p^3(pn+r))\equiv 0 \pmod{16},$$
$$\overline{pp}_o(8n+7)\equiv 0 \pmod{32}.$$

In the same paper, he also found that for all $\alpha,n\ge 0$,

$$\overline{pp}_o(9^\alpha(9n+6))\equiv 0 \pmod{3},$$
$$\overline{pp}_o(9^\alpha(27n+18))\equiv 0 \pmod{3}.$$

In this article, we prove some infinite families of congruences for $\overline{pp}_o(n)$ modulo $32, 64$ and 128. In the proof, we use Ramamnujan's theta function identities and some known identities of $t_k(n)$, for $k=6,\ 7$, where $t_k(n)$ denotes the number of representations of n as a sum of k triangular numbers. We also apply series representations of certain theta functions. In the following two theorems we find infinite families of congruences for $\overline{pp}_o(n)$ modulo 32 and 64.

Theorem 1.1. *Let p be an odd prime such that $p\equiv 3 \pmod 4$. Then for $\alpha,n\ge 0$, we have*

$$\overline{pp}_o\left(8p^{2\alpha+1}(pn+r)+p^{2(\alpha+1)}\right)\equiv 0 \pmod{32}, \qquad (2)$$

$$\overline{pp}_o\left(8p^{2\alpha+1}(pn+r)+5p^{2(\alpha+1)}\right)\equiv 0 \pmod{64}, \qquad (3)$$

where $1\le r<p$.

Theorem 1.2. *Let p be an odd prime such that $p \equiv 3 \pmod 4$. Then for $\alpha, n \geq 0$, we have*

$$\overline{pp}_o\left(8p^{2\alpha+1}(pn+r)+2p^{2(\alpha+1)}\right) \equiv 0 \pmod{64}, \tag{4}$$

where $1 \leq r < p$.

In the following results, we find three infinite families of congruences for $\overline{pp}_o(n)$ modulo 128.

Theorem 1.3. *Let p be an odd prime such that $p \equiv 5 \pmod 8$ or $p \equiv 7 \pmod 8$. Then for $\alpha, n \geq 0$, we have*

$$\overline{pp}_o\left(16p^{2\alpha+1}(pn+r)+6p^{2(\alpha+1)}\right) \equiv 0 \pmod{128}, \tag{5}$$

where $1 \leq r < p$.

Theorem 1.4. *Let p be an odd prime such that $p \equiv 1 \pmod 4$. Then for $\alpha, n \geq 0$, we have*

$$\overline{pp}_o\left(6p^{4\alpha+3}\right) \equiv 0 \pmod{128}.$$

Theorem 1.5. *Let p be an odd prime such that $p \nmid n$. Then for $\alpha, n \geq 0$, we have*

$$\overline{pp}_o(16p^{4\alpha+3}\cdot n+14p^{4\alpha+4}) \equiv 0 \pmod{128}. \tag{6}$$

We also find two Ramanujan-like congruences for $\overline{pp}_o(n)$ modulo 128, namely:

Theorem 1.6. *For any integer $n \geq 0$, we have*

$$\overline{pp}_o(72n+42) \equiv 0 \pmod{128}, \tag{7}$$

$$\overline{pp}_o(72n+66) \equiv 0 \pmod{128}. \tag{8}$$

We end this section by giving two internal congruences for $\overline{pp}_o(n)$ as listed below.

Theorem 1.7. *For any integer $n \geq 0$, we have*

$$\overline{pp}_o(8n+6) \equiv 6\overline{pp}_o(4n+3) \pmod{128}. \tag{9}$$

Theorem 1.8. *For any integer $n, \alpha \geq 0$, we have*

$$\overline{pp}_o(12\times 3^\alpha n+6\times 3^\alpha) \equiv (-1)^\alpha\overline{pp}_o(12n+6) \pmod{72}. \tag{10}$$

The rest of this paper is organized as follows. In Section 2 we recall Ramanujan's theta functions and also give some lemmas which will be used to prove our main results. In Section 3 we prove Theorems 1.1 and 1.2. We prove Theorems 1.3, 1.4 and 1.5 in Section 4. We apply series representations of certain theta functions to prove these theorems. We also use certain known identities for $t_k(n)$. Finally, in Section 5 we prove Theorems 1.6, 1.7 and 1.8.

