J. Korean Math. Soc. **57** (2020), No. 2, pp. 451–470
https://doi.org/10.4134/JKMS.j190133
pISSN: 0304-9914 / eISSN: 2234-3008

AN EVALUATION FORMULA FOR A GENERALIZED CONDITIONAL EXPECTATION WITH TRANSLATION THEOREMS OVER PATHS

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Abstract. Let $C[0,T]$ denote an analogue of Wiener space, the space of real-valued continuous functions on the interval $[0,T]$. For a partition $0=t_0<t_1<\cdots<t_n<t_{n+1}=T$ of $[0,T]$, define $X_n : C[0,T]\to\mathbb{R}^{n+1}$ by $X_n(x)=(x(t_0),x(t_1),\ldots,x(t_n))$. In this paper we derive a simple evaluation formula for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ with the conditioning function X_n which has a drift and does not contain the present position of paths. As applications of the formula with X_n, we evaluate the Radon-Nikodym derivatives of the functions $\int_0^T[x(t)]^m d\lambda(t)(m\in\mathbb{N})$ and $[\int_0^T x(t)d\lambda(t)]^2$ on $C[0,T]$, where λ is a complex-valued Borel measure on $[0,T]$. Finally we derive two translation theorems for the Radon-Nikodym derivatives of the functions on $C[0,T]$.

1. Introduction and an analogue of the Wiener space

Let $C_0[0,T]$ denote the Wiener space, the space of real-valued continuous functions x on the interval $[0,T]$ with $x(0)=0$. Calculations involving the conditional Wiener integrals of the functions on $C_0[0,T]$ are important in the study of Feynman integral. In particular, when $0=t_0<t_1<\cdots<t_n<t_{n+1}=T$ is a partition of $[0,T]$ and $\xi_j\in\mathbb{R}$ for $j=0,1,\ldots,n$, the conditional Wiener integral of a time integral in which the paths pass through the point ξ_j at each time t_j for $j=0,1,\ldots,n$, where t_j is not the present time T, is very useful in the Feynman integration theory. But, in general, the Wiener integral and the conditional Wiener integral on $C_0[0,T]$ is not invariant under translation [1, 11]. In [7], Park and Skoug derived a simple formula for conditional Wiener integrals containing the time integral with the conditioning function $(x(t_1),\ldots,x(t_n),x(t_{n+1}))$ for $x\in C_0[0,T]$ which contains the present positions of the paths in $C_0[0,T]$. In their simple formula, they expressed the

Received February 11, 2019; Accepted May 16, 2019.
2010 *Mathematics Subject Classification.* 28C20, 60G05, 60G15.
Key words and phrases. Analogue of Wiener measure, conditional Wiener integral, Feynman integral, translation theorem, Wiener integral, Wiener space.
This work was supported by Kyonggi University Research Grant 2018.

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conditional Wiener integral directly in terms of ordinary Wiener integral and derived a translation theorem for the conditional Wiener integrals on $C_0[0,T]$.

More generally, let $C[0,T]$ denote the space of continuous real-valued functions on $[0,T]$. Im and Ryu [6, 9] introduced a finite positive measure w_φ on $C[0,T]$ which generalizes the Wiener space $C_0[0,T]$. When w_φ is a probability measure, the author [2] and Ryu [8] derived separately the same simple formula for a generalized conditional Wiener integral of the functions on $C[0,T]$ with the conditioning function $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$ for $x \in C[0,T]$. They then evaluated the generalized conditional Wiener integrals of various functions which are interested in both Feynman integral and quantum mechanics. In particular, Ryu [10] derived a translation theorem for the generalized analogue of Wiener integral and established properties of the generalized analogue of Wiener measure from it. Furthermore, the author [3] derived another simple formula for the generalized conditional Wiener integrals with the conditioning function $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ for $x \in C[0,T]$ and he [2, 3] derived translation theorems for the generalized conditional Wiener integrals on $C[0,T]$ with two conditioning functions X_n and X_{n+1}. Note that X_{n+1} contains the present positions of paths in $C[0,T]$ and X_n does not. In both cases, the motion in the formulas has the mean zero with the variance function $\beta(t) = t$ on $[0,T]$, and it has no drifts. In addition, the author [5] derived a simple evaluation formula for Radon-Nikodym derivatives similar to the generalized conditional Wiener integrals of functions on $C[0,T]$ with the conditioning function X_{n+1} which has a drift and an initial weight. Using the formula, he evaluated various Radon-Nikodym derivatives of the functions on $C[0,T]$ containing the time integral.

In this paper, using the formula with X_{n+1} in [5], we derive another simple evaluation formula for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ with the conditioning function X_n which has a drift and does not contain the present position of paths. As applications of the formula with X_n, we evaluate the Radon-Nikodym derivatives of the functions given by $\int_0^T [x(t)]^m d\lambda(t) (m \in \mathbb{N})$ and $[\int_0^T x(t) d\lambda(t)]^2$ on $C[0,T]$, where λ is a complex-valued Borel measure on $[0,T]$. Finally we derive two translation theorems for the Radon-Nikodym derivatives of the functions on $C[0,T]$ with the conditioning functions X_n and X_{n+1}.

We now introduce a finite measure over paths with its properties. Let $\alpha, \beta : [0,T] \to \mathbb{R}$ be two functions, where β is continuous and strictly increasing. Let φ be a positive finite measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ and m_L be the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. For $\vec{t_n} = (t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le T$, let $J_{\vec{t_n}} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t_n}}(x) = (x(t_0), x(t_1), \dots, x(t_n))$. For $\prod_{j=0}^n B_j \in \mathcal{B}(\mathbb{R}^{n+1})$, the subset $J_{\vec{t_n}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called a cylinder set I and let $\mathcal{I}$ be the set of all such cylinder sets I.

Define a pre-measure $m_{\alpha,\beta;\varphi}$ on $\mathcal{I}$ by

$$m_{\alpha,\beta;\varphi}(I) = \int_{B_0}\int_{\prod_{j=1}^n B_j} W_n(\vec{t}_n, \vec{u}_n, u_0)\,dm_L^n(\vec{u}_n)\,d\varphi(u_0),$$

where for $\vec{u}_n = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}$,

$$(1)\qquad W_n(\vec{t}_n, \vec{u}_n, u_0) = \left[\frac{1}{\prod_{j=1}^n 2\pi[\beta(t_j) - \beta(t_{j-1})]}\right]^{\frac{1}{2}} \times \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{[u_j - \alpha(t_j) - u_{j-1} + \alpha(t_{j-1})]^2}{\beta(t_j) - \beta(t_{j-1})}\right\}.$$

Let $\mathcal{B}(C[0,T])$ denote the Borel σ-algebra of $C[0,T]$ with the supremum norm. Then $\mathcal{B}(C[0,T])$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for all $I \in \mathcal{I}$. This measure $w_{\alpha,\beta;\varphi}$ is called an analogue of a generalized Wiener measure on $(C[0,T], \mathcal{B}(C[0,T]))$ according to φ [9,10]. From the definition of $w_{\alpha,\beta;\varphi}$, we have the following theorem which is useful in the next sections [6].