2. Preliminaries

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Three special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{11}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{12}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = (q;q)_\infty, \tag{13}$$

where the product representations arise from Jacobi's famous triple product identity [2, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty. \tag{14}$$

By manipulating the q-products, one can easily arrive at the following representations:

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \ \psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

$$\chi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty}, \ (-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty}. \tag{15}$$

We now recall two definitions from [9, p. 225]. Let Π represent a pentagonal number (a number of the form $\frac{3n^2+n}{2}$) and Ω represent an octagonal number (a number of the form $3n^2+2n$). Let $\Pi(q) = \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}}$ and $\Omega(q) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n}$. Then,

$$\Pi(q) = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^3}{(q;q)_\infty (q^6;q^6)_\infty}, \tag{16}$$

and

$$\Omega(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}$$

$$= \prod_{n\geq 1} (1-q^{6n-5})(1-q^{6n-1})(1-q^{6n}) = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}. \tag{17}$$

We recall the following properties of theta functions.

Lemma 2.1. *From* [2, p. 40, Entry 25(i) and (ii)], *we have*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{18}$$

Lemma 2.2. *From* [2, p. 40, Entry 25(v) and (vi)], *we have*

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2. \tag{19}$$

Lemma 2.3 ([2, p. 114, Entry 8(ii) and p. 49, Corollary (i)])**.**

$$\varphi(-q)^4 = 1 + 8\sum_{k=1}^{\infty} \frac{(-1)^k k q^k}{1+q^k}. \tag{20}$$

Lemma 2.4. *By binomial expansion, for any non negative integer k, we have*

$$\varphi(-q)^{2^n} \equiv 1 \pmod{2^{n+1}}, \tag{21}$$

or

$$(q;q)_\infty^{2^k} \equiv (q^2;q^2)_\infty^{2^{k-1}} \pmod{2^k}. \tag{22}$$

We need the following congruences:

Lemma 2.5.

$$\varphi(-q)^4 \equiv 3 - 2\varphi(-q)^2 \pmod{16}. \tag{23}$$

Proof. We have,

$$\varphi(q)^2 = \left(1 + 2\sum_{k=1}^{\infty} q^{k^2}\right)^2 \equiv 1 \pmod 4,$$

so,

$$\varphi(q)^2 - 1 \equiv 0 \pmod 4,$$

i.e.,

$$\varphi(q)^2 + 3 \equiv 0 \pmod 4.$$

Multiplying these above two equations we get,

$$\varphi(q)^4 + 2\varphi(q)^2 \equiv 3 \pmod{16}.$$

Now substituting $-q$ for q, we obtain the required result. □

Recall these 3-dissection formulas [9, p. 225] and the following lemma.

Lemma 2.6.

$$\varphi(q) = \varphi(q^9) + 2q\Omega(q^3), \tag{24}$$

$$\psi(q) = \Pi(q^3) + 2q\psi(q^9). \tag{25}$$

We also require the following p-dissections for some theta functions.

Lemma 2.7 ([2, p. 49])**.** For any prime p,

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}). \tag{26}$$

Lemma 2.8 (Cui and Gu [8, Theorem 2.1]). *If p is an odd prime, then*

$$\psi(q)=q^{\frac{p^2-1}{8}}\psi(q^{p^2})+\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}}). \tag{27}$$

Furthermore, for $0\leq k\leq \frac{p-3}{2}$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod{p}.$$

Lemma 2.9 ([1, Ahmed and Baruah]). *If $p\geq 3$ is a prime, then*

$$\begin{aligned}(q;q)_\infty^3&=p(-1)^{\frac{p-1}{2}}q^{\frac{p^2-1}{8}}(q^{p^2};q^{p^2})_\infty^3\\&\quad+\sum_{\substack{k=0\\k\neq\frac{p-1}{2}}}^{p-1}(-1)^kq^{\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}}.\end{aligned} \tag{28}$$

Furthermore, if $k\neq\frac{p-1}{2}$, $0\leq k\leq p-1$, then

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod{p}.$$

3. Proofs of Theorems 1.1 and 1.2

In this section, we first prove the following two lemmas from which Theorem 1.1 readily follows.