Theorem 1.1. *If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then*

$$\int_{C[0,T]} f(x(t_0), x(t_1), \ldots, x(t_n))\,dw_{\alpha,\beta;\varphi}(x) \stackrel{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \ldots, u_n) W_n(\vec{t}_n, \vec{u}_n, u_0)\,dm_L^n(\vec{u}_n)\,d\varphi(u_0),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let m be a positive integer, let $X : C[0,T] \to \mathbb{R}^m$ be Borel measurable, let $F : C[0,T] \to \mathbb{C}$ be integrable and let m_X be the measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^m)$ of $\mathbb{R}^m$ induced by X. Let $\mathcal{D} = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^m)\}$ and let $w_{\mathcal{D}}(E) = w_{\alpha,\beta;\varphi}(E)$ for each $E \in \mathcal{D}$. In view of the Radon-Nikodym theorem, there exists a $\mathcal{D}$-measurable and $w_{\mathcal{D}}$-integrable function Ψ on $C[0,T]$ which is unique up to $w_{\mathcal{D}}$ a.e. such that the relation $\int_E \Psi(x)dw_{\mathcal{D}}(x) = \int_E F(x)dw_{\alpha,\beta;\varphi}(x)$ holds for every $E \in \mathcal{D}$. Moreover, there exists a Borel measurable and m_X-integrable function ψ on $\mathbb{R}^m$ which is unique up to m_X a.e. such that $\Psi(x) = (\psi \circ X)(x)$ for $w_{\mathcal{D}}$ a.e. x in $C[0,T]$. ψ is called a generalized conditional expectation of F given X and it is denoted by $GE[F|X]$. We note that if φ is a probability measure on $\mathbb{R}$, then m_X is also a probability measure on $\mathbb{R}^m$, so that $GE[F|X]$ is in fact the conditional expectation of F given X.

2. A simple formula for the generalized conditional expectation

In this section, we derive a simple evaluation formula for the generalized conditional expectations of functions on $C[0,T]$ with an appropriate conditioning function.

Throughout the remainder of this paper, we assume that $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ is an arbitrary fixed partition of $[0,T]$ unless otherwise specified. To derive the desired simple evaluation formula for a generalized conditional expectation, we begin with letting for $t \in [0,T]$

$$\gamma_{1j}(t) = \frac{\beta(t_j) - \beta(t)}{\beta(t_j) - \beta(t_{j-1})} \text{ and } \gamma_{2j}(t) = \frac{\beta(t) - \beta(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})}. \tag{2}$$

For a function $f : [0,T] \to \mathbb{R}$, define the polygonal function $P_\beta^{n+1}(f)$ of f by

$$\begin{aligned} &P_\beta^{n+1}(f)(t) \\ &= \sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(t)[f(t_{j-1}) + \gamma_{2j}(t)[f(t_j) - f(t_{j-1})]] + \chi_{\{0\}}(t)f(0) \end{aligned} \tag{3}$$

for $t \in [0,T]$, where χ denotes the characteristic function. Similarly, for $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$, the polygonal function $P_\beta^{n+1}(\vec{\eta}_{n+1})$ of $\vec{\eta}_{n+1}$ on $[0,T]$ is defined by (3) with replacing $f(t_j)$ by η_j. Then both $P_\beta^{n+1}(f)$ and $P_\beta^{n+1}(\vec{\eta}_{n+1})$ belong to $C[0,T]$, and $P_\beta^{n+1}(f)(t_j) = f(t_j)$, $P_\beta^{n+1}(\vec{\eta}_{n+1})(t_j) = \eta_j$ at each t_j. For $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, the polygonal function $P_\beta^n(\vec{\eta}_n)$ of $\vec{\eta}_n$ on $[0,t_n]$ is defined by (3) with replacing $f(t_j)$ by η_j for $j = 0, 1, \ldots, n$.

For $s_1, s_2 \in [0,T]$, let

$$\Gamma_j(s_1,s_2) = \gamma_{1j}(s_1)\gamma_{2j}(s_2)[\beta(t_j) - \beta(t_{j-1})]. \tag{4}$$

For $t \in [0,T]$, let

$$\Gamma(t) = \sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(t)\Gamma_j(t,t) \tag{5}$$

and let $Z_t(f) = f(t) - P_\beta^{n+1}(f)(t)$ for a function $f : [0,T] \to \mathbb{R}$.

We now have the following two theorems from [5].

Theorem 2.1. *Suppose that Z_t is defined on $C[0,T]$ and let $X_{n+1} : C[0,T] \to \mathbb{R}^{n+2}$ be given by*

$$X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1})). \tag{6}$$

Then the process $\{Z_t : 0 \le t \le T\}$ and X_{n+1} are independent if $\varphi(\mathbb{R}) = 1$.

Theorem 2.2. *Let $F : C[0,T] \to \mathbb{C}$ be integrable and X_{n+1} be given by (6) of Theorem 2.1. Then we have for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$,*

$$GE[F|X_{n+1}](\vec{\eta}_{n+1}) = \frac{1}{\varphi(\mathbb{R})} \int_{C[0,T]} F(x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x),$$

where $m_{X_{n+1}}$ denotes the measure on $\mathcal{B}(\mathbb{R}^{n+2})$ induced by X_{n+1}.

Theorem 2.3. *Let $X_n : C[0,T] \to \mathbb{R}^{n+1}$ be given by*

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n))$$

and let $F : C[0,T] \to \mathbb{C}$ be integrable. Let m_{X_n} be the measure induced by X_n on $\mathcal{B}(\mathbb{R}^{n+1})$. Then we have for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$

$$(7)\qquad GE[F|X_n](\vec{\eta}_n) = \frac{1}{\varphi(\mathbb{R})}\int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1}) \int_{C[0,T]} F(x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x) dm_L(\eta_{n+1}),$$

where $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \dots, \eta_n, \eta_{n+1})$ and

$$W_T(\eta_n, \eta_{n+1}) = \left[\frac{1}{2\pi[\beta(T) - \beta(t_n)]}\right]^{\frac{1}{2}} \exp\left\{-\frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2[\beta(T) - \beta(t_n)]}\right\}.$$

Proof. Let $m_{X_{n+1}}$ be the measure as in Theorem 2.2. Then for any Borel subset B of $\mathbb{R}^{n+1}$, we have $X_n^{-1}(B) = X_{n+1}^{-1}(B \times \mathbb{R})$ so that we have by Theorem 2.2

$$\begin{aligned}\int_{X_n^{-1}(B)} F(x)dw_{\alpha,\beta;\varphi}(x) &= \int_{B\times\mathbb{R}} GE[F|X_{n+1}](\vec{\eta}_{n+1})dm_{X_{n+1}}(\vec{\eta}_{n+1}) \\ &= \frac{1}{\varphi(\mathbb{R})}\int_{B\times\mathbb{R}}\int_{C[0,T]} F(x - P_\beta^{n+1}(x) \\ &\quad + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x)dm_{X_{n+1}}(\vec{\eta}_{n+1}).\end{aligned}$$

By Theorem 1.1 and the Fubini's theorem, we have for $\vec{t}_n = (t_0, t_1, \dots, t_n)$

$$\begin{aligned}&\int_{X_n^{-1}(B)} F(x)dw_{\alpha,\beta;\varphi}(x) \\ &= \frac{1}{\varphi(\mathbb{R})}\int_{\mathbb{R}^{n+2}} \chi_{B\times\mathbb{R}}(\vec{\eta}_{n+1}) W_{n+1}((\vec{t}_n, t_{n+1}), (\eta_1, \dots, \eta_n, \eta_{n+1}), \eta_0)\int_{C[0,T]} \\ &\quad F(x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x)dm_L^{n+1}(\eta_1, \dots, \eta_n, \eta_{n+1})d\varphi(\eta_0) \\ &= \frac{1}{\varphi(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \chi_B(\vec{\eta}_n) W_n(\vec{t}_n, (\eta_1, \dots, \eta_n), \eta_0)\int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\int_{C[0,T]} F(x \\ &\quad - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x)dm_L(\eta_{n+1})dm_L^n(\eta_1, \dots, \eta_n)d\varphi(\eta_0) \\ &= \frac{1}{\varphi(\mathbb{R})}\int_B\int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\int_{C[0,T]} F(x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1})) \\ &\quad dw_{\alpha,\beta;\varphi}(x)dm_L(\eta_{n+1})dm_{X_n}(\vec{\eta}_n),\end{aligned}$$

where W_{n+1} is given by (1) with replacing n by $n+1$. Now, (7) follows by the definition of $GE[F|X_n]$. □

Remark 2.4. (a) Let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})}\varphi$. Let P_{X_n} be the probability distribution of X_n on $\mathbb{R}^{n+1}$ and let $GE_{\varphi_0}[F|X_n]$ denote the conditional expectation of F with respect to $w_{\alpha,\beta;\varphi_0}$. Since B is a P_{X_n} null-set if and only if B is an m_{X_n} null-set, (7) can be rewritten by

$$GE[F|X_n](\vec{\eta}_n) = \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\int_{C[0,T]} F(x - P_\beta^{n+1}(x)$$

$$+ P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi_0}(x)dm_L(\eta_{n+1})$$
$$= GE_{\varphi_0}[F|X_n](\vec{\eta}_n)$$

for P_{X_n} a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$ (or equivalently, for m_{X_n} a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$).

(b) Theorem 2.3 is an extension of Theorem 2.5 in [3].

3. Evaluations of the generalized conditional expectations

In this section, using Theorem 2.3, we evaluate the generalized conditional expectations of various functions which are useful in the Feynman integration theory.

The following theorem is needed to prove various results in this section [5].

Theorem 3.1. *Let $s_1 \in [t_{j-1}, t_j]$ and $s_2 \in [t_{k-1}, t_k]$ with $1 \le j \le n+1$ and $1 \le k \le n+1$. For $x \in C[0,T]$, let $G(x) = x(s_1)x(s_2)$. Suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$. Then G is $w_{\alpha,\beta;\varphi}$-integrable and we have the followings*:

(a) *If $j \ne k$, then for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$, we have*

$$GE[G|X_{n+1}](\vec{\eta}_{n+1}) = [Z_{s_1}(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1)][Z_{s_2}(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2)].$$

(b) *If $j = k$, then for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$, we have*

$$GE[G|X_{n+1}](\vec{\eta}_{n+1}) = [Z_{s_1}(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1)][Z_{s_2}(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2)] + \Gamma_j(s_1 \vee s_2, s_1 \wedge s_2),$$

where $s_1 \vee s_2 = \max\{s_1, s_2\}$, $s_1 \wedge s_2 = \min\{s_1, s_2\}$ and Γ_j is given by (4), so that $Cov(Z_{s_1}, Z_{s_2}) = \Gamma_j(s_1 \vee s_2, s_1 \wedge s_2)$ if $\varphi(\mathbb{R}) = 1$.

Theorem 3.2. *Under the assumptions as in Theorem 3.1, we have the followings*:

(a) *If $1 \le j \le n$ and $1 \le k \le n$ with $j \ne k$, then for m_{X_n} a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$GE[G|X_n](\vec{\eta}_n) = [Z_{s_1}(\alpha) + P_\beta^n(\vec{\eta}_n)(s_1)][Z_{s_2}(\alpha) + P_\beta^n(\vec{\eta}_n)(s_2)].$$

(b) *If $1 \le j \le n$ and $1 \le k \le n$ with $j = k$, then for m_{X_n} a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$GE[G|X_n](\vec{\eta}_n) = [Z_{s_1}(\alpha) + P_\beta^n(\vec{\eta}_n)(s_1)][Z_{s_2}(\alpha) + P_\beta^n(\vec{\eta}_n)(s_2)] + \Gamma_j(s_1 \vee s_2, s_1 \wedge s_2).$$

(c) *If $1 \le j \le n$ and $k = n+1$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$*

$$GE[G|X_n](\vec{\eta}_n) = [Z_{s_1}(\alpha) + P_\beta^n(\vec{\eta}_n)(s_1)][\alpha(s_2) - \alpha(t_n) + \eta_n].$$

(d) *If $j = n+1$ and $k = n+1$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$*

$$GE[G|X_n](\vec{\eta}_n) = [\alpha(s_1) - \alpha(t_n) + \eta_n][\alpha(s_2) - \alpha(t_n) + \eta_n] + \gamma_{2(n+1)}(s_1)[\beta(s_2) - \beta(t_n)] + \Gamma_j(s_1 \vee s_2, s_1 \wedge s_2),$$

where $\gamma_{2(n+1)}$ is given by (2) with $j = n+1$.

Proof. If $s_1 \in [t_{j-1}, t_j]$ and $s_2 \in [t_{k-1}, t_k]$ with $1 \le j \le n$ and $1 \le k \le n$, then we have $P_\beta^{n+1}(\vec{\eta}_{n+1})(s_l) = P_\beta^n(\vec{\eta}_n)(s_l)$ for $l = 1, 2$, where $\vec{\eta}_{n+1} = (\vec{\eta}_n, \eta_{n+1})$, so that we have (a) and (b) in this theorem by (7), (a) and (b) of Theorem 3.1. If $s_2 \in [t_n, T]$, then we have

$$\begin{aligned}&\int_{\mathbb{R}} P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2) W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1})\\
&= \left[\frac{1}{2\pi[\beta(T) - \beta(t_n)]}\right]^{\frac{1}{2}} \int_{\mathbb{R}} [\eta_n + \gamma_{2(n+1)}(s_2)(\eta_{n+1} - \eta_n)]\\
&\quad \times \exp\left\{-\frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2[\beta(T) - \beta(t_n)]}\right\} dm_L(\eta_{n+1})\\
&= \eta_n + \gamma_{2(n+1)}(s_2)[\alpha(T) - \alpha(t_n)] = \eta_n - \alpha(t_n) + P_\beta^{n+1}(\alpha)(s_2).\end{aligned}$$

Since $Z_{s_2}(\alpha) + \eta_n - \alpha(t_n) + P_\beta^{n+1}(\alpha)(s_2) = \alpha(s_2) - \alpha(t_n) + \eta_n$, we have (c) by Theorem 2.3 and (a) of Theorem 3.1. If $s_1, s_2 \in [t_n, T]$, then we have

$$\begin{aligned}&\int_{\mathbb{R}} P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1) P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2) W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1})\\
&= \left[\frac{1}{2\pi[\beta(T) - \beta(t_n)]}\right]^{\frac{1}{2}} \int_{\mathbb{R}} [\eta_n + \gamma_{2(n+1)}(s_1)(\eta_{n+1} - \eta_n)][\eta_n + \gamma_{2(n+1)}(s_2)\\
&\quad \times (\eta_{n+1} - \eta_n)] \exp\left\{-\frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2[\beta(T) - \beta(t_n)]}\right\} dm_L(\eta_{n+1})\\
&= \eta_n^2 + \eta_n[\alpha(T) - \alpha(t_n)][\gamma_{2(n+1)}(s_1) + \gamma_{2(n+1)}(s_2)] + \gamma_{2(n+1)}(s_1)\\
&\quad \times \gamma_{2(n+1)}(s_2)[[\alpha(T) - \alpha(t_n)]^2 + \beta(T) - \beta(t_n)]\\
&= [\eta_n + \gamma_{2(n+1)}(s_1)[\alpha(T) - \alpha(t_n)]][\eta_n + \gamma_{2(n+1)}(s_2)[\alpha(T) - \alpha(t_n)]]\\
&\quad + \gamma_{2(n+1)}(s_1)[\beta(s_2) - \beta(t_n)]\\
&= [\eta_n - \alpha(t_n) + P_\beta^{n+1}(\alpha)(s_1)][\eta_n - \alpha(t_n) + P_\beta^{n+1}(\alpha)(s_2)] + \gamma_{2(n+1)}(s_1)\\
&\quad \times [\beta(s_2) - \beta(t_n)]\end{aligned}$$