Lemma 3.1. *We have*

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+1)q^n=4\frac{\varphi(q)^3\psi(q^2)}{\varphi(-q)^4}, \tag{29}$$

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+2)q^n=8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4}, \tag{30}$$

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+3)q^n=16\frac{\varphi(q)\psi(q^2)^3}{\varphi(-q)^4}. \tag{31}$$

Proof. From (1) and using (18) and (19), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_o(n)q^n&=\frac{\varphi(q)}{\varphi(-q)}=\frac{\varphi(q)^2}{\varphi(-q^2)^2}=\frac{\varphi(q)^2\varphi(q^2)^2}{\varphi(-q^4)^4}\\&=\frac{(\varphi(q^4)+2q\psi(q^8))^2(\varphi(q^4)^2+4q^2\psi(q^8)^2)}{\varphi(-q^4)^4}\\&=\frac{\varphi(q^4)^4+4q\varphi(q^4)^3\psi(q^8)+8q^2\varphi(q^4)^2\psi(q^8)^2+16q^3\varphi(q^4)\psi(q^8)^3+16q^4\psi(q^8)^4}{\varphi(-q^4)^4}.\end{aligned}$$

Extracting the terms containing q^{4n+r} for $r=1,2,3$, we complete the proof. □

Lemma 3.2. *Let p be an odd prime such that $p \equiv 3 \pmod 4$. Then for $n, \alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{pp}_o\left(8p^{2\alpha} \cdot n + p^{2\alpha}\right) q^n \equiv 4(q;q)_\infty^3 \varphi(q^2) \pmod{32}, \tag{32}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o\left(8p^{2\alpha} \cdot n + 5p^{2\alpha}\right) q^n \equiv 56(q;q)_\infty^3 \psi(q^4) \pmod{64}. \tag{33}$$

Proof. From (29), (15) and using (18) and (21), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(4n+1)q^n &= 4\frac{(q^2;q^2)_\infty^{18}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^4} \\ &= 4\frac{(q;q)_\infty^2(q^2;q^2)_\infty^{10}}{(q^4;q^4)_\infty^4}\frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^{16}} \\ &= 4\frac{(q;q)_\infty^2(q^2;q^2)_\infty^{10}}{(q^4;q^4)_\infty^4}\frac{1}{\varphi(-q)^8} \\ &\equiv 4\frac{(q;q)_\infty^2(q^2;q^2)_\infty^{10}}{(q^4;q^4)_\infty^4} \pmod{64} \\ &\equiv 4\frac{(q^2;q^2)_\infty^{11}}{(q^4;q^4)_\infty^4}\varphi(-q) \pmod{64} \\ &\equiv 4\frac{(q^2;q^2)_\infty^{11}}{(q^4;q^4)_\infty^4}\left(\varphi(q^4) - 2q\psi(q^8)\right) \pmod{64}.\end{aligned}$$

Extracting the terms containing q^{2n} and q^{2n+1} from the above congruence and using (21), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+1)q^n \equiv 4(q;q)_\infty^3 \varphi(q^2) \pmod{32}, \tag{34}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+5)q^n \equiv 56(q;q)_\infty^3 \psi(q^4) \pmod{64}, \tag{35}$$

which are the cases $\alpha = 0$ in (32) and (33), respectively.