so that we have (d) by Theorem 2.3 and (b) of Theorem 3.1. □

Theorem 3.3. *For $x \in C[0, T]$, let $G_1(x) = [\int_0^T x(t) d\lambda(t)]^2$, where λ is a finite complex measure on the Borel class of $[0, T]$. Suppose that*

$$\int_0^T [\alpha(t)]^2 d|\lambda|(t) < \infty \text{ and } \int_{\mathbb{R}} u^2 d\varphi(u) < \infty. \tag{8}$$

Then G_1 is $w_{\alpha,\beta;\varphi}$-integrable and for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$, we have

$$\begin{aligned}GE[G_1|X_{n+1}](\vec{\eta}_{n+1}) = &\left[\int_0^T [Z_t(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(t)] d\lambda(t)\right]^2\\
&+ \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2),\end{aligned}$$

where $\Lambda(s,t) = \sum_{j=1}^{n+1}\chi_{[t_{j-1},t_j]^2}(s,t)\Gamma_j(s,t)$ *for* $(s,t)\in[0,T]^2$. *In particular, if the support of* λ *is contained in* $\{t_0,t_1,\dots,t_n,t_{n+1}\}$, *then for* $m_{X_{n+1}}$ *a.e.* $\vec{\eta}_{n+1}=(\eta_0,\eta_1,\dots,\eta_n,\eta_{n+1})\in\mathbb{R}^{n+2}$, *we have* $GE[G_1|X_{n+1}](\vec{\eta}_{n+1}) = [\sum_{j=0}^{n+1}\eta_j\lambda(\{t_j\})]^2$.

Proof. Using the same method as used in the proof of [5, Theorem 4.3], we can prove the integrability of G_1 by (8). Now we evaluate $GE[G_1|X_{n+1}]$. For $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1}=(\eta_0,\eta_1,\dots,\eta_n,\eta_{n+1})\in\mathbb{R}^{n+2}$, we have by Theorem 2.2

$$\begin{aligned}
&GE[G_1|X_{n+1}](\vec{\eta}_{n+1})\\
&= \int_{C[0,T]}\bigg[\int_0^T [Z_t(x)+P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\bigg]^2 dw_{\alpha,\beta;\varphi_0}(x)\\
&= \int_{C[0,T]}\bigg[\eta_0\lambda(\{t_0\})+\sum_{j=1}^{n+1}\int_{(t_{j-1},t_j]}[Z_t(x)+P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\bigg]^2\\
&\quad dw_{\alpha,\beta;\varphi_0}(x),
\end{aligned}$$

where $\varphi_0=\frac{1}{\varphi(\mathbb{R})}\varphi$. Now we have by Theorem 3.1

$$\begin{aligned}
&GE[G_1|X_{n+1}](\vec{\eta}_{n+1})\\
&= [\eta_0\lambda(\{t_0\})]^2 + 2\eta_0\lambda(\{t_0\})\sum_{j=1}^{n+1}\int_{(t_{j-1},t_j]}[Z_t(\alpha)+P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\\
&\quad + 2\sum_{1\le j<k\le n+1}\int_{(t_{j-1},t_j]\times(t_{k-1},t_k]}[Z_{s_1}(\alpha)+P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1)][Z_{s_2}(\alpha)\\
&\quad + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2)]d\lambda^2(s_1,s_2)+\sum_{j=1}^{n+1}\int_{(t_{j-1},t_j]^2}\int_{C[0,T]}[Z_{s_1}(x)\\
&\quad + P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1)][Z_{s_2}(x)+P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2)]dw_{\alpha,\beta;\varphi_0}(x)d\lambda^2(s_1,s_2)\\
&= \int_0^T\int_0^T [Z_{s_1}(\alpha)+P_\beta^{n+1}(\vec{\eta}_{n+1})(s_1)][Z_{s_2}(\alpha)+P_\beta^{n+1}(\vec{\eta}_{n+1})(s_2)]\\
&\quad d\lambda^2(s_1,s_2)+\sum_{j=1}^{n+1}\int_{(t_{j-1},t_j]^2}\Gamma_j(s_1\vee s_2, s_1\wedge s_2)d\lambda^2(s_1,s_2)\\
&= \bigg[\int_0^T [Z_t(\alpha)+P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\bigg]^2 + \int_0^T\int_0^T \Lambda(s_1\vee s_2, s_1\wedge s_2)\\
&\quad d\lambda^2(s_1,s_2),
\end{aligned}$$

since $\Gamma_j(s_1,s_2)=0$ if $s_l=t_j$ or $s_l=t_{j-1}$ for $l=1,2$. □

Remark 3.4. (1) The expressions for the results of [5, Theorem 4.3] and Theorem 3.3 are similar, but Theorem 3.3 is an extension of [5, Theorem 4.3] since the measure λ in [5, Theorem 4.3] is a continuous complex measure and it in Theorem 3.3 is an arbitrary complex measure. In addition, the proof of Theorem 3.3 is more complicated than that of [5, Theorem 4.3].

(2) Let $\varphi(\mathbb{R}) = 1$ and $Z(x) = \int_0^T Z_t(x)d\lambda(t)$ for $x \in C[0,T]$. Under the assumptions as in Theorem 3.3, we have $E[Z] = \int_0^T Z_t(\alpha)d\lambda(t)$ and $Var(Z) = \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2)]d\lambda^2(s_1, s_2)$.

Theorem 3.5. *Under the assumptions as in Theorem 3.3, we have for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$,*

$$
\begin{aligned}
&GE[G_1|X_n](\vec{\eta}_n) \\
&= \left[\int_0^{t_n}[Z_t(\alpha) + P_\beta^n(\vec{\eta}_n)(t)]d\lambda(t) + \int_{(t_n,T]}[\alpha(t) - \alpha(t_n) + \eta_n]d\lambda(t)\right]^2 \\
&\quad + \int_0^{t_n}\int_0^{t_n}\Lambda(s_1 \vee s_2, s_1 \wedge s_2)d\lambda^2(s_1, s_2) + \int_{t_n}^T\int_{t_n}^T[\beta(s_1 \wedge s_2) \\
&\quad - \beta(t_n)]d\lambda^2(s_1, s_2).
\end{aligned}
$$

In particular, if the support of λ is contained in $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, we have $GE[G_1|X_n](\vec{\eta}_n) = [\sum_{j=0}^n \eta_j\lambda(\{t_j\}) + [\alpha(T) - \alpha(t_n) + \eta_n]\lambda(\{T\})]^2 + [\beta(T) - \beta(t_n)][\lambda(\{T\})]^2$.