Using the p-dissections of $\varphi(q)$ and $(q;q)_\infty^3$ from (26) and (28) in (34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(8n+1)q^n \equiv 4\Big\{ &\sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn \cdot \frac{pn+2k+1}{2}} \\ &+ p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3 \Big\} \\ &\times \left\{ \varphi(q^{2p^2}) + \sum_{r=1}^{p-1} q^{2r^2} f(q^{2p(p-2r)}, q^{2p(p+2r)}) \right\} \pmod{32}.\end{aligned} \tag{36}$$

Now consider the congruence

$$\frac{k(k+1)}{2} + 2r^2 \equiv \frac{p^2-1}{8} \pmod{p},$$

which is equivalent to

$$(2k+1)^2 + 16r^2 \equiv 0 \pmod{p}.$$

If $p \equiv 1 \pmod{4}$, then the only solution of the above congruence is $k = \frac{p-1}{2}$ and $r = 0$. Hence extracting the terms containing $q^{pn+\frac{p^2-1}{8}}$ from (36), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}_o\left(8\left(pn + \frac{p^2-1}{8}\right)+1\right)q^n \equiv 4(q^p;q^p)_\infty^3\varphi(q^{2p}) \pmod{32}. \tag{37}$$

Again, extracting the terms containing q^{pn} from the above congruence, we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o\left(p(8n+p)\right)q^n \equiv 4(q;q)_\infty^3\varphi(q^2) \pmod{32}. \tag{38}$$

Now using mathematical induction, we can easily arrive at (32). Proceeding similarly as shown in the proof of (32), we readily arrive at (33). □

Proof of Theorem 1.1. With the aid of (32), and the p-dissections of $\varphi(q)$ and $(q;q)_\infty^3$ from (26) and (28), respectively, we arrive at (2). Also using (33), and the p-dissections of $\psi(q)$ and $(q;q)_\infty^3$ from (27) and (28), respectively, we readily obtain (3). □

To prove Theorem 1.2, we first establish the following lemma.

Lemma 3.3. *Let p be an odd prime such that $p \equiv 3 \pmod{4}$. Then for any non-negative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{pp}_o\left(8p^{2\alpha}\cdot n + 2p^{2\alpha}\right) \equiv 8(q;q)_\infty^3\psi(q) \pmod{64}. \tag{39}$$

Proof. From (30), we have

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n = 8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4}. \tag{40}$$

Applying (21), (40) yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n \equiv 8\varphi(q)^2\psi(q^2)^2 \pmod{64}. \tag{41}$$

Since, [2, p. 40, Entry 25(v) and (vi)],

$$\varphi^2(q) = \varphi(q^2)^2 + 4q\psi(q^4)^2, \tag{42}$$

extracting the terms containing q^{2n} from (41), we obtain

$$\sum_{n=0}^{\infty}\overline{pp}_o(8n+2)q^n \equiv 8\psi^2(q)\varphi^2(q) \pmod{64}. \tag{43}$$

Now if we apply (15) and (22), then (43) yields

$$\sum_{n=0}^{\infty}\overline{pp}_o(8n+2)q^n \equiv 8(q;q)_\infty^3\psi(q) \pmod{64},$$

which is the case $\alpha = 0$ in (39). Using the p-dissections of $\psi(q)$ and $(q;q)_\infty^3$ from (27) and (28), respectively, and then proceeding similarly as shown in the proof of (32), we obtain the required result. □

Proof of Theorem 1.2. Employing the p-dissection of $(q;q)_\infty^3$ and $\psi(q)$ from (28) and (27), respectively in (39), we easily arrive at (4). □

4. Proofs of Theorems 1.3, 1.4 and 1.5

Lemma 4.1. *Let p be an odd prime such that $p \equiv 5 \pmod 8$ or $p \equiv 7 \pmod 8$. Then for any non-negative integers n and α, we have*

$$\sum_{n=0}^{\infty}\overline{pp}_o\left(16p^{2\alpha}\cdot n + 6p^{2\alpha}\right) \equiv 96(q^2;q^2)_\infty^3\psi(-q) \pmod{128}. \tag{44}$$

Proof. From (30), we have

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+2)q^n = 8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4}.$$