Proof. By Theorems 2.3 and 3.3, we have for $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$
\begin{aligned}
&GE[G_1|X_n](\vec{\eta}_n) \\
&= \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\left[\int_0^T[Z_t(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\right]^2 dm_L(\eta_{n+1}) \\
&\quad + \int_0^T\int_0^T\Lambda(s_1 \vee s_2, s_1 \wedge s_2)d\lambda^2(s_1, s_2),
\end{aligned}
$$

where $\vec{\eta}_{n+1} = (\vec{\eta}_n, \eta_{n+1})$. Since W_T is Gaussian, we have by Theorem 3.2

$$
\begin{aligned}
&\int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\left[\int_0^T[Z_t(\alpha) + P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\right]^2 dm_L(\eta_{n+1}) \\
&= \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1})\left[\int_0^{t_n}[Z_t(\alpha) + P_\beta^n(\vec{\eta}_n)(t)]d\lambda(t) + \int_{(t_n,T]}[Z_t(\alpha) \right. \\
&\quad \left. + P_\beta^{n+1}(\vec{\eta}_{n+1})(t)]d\lambda(t)\right]^2 dm_L(\eta_{n+1}) \\
&= \left[\int_0^{t_n}[Z_t(\alpha) + P_\beta^n(\vec{\eta}_n)(t)]d\lambda(t) + \int_{(t_n,T]}[\alpha(t) - \alpha(t_n) + \eta_n]d\lambda(t)\right]^2 \\
&\quad + [\beta(T) - \beta(t_n)]\left[\int_{t_n}^T \gamma_{2(n+1)}(t)d\lambda(t)\right]^2.
\end{aligned}
$$

We also have

$$[\beta(T)-\beta(t_n)]\bigg[\int_{t_n}^T \gamma_{2(n+1)}(t)d\lambda(t)\bigg]^2+\int_{[t_n,T]^2}\Gamma_{n+1}(s_1\vee s_2,s_1\wedge s_2)$$
$$d\lambda^2(s_1,s_2)$$
$$=\frac{1}{\beta(T)-\beta(t_n)}\bigg[\int_{\Delta_1}[\beta(s_1)-\beta(t_n)][\beta(s_2)-\beta(t_n)+\beta(T)-\beta(s_2)]d\lambda^2(s_1,$$
$$s_2)+\int_{\Delta_2}[\beta(s_2)-\beta(t_n)][\beta(s_1)-\beta(t_n)+\beta(T)-\beta(s_1)]d\lambda^2(s_1,s_2)\bigg]$$
$$=\int_{t_n}^T\int_{t_n}^T[\beta(s_1\wedge s_2)-\beta(t_n)]d\lambda^2(s_1,s_2),$$

where $\Delta_1=\{(s_1,s_2):t_n\le s_1\le s_2\le T_n\}$ and $\Delta_2=\{(s_1,s_2):t_n\le s_2< s_1\le T_n\}$. Now we have

$$GE[G_1|X_n](\vec{\eta}_n)$$
$$=\bigg[\int_0^{t_n}[Z_t(\alpha)+P_\beta^n(\vec{\eta}_n)(t)]d\lambda(t)+\int_{(t_n,T]}[\alpha(t)-\alpha(t_n)+\eta_n]d\lambda(t)\bigg]^2$$
$$+\int_0^{t_n}\int_0^{t_n}\Lambda(s_1\vee s_2,s_1\wedge s_2)d\lambda^2(s_1,s_2)+[\beta(T)-\beta(t_n)]$$
$$\times\bigg[\int_{t_n}^T\gamma_{2(n+1)}(t)d\lambda(t)\bigg]^2+\int_{[t_n,T]^2}\Gamma_{n+1}(s_1\vee s_2,s_1\wedge s_2)d\lambda^2(s_1,s_2)$$
$$=\bigg[\int_0^{t_n}[Z_t(\alpha)+P_\beta^n(\vec{\eta}_n)(t)]d\lambda(t)+\int_{(t_n,T]}[\alpha(t)-\alpha(t_n)+\eta_n]d\lambda(t)\bigg]^2$$
$$+\int_0^{t_n}\int_0^{t_n}\Lambda(s_1\vee s_2,s_1\wedge s_2)d\lambda^2(s_1,s_2)+\int_{t_n}^T\int_{t_n}^T[\beta(s_1\wedge s_2)-\beta(t_n)]$$
$$d\lambda^2(s_1,s_2)$$

which completes the proof. □

The following theorem is useful for the remainder results of this section [5].

Theorem 3.6. *For $m\in\mathbb{N}$ and $t\in[0,T]$, let $F_t(x)=[x(t)]^m$ for $x\in C[0,T]$ and suppose that $\int_{\mathbb{R}}|u|^m d\varphi(u)<\infty$. Then F_t is $w_{\alpha,\beta;\varphi}$-integrable and for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1}\in\mathbb{R}^{n+2}$, $GE[F_t|X_{n+1}](\vec{\eta}_{n+1})$ is given by*

$$GE[F_t|X_{n+1}](\vec{\eta}_{n+1})$$
$$=\sum_{k=0}^{[\frac{m}{2}]}\frac{m!}{2^k k!(m-2k)!}[P_\beta^{n+1}(\vec{\eta}_{n+1})(t)+Z_t(\alpha)]^{m-2k}[\Gamma(t)]^k, \tag{9}$$

where $\Gamma(t)$ is given by (5) and $[\frac{m}{2}]$ denotes the greatest integer less than or equal to $\frac{m}{2}$. In particular, if $t=t_j$ for some $j\in\{0,1,\dots,n,n+1\}$, then we *have* $GE[F_t|X_{n+1}](\vec{\eta}_{n+1}) = \eta_j^m$ *for* $m_{X_{n+1}}$ *a.e.* $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \dots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$.

Theorem 3.7. *Under the assumptions as in Theorem 3.6, we have the followings*:

(a) *If* $t \in [0, t_n]$, *then for* m_{X_n} *a.e.* $\vec{\eta}_n \in \mathbb{R}^{n+1}$,

$$GE[F_t|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!} [P_\beta^n(\vec{\eta}_n)(t) + Z_t(\alpha)]^{m-2k} [\Gamma(t)]^k.$$

In particular, if $t = t_j$ *for some* $j \in \{0, 1, \dots, n\}$, *then* $GE[F_t|X_n](\vec{\eta}_n) = \eta_j^m$ *for* m_{X_n} *a.e.* $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$.

(b) *If* $t \in (t_n, T]$, *then for* m_{X_n} *a.e.* $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$,

$$\begin{aligned} GE[F_t|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}-k]} & \frac{m!}{2^{k+l} k! l! (m-2k-2l)!} [\eta_n + \alpha(t) \\ & - \alpha(t_n)]^{m-2k-2l} [\gamma_{2(n+1)}(t)[\beta(t) - \beta(t_n)]]^l [\Gamma(t)]^k. \end{aligned}$$

In particular, if $t = T$, *then* $GE[F|X_n](\vec{\eta}_n)$ *is reduced to*

$$GE[F|X_n](\vec{\eta}_n) = \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l! (m-2l)!} [\eta_n + \alpha(T) - \alpha(t_n)]^{m-2l} [\beta(T) - \beta(t_n)]^l.$$

Proof. (a) follows from (9) of Theorem 3.6 since W_T is a probability density. To prove (b), let $t \in (t_n, T]$. Then we have by the change of variable theorem

$$\begin{aligned} & \int_{\mathbb{R}} [P_\beta^{n+1}(\vec{\eta}_{n+1})(t) + Z_t(\alpha)]^{m-2k} W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1}) \\ = & \left[\frac{1}{2\pi[\beta(T) - \beta(t_n)]}\right]^{\frac{1}{2}} \int_{\mathbb{R}} [\gamma_{2(n+1)}(t)(\eta_{n+1} - \eta_n) + \eta_n + \alpha(t) - \alpha(t_n) \\ & - \gamma_{2(n+1)}(t)[\alpha(T) - \alpha(t_n)]]^{m-2k} \exp\left\{-\frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2[\beta(T) - \beta(t_n)]}\right\} \\ & dm_L(\eta_{n+1}) \\ = & \left[\frac{1}{2\pi\gamma_{2(n+1)}(t)[\beta(t) - \beta(t_n)]}\right]^{\frac{1}{2}} \int_{\mathbb{R}} [u + \eta_n + \alpha(t) - \alpha(t_n)]^{m-2k} \\ & \times \exp\left\{-\frac{u^2}{2\gamma_{2(n+1)}(t)[\beta(t) - \beta(t_n)]}\right\} dm_L(u). \end{aligned}$$

Using the same process as used in the proof of [2, Theorem 3.1], we have by Theorem 3.6

$$\begin{aligned} GE[F_t|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}-k]} & \frac{m!}{2^{k+l} k! l! (m-2k-2l)!} [\eta_n + \alpha(t) \\ & - \alpha(t_n)]^{m-2k-2l} [\gamma_{2(n+1)}(t)[\beta(t) - \beta(t_n)]]^l [\Gamma(t)]^k \end{aligned}$$

which proves (b). □

By Theorem 3.7 and [5, Theorem 4.5], we can prove the following theorem.

Theorem 3.8. *For $m \in \mathbb{N}$, let $F(x) = \int_0^T [x(t)]^m d\lambda(t)$ for $x \in C[0,T]$, where λ is a finite complex measure on the Borel class of $[0,T]$, and suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$ and $\int_0^T |\alpha(t)|^m d|\lambda|(t) < \infty$. Then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, $GE[F|X_n](\vec{\eta}_n)$ is given by*

$$\begin{aligned} GE[F|X_n](\vec{\eta}_n) = &\sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!} \int_0^{t_n} [P_\beta^n(\vec{\eta}_n)(t) + Z_t(\alpha)]^{m-2k} [\Gamma(t)]^k d\lambda(t) \\ &+ \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}-k]} \frac{m!}{2^{k+l} k! l! (m-2k-2l)!} \int_{(t_n,T]} [\eta_n + \alpha(t) \\ &- \alpha(t_n)]^{m-2k-2l} [\gamma_{2(n+1)}(t)[\beta(t) - \beta(t_n)]]^l [\Gamma(t)]^k d\lambda(t). \end{aligned}$$

In particular, if the support of λ is contained in $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$, then $GE[F|X_n](\vec{\eta}_n)$ is reduced to

$$\begin{aligned} GE[F|X_n](\vec{\eta}_n) = &\sum_{j=0}^{n} \lambda(\{t_j\}) \eta_j^m + \lambda(\{T\}) \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l!(m-2l)!} [\eta_n + \alpha(T) \\ &- \alpha(t_n)]^{m-2l} [\beta(T) - \beta(t_n)]^l. \end{aligned}$$

Applying the integration by parts formula to the result of Theorem 3.8 repeatedly, we have the following corollary [3].

Corollary 3.9. *Suppose that $\alpha(t) = P_\beta^n(\alpha)(t)$ for $t \in [0, t_n]$ and $\alpha(t) = \alpha(t_n)$ for $t \in (t_n T]$. For $m \in \mathbb{N}$, let $F(x) = \int_0^T [x(t)]^m d\beta(t)$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then, for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, $GE[F|X_n](\vec{\eta}_n)$ is given by*

$$\begin{aligned} &GE[F|X_n](\vec{\eta}_n) \\ &= \sum_{j=1}^{n} \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \frac{m!(l+k)![\beta(t_j) - \beta(t_{j-1})]^{k+1} \eta_{j-1}^{m-2k-l} (\eta_j - \eta_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!} \\ &\quad + \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}-k]} \frac{m!(2l+k)! \eta_n^{m-2k-2l} [\beta(T) - \beta(t_n)]^{l+k+1}}{2^{l+k} l!(m-2k-2l)!(2l+2k+1)!}. \end{aligned}$$

Example 3.10. For $l = 1, 2, 3$, let $F_l(x) = \int_0^T [x(t)]^l d\beta(t)$ for $x \in C[0,T]$. By [5, Example 4.6], we have the followings:

(a) If $\int_{\mathbb{R}} |u| d\varphi(u) < \infty$ and $\int_0^T |\alpha(t)| d\beta(t) < \infty$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} GE[F_1|X_n](\vec{\eta}_n) = & \int_0^T \alpha(t) d\beta(t) + \frac{1}{2}\sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\eta_j - \alpha(t_j) \\ & + \eta_{j-1} - \alpha(t_{j-1})] + [\beta(T) - \beta(t_n)][\eta_n - \alpha(t_n)]. \end{aligned}$$

(b) If $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$ and $\int_0^T [\alpha(t)]^2 d\beta(t) < \infty$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & GE[F_2|X_n](\vec{\eta}_n) \\ = & \int_0^T \alpha(t) \int_{\mathbb{R}} [\alpha(t) + 2P_\beta^{n+1}(\vec{\eta}_{n+1} - \alpha)(t)] W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1}) d\beta(t) \\ & + \frac{1}{6}\sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1}) + 2[[\eta_j - \alpha(t_j)]^2 + [\eta_j - \alpha(t_j)] \\ & \times [\eta_{j-1} - \alpha(t_{j-1})] + [\eta_{j-1} - \alpha(t_{j-1})]^2]] + \frac{1}{2}[\beta(T) - \beta(t_n)][\beta(T) - \beta(t_n) \\ & + 2[\eta_n - \alpha(t_n)]^2] \\ = & \int_0^{t_n} \alpha(t)[\alpha(t) + 2P_\beta^n(\vec{\eta}_n - \alpha)(t)] d\beta(t) + \int_{t_n}^T \alpha(t)[\alpha(t) + 2[\eta_n - \alpha(t_n)]] \\ & d\beta(t) + \frac{1}{6}\sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1}) + 2[[\eta_j - \alpha(t_j)]^2 + [\eta_j \\ & - \alpha(t_j)][\eta_{j-1} - \alpha(t_{j-1})] + [\eta_{j-1} - \alpha(t_{j-1})]^2]] + \frac{1}{2}[\beta(T) - \beta(t_n)][\beta(T) \\ & - \beta(t_n) + 2[\eta_n - \alpha(t_n)]^2] \end{aligned}$$

since $\int_{\mathbb{R}} P_\beta^{n+1}(\vec{\eta}_{n+1} - \alpha)(t) W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1}) = \eta_n - \alpha(t_n)$, where $\vec{\eta}_{n+1} = (\vec{\eta}_n, \eta_{n+1})$.