Employing (20), the above equation can be rewritten as

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+2)q^n = 8\frac{\varphi(q)^2\psi(q^2)^2}{1-x}, \tag{45}$$

where, $x = 8\sum_{k=1}^{\infty}\frac{(-1)^{k+1}kq^k}{1+q^k}$, i.e., $\varphi^4(-q) = 1-x$. Now expanding the term $\frac{1}{1-x}$, we obtain

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+2)q^n \equiv 8\varphi(q)^2\psi(q^2)^2(2-\varphi(-q)^4) \pmod{128}.$$

Since $\varphi(q)\varphi(-q) = \varphi(-q^2)^2$, the above congruence reduces to

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+2)q^n \equiv 16\psi(q^2)^2\varphi(q)^2 - 8\psi(q^2)^2\varphi(-q^2)^4\varphi(-q)^2 \pmod{128}. \tag{46}$$

Employing (42) and then extracting the terms containing q^{2n+1} from (46), we find that

$$\sum_{n=0}^{\infty}\overline{pp}_o(8n+6)q^n \equiv 64\psi(q)^2\psi(q^2)^2 + 32\psi(q)^2\varphi(-q)^4\psi(q^2)^2 \pmod{128}. \tag{47}$$

Using (15) and (22), (47) yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 64\psi(q^2)^3 + 32\frac{(q^4;q^4)_\infty^4(q^2;q^2)_\infty}{\varphi(-q)} \pmod{128}.$$

Now using (11) for $\varphi(-q)$ and then expanding the series as shown in (45), the above congruence reduces to

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 64\psi(q^2)^3 + 32(q^4;q^4)_\infty^4(q^2;q^2)_\infty(2-\varphi(-q)) \pmod{128}.$$

Employing (15) and (22) in the above congruence, we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 64\psi(q^2)^3 + 64\psi(q^2)^3 - 32(q^4;q^4)_\infty^4(q^2;q^2)_\infty\varphi(-q) \pmod{128},$$

which is equivalent to

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 96(q^4;q^4)_\infty^4(q^2;q^2)_\infty\varphi(-q) \pmod{128}. \tag{48}$$

Employing (18) with q replaced by $-q$, and then extracting the terms containing q^{2n} from (48), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+6)q^n \equiv 96(q^2;q^2)_\infty^4(q;q)_\infty\varphi(q^2) \pmod{128}.$$

Again, by using (15) and (22), the above congruence reduces to

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+6)q^n \equiv 96(q^2;q^2)_\infty^3\psi(-q) \pmod{128},$$

which is the case $\alpha = 0$ in (44). With the aid of p-dissections from (27) and (28), and using mathematical induction we easily arrive at (44). □

Proof of Theorem 1.3. Employing (27) and (28) in (44), we readily obtain (5). □

Proof of Theorem 1.4. From (30), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n &= 8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4} = 8\frac{\varphi(q)^6\psi(q^2)^2}{\varphi(-q^2)^8} \\ &= 8\frac{\psi(q^2)^2(\varphi(q^2)^2+4q\psi(q^4)^2)^3}{\varphi(-q^2)^8} \\ &= 8\frac{\psi(q^2)^2(\varphi(q^2)^6+12q\varphi(q^2)^4\psi(q^4)^2+48q^2\varphi(q^2)^2\psi(q^4)^4+64q^3\psi(q^4)^6)}{\varphi(-q^2)^8}.\end{aligned} \tag{49}$$

Now extracting the terms containing q^{2n+1} and using (21), it follows that, modulo 128,

$$\sum_{n=0}^{\infty}\overline{pp}_o(8n+6)q^n = \frac{96\varphi(q)^4\psi(q)^2\psi(q^2)^2+512q\psi(q)^2\psi(q^2)^6}{\varphi(-q)^8}$$

$$\equiv \frac{96\varphi(q)^2\psi(q)^6}{\varphi(-q)^8}$$

$$\equiv 96\psi(q)^6 \tag{50}$$

$$= 96\sum_{n=0}^{\infty}t_6(n)q^n, \tag{51}$$

where $t_k(n)$ denotes the number of representations of n as a sum of k triangular numbers.