(c) If $\int_{\mathbb{R}} u^3 d\varphi(u) < \infty$ and $\int_0^T [\alpha(t)]^3 d\beta(t) < \infty$, then for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & GE[F_3|X_n](\vec{\eta}_n) \\ = & \int_0^T \int_{\mathbb{R}} \alpha(t)[[\alpha(t)]^2 + 3[\Gamma(t) + P_\beta^{n+1}(\vec{\eta}_{n+1} - \alpha)(t)[P_\beta^{n+1}(\vec{\eta}_{n+1} - \alpha)(t) \\ & + \alpha(t)]]] W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1}) d\beta(t) + \frac{1}{4}\sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][[\beta(t_j) \\ & - \beta(t_{j-1})][\eta_j - \alpha(t_j) + \eta_{j-1} - \alpha(t_{j-1})] + [\eta_j - \alpha(t_j)]^3 + [\eta_j - \alpha(t_j)]^2 \\ & \times [\eta_{j-1} - \alpha(t_{j-1})] + [\eta_j - \alpha(t_j)][\eta_{j-1} - \alpha(t_{j-1})]^2 + [\eta_{j-1} - \alpha(t_{j-1})]^3] \end{aligned}$$

$$+\frac{1}{2}[\beta(T)-\beta(t_n)][\eta_n-\alpha(t_n)][3[\beta(T)-\beta(t_n)]+2[\eta_n-\alpha(t_n)]^2].$$

For $t\in(t_n,T]$, we have by the change of variable theorem

$$\begin{aligned}&\int_{\mathbb{R}}[P_\beta^{n+1}(\vec{\eta}_{n+1}-\alpha)(t)]^2W_T(\eta_n,\eta_{n+1})dm_L(\eta_{n+1})\\&=\left[\frac{1}{2\pi[\beta(T)-\beta(t_n)]}\right]^{\frac{1}{2}}\int_{\mathbb{R}}[\gamma_{2(n+1)}(t)[\eta_{n+1}-\eta_n-\alpha(T)+\alpha(t_n)]\\&\quad+\eta_n-\alpha(t_n)]^2\exp\left\{-\frac{[\eta_{n+1}-\eta_n-\alpha(T)+\alpha(t_n)]^2}{2[\beta(T)-\beta(t_n)]}\right\}dm_L(\eta_{n+1})\\&=\gamma_{2(n+1)}(t)[\beta(t)-\beta(t_n)]+[\eta_n-\alpha(t_n)]^2\end{aligned}$$

and $\Gamma(t)+\gamma_{2(n+1)}(t)[\beta(t)-\beta(t_n)]=\beta(t)-\beta(t_n)$, so that

$$\begin{aligned}&GE[F_3|X_n](\vec{\eta}_n)\\&=\int_0^{t_n}\alpha(t)[[\alpha(t)]^2+3[\Gamma(t)+P_\beta^n(\vec{\eta}_n-\alpha)(t)[P_\beta^n(\vec{\eta}_n-\alpha)(t)+\alpha(t)]]]d\beta(t)\\&\quad+\int_{t_n}^T\alpha(t)[[\alpha(t)]^2+3[\beta(t)-\beta(t_n)+[\eta_n-\alpha(t_n)][\eta_n+\alpha(t)-\alpha(t_n)]]]\\&\quad d\beta(t)+\frac{1}{4}\sum_{j=1}^n[\beta(t_j)-\beta(t_{j-1})][[\beta(t_j)-\beta(t_{j-1})][\eta_j-\alpha(t_j)+\eta_{j-1}\\&\quad-\alpha(t_{j-1})]+[\eta_j-\alpha(t_j)]^3+[\eta_j-\alpha(t_j)]^2[\eta_{j-1}-\alpha(t_{j-1})]+[\eta_j-\alpha(t_j)]\\&\quad\times[\eta_{j-1}-\alpha(t_{j-1})]^2+[\eta_{j-1}-\alpha(t_{j-1})]^3]+\frac{1}{2}[\beta(T)-\beta(t_n)][\eta_n-\alpha(t_n)]\\&\quad\times[3[\beta(T)-\beta(t_n)]+2[\eta_n-\alpha(t_n)]^2].\end{aligned}$$

Theorem 3.11. *Let $F(x)=\exp\{\int_0^T x(t)d\beta(t)\}$ for $x\in C[0,T]$. Suppose that $\lim_{t\to T^-}\alpha(t)=\alpha(T)$. Let $\tau:0=t_0<t_1<\cdots<t_n<t_{n+1}=T$ be any partition of $[0,T]$ and let $X_\tau(x)=(x(t_0),x(t_1),\ldots,x(t_n))$ for $x\in C[0,T]$. Then, for $w_{\alpha,\beta;\varphi}$ a.e. $y\in C[0,T]$, we have*

$$\lim_{\|\tau\|\to0}GE[F|X_\tau](X_\tau(y))=F(y).$$

Proof. For $w_{\alpha,\beta;\varphi}$ a.e. $y\in C[0,T]$, we have

$$\begin{aligned}&GE[F|X_\tau](X_\tau(y))\\&=\exp\left\{\frac{1}{2}\sum_{j=1}^n[\beta(t_j)-\beta(t_{j-1})][y(t_{j-1})+y(t_j)]\right\}\int_{C[0,T]}\exp\left\{\int_0^T Z_t(x)\right.\\&\quad\left.d\beta(t)\right\}dw_{\alpha,\beta;\varphi_0}(x)\int_{\mathbb{R}}W_T(y(t_n),\eta_{n+1})\exp\left\{\frac{1}{2}[\beta(T)-\beta(t_n)][y(t_n)\right.\\&\quad\left.+\eta_{n+1}]\right\}dm_L(\eta_{n+1})\end{aligned}$$

$$= \exp\bigg\{\frac{1}{2}\bigg[\sum_{j=1}^{n}[\beta(t_j)-\beta(t_{j-1})][y(t_{j-1})+y(t_j)]+[\beta(T)-\beta(t_n)][2y(t_n) + \alpha(T)-\alpha(t_n)]\bigg]+\frac{1}{8}[\beta(T)-\beta(t_n)]^3\bigg\}\int_{C[0,T]}\exp\bigg\{\int_0^T Z_t(x)d\beta(t)\bigg\} dw_{\alpha,\beta;\varphi_0}(x)$$

by Theorem 2.3 and the same process used in Example 3.10, where $\varphi_0=\frac{1}{\varphi(\mathbb{R})}\varphi$. Letting $\|\tau\|\to 0$, we have

$$\lim_{\|\tau\|\to 0} GE[F|X_\tau](X_\tau(y))=F(y)$$

because $\lim_{\|\tau\|\to 0} Z_t(x)=0$ for $x\in C[0,T]$ and both α and β are left-continuous at T. □

4. Translation theorems for the generalized conditional expectation

In this section we derive translation theorems for the Radon-Nikodym derivatives of the functions on $C[0,T]$ with the conditioning functions X_n and X_{n+1}. To do this, we need the following translation theorem.

Theorem 4.1. *Let h be continuous and of bounded variation on $[0,T]$. Suppose that α is of bounded variation or continuous. Let $a\in\mathbb{R}$ and define x_0 by $x_0(t)=\int_0^t h(u)d\beta(u)+a$ for $t\in[0,T]$. Let φ_a be the measure defined by $\varphi_a(B)=\varphi(B+a)$ for $B\in\mathcal{B}(\mathbb{R})$. If $F:C[0,T]\to\mathbb{C}$ is $w_{\alpha,\beta;\varphi}$-integrable, then $F(\cdot+x_0)$ is $w_{\alpha,\beta;\varphi_a}$-integrable and*

$$\int_{C[0,T]}F(x)dw_{\alpha,\beta;\varphi}(x)=J_1(h)\int_{C[0,T]}F(x+x_0)J_2(h,x)dw_{\alpha,\beta;\varphi_a}(x), \tag{10}$$

where $J_1(h)=\exp\{-\frac{1}{2}\int_0^T[h(t)]^2d\beta(t)+\int_0^T h(t)d\alpha(t)\}$ and $J_2(h,x)=\exp\{-\int_0^T h(t)dx(t)\}$ for $x\in C[0,T]$.