From Xia [15], we have, for $n,\ \alpha \geq 0$,

$$t_6\left(\frac{3(p^\alpha-1)}{4}\right) = \frac{p^{2\alpha+2}-1}{p^2-1}, \tag{52}$$

where $p \equiv 1 \pmod 4$.

It is easy to check that

$$\frac{p^{8\alpha+8}-1}{p^2-1} = \sum_{i=0}^{4(\alpha+1)-1} p^{2i} \equiv \sum_{i=0}^{4(\alpha+1)-1} 1 \equiv 0 \pmod 4.$$

Hence, by replacing α by $4\alpha+3$ in (52) and using the above result, we find that

$$t_6\left(\frac{3(p^{4\alpha+3}-1)}{4}\right) \equiv 0 \pmod 4.$$

Now employing the above in (51), we complete the proof. □

Proof of Theorem 1.5. From (50), we have, modulo 128

$$\sum_{n=0}^{\infty}\overline{pp}_o(8n+6)q^n \equiv 96\psi(q)^6$$

$$\equiv 96\varphi(q)^3\psi(q^2)^3$$

$$\equiv 96\psi(q^2)^3(\varphi(q^4)+2q\psi(q^8))^3.$$

Extracting the terms containing q^{2n+1}, it follows that, modulo 128,

$$\sum_{n=0}^{\infty}\overline{pp}_o(16n+14)q^n \equiv 96\psi(q)^3(6\varphi(q^2)^2\psi(q^4)+8q\psi(q^2)^3)$$

$$\equiv 64\psi(q)^3\psi(q^4)\varphi(q^2)^2$$

$$\equiv 64\psi(q)^3\psi(q^4)$$

$$\equiv 64\psi(q)^7$$

$$\equiv 64\sum_{n=0}^{\infty} t_7(n)q^n. \tag{53}$$

From Yao [16], we have

$$t_7\left(p^{2\alpha}\cdot n+\frac{7(p^{2\alpha}-1)}{8}\right)=\lambda_2(p,\alpha,8n+7)t_7(n)-\frac{p^5(p^{5\alpha}-1)}{p^5-1}t_7\left(\frac{n-\frac{7(p^2-1)}{8}}{p^2}\right),$$

where p is an odd prime, $\alpha \geq 0$ and

$$\lambda_2(p,\alpha,n)=\frac{p^{5\alpha+5}-p^{5\alpha+2}\left(\frac{-n}{p}\right)+p^2\left(\frac{-n}{p}\right)-1}{p^5-1}.$$

Replacing n by $pn+\frac{7(p^2-1)}{8}$ and employing the above for λ_2, we obtain

$$\begin{aligned}t_7\left(p^{2\alpha+1}\cdot n+\frac{7(p^{2\alpha+2}-1)}{8}\right) &= \frac{p^{5\alpha+5}-1}{p^5-1}t_7\left(pn+\frac{7(p^2-1)}{8}\right)\\ &\quad -\frac{p^5(p^{5\alpha}-1)}{p^5-1}t_7\left(\frac{n}{p}\right).\end{aligned}$$

It is easy to show that for any odd prime p and $\alpha \geq 0$,

$$\frac{p^{10(\alpha+1)}-1}{p^5-1}=\sum_{i=0}^{10(\alpha+1)-1} p^{5i}\equiv\sum_{i=0}^{10(\alpha+1)-1} 1\equiv 0 \pmod 2.$$

Now replacing α by $2\alpha+1$, we find that

$$t_7\left(p^{4\alpha+3}\cdot n+\frac{7(p^{4\alpha+4}-1)}{8}\right)\equiv-\frac{p^5(p^{10\alpha+5}-1)}{p^5-1}t_7\left(\frac{n}{p}\right) \pmod 2.$$

If $p \nmid n$, then $t_7\left(\frac{n}{p}\right)=0$. Hence from (53) and the above we easily arrive at (6). □

5. Proofs of Theorems 1.6, 1.7 and 1.8

Proof of Theorem 1.6. Extracting the terms containing q^{2n} from (49), and using (21), (24) and (25), it follows, modulo 128,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_o(8n+2)q^n &= \frac{8\psi(q)^2\varphi(q)^6+384q\psi(q)^2\psi(q^2)^4\varphi(q)^2}{\varphi(-q)^8}\\ &\equiv 8\frac{\psi(q)^2\varphi(q)^6}{\varphi(-q)^8}\\ &\equiv 8\psi(q)^2\varphi(q)^6\\ &= 8\left(\Pi(q^3)+q\psi(q^9)\right)^2\left(\varphi(q^9)+2q\Omega(q^3)\right)^6\\ &= 8\left(\Pi(q^3)^2+2q\Pi(q^3)\psi(q^9)+q^2\psi(q^9)^2\right)(\varphi(q^9)^6\end{aligned}$$

$$+ 12q\varphi(q^9)^5\Omega(q^3) + 60q^2\varphi(q^9)^4\Omega(q^3)^2 + 160q^3\varphi(q^9)^3\Omega(q^3)^3 + 240q^4\varphi(q^9)^2\Omega(q^3)^4 + 192q^5\varphi(q^9)\Omega(q^3)^5 + 64q^6\Omega(q^3)^6).$$

Extracting the terms containing q^{3n+2}, we have, modulo 128,

$$\sum_{n=0}^{\infty}\overline{pp}_o(24n+18)q^n \equiv 96\Pi(q)^2\varphi(q^3)^4\Omega(q)^2 + 64\Pi(q)\varphi(q^3)^5\psi(q^3)\Omega(q) + 8\varphi(q^3)^6\psi(q^3)^2. \tag{54}$$

Now,

$$\begin{aligned}\Pi(q)\Omega(q) &= \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty}\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}\\ &= \frac{(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2(q^4;q^4)_\infty}\\ &= \frac{(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty}\frac{\varphi(-q^2)}{\varphi(-q)}\\ &\equiv \frac{(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty} \pmod 2, \end{aligned} \tag{55}$$

and

$$\Pi(q)^2\Omega(q)^2 \equiv \frac{(q^3;q^3)_\infty^6(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^2} \pmod 4. \tag{56}$$

From (54) and using (55) and (56), it follows, modulo 128,

$$\sum_{n=0}^{\infty}\overline{pp}_o(24n+18)q^n \equiv 96\varphi(q^3)^4\frac{(q^3;q^3)_\infty^6(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^2} + 64\varphi(q^3)^5\psi(q^3)\frac{(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty} + 8\varphi(q^3)^6\psi(q^3)^2.$$

Hence it follows that,

$$\sum_{n=0}^{\infty}\overline{pp}_o(72n+42)q^n \equiv 0 \pmod{128},$$

and

$$\sum_{n=0}^{\infty}\overline{pp}_o(72n+66)q^n \equiv 0 \pmod{128}.$$

□

Proof of Theorem 1.7. From (31), we have

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n+3)q^n = 16\frac{\varphi(q)\psi(q^2)^3}{\varphi(-q)^4}. \tag{57}$$

From (21), we have $\varphi^4(-q) \equiv 1 \pmod 8$. Therefore, (57) yields, modulo 128,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(4n+3)q^n &\equiv 16\varphi(q)\psi(q^2)^3 \\ &= 16\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^3} \\ &= 16\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2}\frac{(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^2} \\ &= 16\left(\psi(q)\psi(q^2)\right)^2. \qquad (58)\end{aligned}$$

From (47) and (21), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 96\left(\psi(q)\psi(q^2)\right)^2 \pmod{128}. \qquad (59)$$

Now, from (58) and (59), we readily arrive at (9). □

We now prove a lemma which will be used to prove Theorem 1.8.

Lemma 5.1.

$$\frac{1}{\chi(-q)^3} = \frac{1}{\chi(-q^3)} + 3q\frac{\psi(q^9)}{\varphi(-q)}. \qquad (60)$$

Proof. Using (15) and (25), we have

$$\begin{aligned}\frac{1}{\chi(-q)^3} &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3} = \frac{\psi(q)}{\varphi(-q)} = \frac{\Pi(q^3)+q\psi(q^9)}{\varphi(-q)} \\ &= \frac{(\Pi(q^3)-2q\psi(q^9))+3q\psi(q^9)}{\varphi(-q)} \\ &= \frac{\left(\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} - 2q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}\right)+3q\psi(q^9)}{\varphi(-q)} \\ &= \frac{\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty}\left(\frac{(q^9;q^9)_\infty^2}{(q^{18};q^{18})_\infty} - 2q\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2}{(q^6;q^6)_\infty(q^9;q^9)_\infty}\right)+3q\psi(q^9)}{\varphi(-q)} \\ &= \frac{1}{\chi(-q^3)}\frac{\varphi(-q^9)-2q\Omega(-q^3)}{\varphi(-q)} + 3q\frac{\psi(q^9)}{\varphi(-q)} = \frac{1}{\chi(-q^3)} + 3q\frac{\psi(q^9)}{\varphi(-q)}.\end{aligned}$$ □

Proof of Theorem 1.8. From (30) and using (60), we have, modulo 72,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n &= 8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4} = 8\frac{\psi(q)^4}{\varphi(-q)^4} \\ &= 8\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}} = 8\left(\frac{1}{\chi(-q)^3}\right)^4\end{aligned}$$

$$= 8\left(\frac{1}{\chi(-q^3)} + 3q\frac{\psi(q^9)}{\varphi(-q)}\right)^4$$
$$\equiv \frac{8}{\chi(-q^3)^4} + 24q\frac{\psi(q^9)}{\chi(-q^3)^3\varphi(-q)}$$
$$= \frac{8}{\chi(-q^3)^4} + 24q\frac{\psi(q^9)}{\chi(-q^3)^3}\frac{\varphi(-q^9)}{\varphi(-q^3)^4}$$
$$\times\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right).$$

Now extracting the terms containing q^{3n+1}, we obtain, modulo 72,

$$\sum_{n=0}^{\infty}\overline{pp}_o(12n+6)q^n \equiv 24\frac{\psi(q^3)\varphi(-q^3)^3}{\chi(-q)^3\varphi(-q)^4}$$
$$= 24\frac{(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty}\frac{(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^3}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3}\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^8}$$
$$= 24\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty^5}{(q;q)_\infty^{11}(q^6;q^6)_\infty}$$
$$= 24(q;q)_\infty(q^2;q^2)_\infty\frac{((q^2;q^2)_\infty^3)^2}{((q;q)_\infty^3)^4}\frac{(q^3;q^3)_\infty^5}{(q^6;q^6)_\infty}$$
$$\equiv 24(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty. \tag{61}$$

Recall the 3-dissection formulas deduce by M. D. Hirschhorn [9, 14.3.1] and also by Z. Cao [5].

$$(q;q)_\infty(q^2;q^2)_\infty = (q^9;q^9)_\infty(q^{18};q^{18})_\infty\left(X(q^3)^{-1} - q - 2q^2X(q^3)\right), \tag{62}$$

where

$$X(q) = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}.$$

Now from (61) and (62), we have, modulo 72

$$\sum_{n=0}^{\infty}\overline{pp}_o(12n+6)q^n$$
$$\equiv 24(q^3;q^3)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{18};q^{18})_\infty\left(X(q^3)^{-1} - q - 2q^2X(q^3)\right).$$

Extracting the terms containing q^{3n+1}, we obtain, modulo 72

$$\sum_{n=0}^{\infty}\overline{pp}_o(36n+18)q^n \equiv -24(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty. \tag{63}$$

Therefore by (62), (63), and using induction on α, for $\alpha \geq 0$, we obtain the required result.

This completes the proof. □

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