Proof. Suppose that F is bounded and continuous, and vanishes on the set $\{x\in C[0,T]:\|x\|_\infty>M\}$ for some real number $M>0$. For nonnegative integer n, let $\vec{s}_{n+1}=(s_0,s_1,\dots,s_n,s_{n+1})=(0,\frac{T}{n+1},\frac{2T}{n+1},\dots,\frac{nT}{n+1},\frac{(n+1)T}{n+1})$. Let P_β^{n+1} be the polygonal function given by (3) with replacing t_j by s_j. Then we have by Theorem 1.1

$$\int_{C[0,T]}F(P_\beta^{n+1}(x))dw_{\alpha,\beta;\varphi}(x)$$
$$=\int_{\mathbb{R}^{n+2}}F(P_\beta^{n+1}(\vec{u}_{n+2}))W_{n+1}(\vec{s}_{n+1},\vec{u}_{n+1},u_0)dm_L^{n+1}(\vec{u}_{n+1})d\varphi(u_0)$$
$$=\exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{[x_0(s_j)-x_0(s_{j-1})]^2}{\beta(s_j)-\beta(s_{j-1})}\bigg\}\int_{\mathbb{R}^{n+2}}F(P_\beta^{n+1}(\vec{u}_{n+2}))W_{n+1}(\vec{s}_{n+1},$$

$$\vec{u}_{n+1} - \vec{x}_1, u_0 - x_0(s_0)) \exp\bigg\{ - \sum_{j=1}^{n+1} \frac{x_0(s_j) - x_0(s_{j-1})}{\beta(s_j) - \beta(s_{j-1})} [u_j - u_{j-1} \\ - x_0(s_j) + x_0(s_{j-1}) - \alpha(s_j) + \alpha(s_{j-1})] \bigg\} dm_L^{n+1}(\vec{u}_{n+1}) d\varphi(u_0),$$

where $\vec{u}_{n+1} = (u_1, \ldots, u_n, u_{n+1})$, $\vec{u}_{n+2} = (u_0, \vec{u}_{n+1})$ and $\vec{x}_1 = (x_0(s_1), x_0(s_2), \ldots, x_0(s_n), x_0(s_{n+1}))$. Let $\vec{x}_0 = (x_0(s_0), x_0(s_1), \ldots, x_0(s_n), x_0(s_{n+1}))$. By the change of variable theorem and Theorem 1.1 again, we have

$$\begin{aligned} &\int_{C[0,T]} F(P_\beta^{n+1}(x)) dw_{\alpha,\beta;\varphi}(x) \\ &= \exp\bigg\{ -\frac{1}{2} \sum_{j=1}^{n+1} \frac{[x_0(s_j) - x_0(s_{j-1})]^2}{\beta(s_j) - \beta(s_{j-1})} \bigg\} \int_{\mathbb{R}^{n+2}} F(P_\beta^{n+1}(\vec{u}_{n+2} + \vec{x}_0)) \\ &\quad \times W_{n+1}(\vec{s}_{n+1}, \vec{u}_{n+1}, u_0) \exp\bigg\{ - \sum_{j=1}^{n+1} \frac{x_0(s_j) - x_0(s_{j-1})}{\beta(s_j) - \beta(s_{j-1})} [u_j - u_{j-1} \\ &\quad - \alpha(s_j) + \alpha(s_{j-1})] \bigg\} dm_L^{n+1}(\vec{u}_{n+1}) d\varphi_a(u_0) \\ &= \exp\bigg\{ -\frac{1}{2} \sum_{j=1}^{n+1} \frac{[x_0(s_j) - x_0(s_{j-1})]^2}{\beta(s_j) - \beta(s_{j-1})} \bigg\} \int_{C[0,T]} F(P_\beta^{n+1}(x + x_0)) \exp\bigg\{ - \sum_{j=1}^{n+1} \\ &\quad \frac{x_0(s_j) - x_0(s_{j-1})}{\beta(s_j) - \beta(s_{j-1})} [x(s_j) - x(s_{j-1}) - \alpha(s_j) + \alpha(s_{j-1})] \bigg\} dw_{\alpha,\beta;\varphi_a}(x). \end{aligned}$$

Since h is continuous on $[0, T]$, we have for $j = 1, \ldots, n, n+1$

$$x_0(s_j) - x_0(s_{j-1}) = \int_{s_{j-1}}^{s_j} h(t) d\beta(t) = h(\xi_j)[\beta(s_j) - \beta(s_{j-1})]$$

for some $\xi_j \in [s_{j-1}, s_j]$ by the mean value theorem for integral. Now, we have

$$\begin{aligned} &\int_{C[0,T]} F(P_\beta^{n+1}(x)) dw_{\alpha,\beta;\varphi}(x) \\ &= \exp\bigg\{ -\frac{1}{2} \sum_{j=1}^{n+1} [h(\xi_j)]^2 [\beta(s_j) - \beta(s_{j-1})] \bigg\} \int_{C[0,T]} F(P_\beta^{n+1}(x + x_0)) \\ &\quad \times \exp\bigg\{ - \sum_{j=1}^{n+1} h(\xi_j) [x(s_j) - x(s_{j-1}) - \alpha(s_j) + \alpha(s_{j-1})] \bigg\} dw_{\alpha,\beta;\varphi_a}(x). \end{aligned}$$

Note that $(P_\beta^{n+1})_{n=0}^\infty$ converges uniformly to the identity function on $C[0, T]$. Letting $n \to \infty$, we have (10) by using the same process used in the proof of [6, Theorem 3.1]. For the other cases of F, (10) can be proved by using the same process used in the proof of [10, Theorem 2.1]. □

Remark 4.2. Let β be continuously differentiable with $\beta' > 0$. Let h be in $C[0,T]$ and let $\frac{h}{\beta'}$ be of bounded variation. Replacing h in Theorem 4.1 by $\frac{h}{\beta'}$ we can obtain [10, Theorem 2.1]. In Theorem 4.1, the conditions those $\beta' > 0$ and β' is continuous are removed so that Theorem 4.1 generalizes [10, Theorem 2.1]. Note that, to prove [10, Theorem 2.1], Ryu used the mean value theorem for differentiation, but the mean value theorem for integral is used in the proof of Theorem 4.1.

We now derive translation theorems for the generalized conditional expectations of functions on $C[0,T]$ with the conditioning functions X_n and X_{n+1}.

Theorem 4.3. *Under the assumptions as in Theorem 4.1, we have for* $m_{X_{n+1}}$ *a.e.* $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}&GE[F|X_{n+1}](\vec{\eta}_{n+1})\\ &= J_1(h)J_3(h,\vec{\eta}_{n+1})GE_{w_{\varphi_a}}[F(\cdot + x_0)J_2(h,\cdot)|X_{n+1}](\vec{\eta}_{n+1} - \vec{x}_0),\end{aligned}$$

where $\vec{x}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n), x_0(t_{n+1}))$, $GE_{w_{\varphi_a}}$ *denotes the generalized conditional expectation with respect to* $w_{\alpha,\beta;\varphi_a}$, *and for* $m \in \mathbb{N}$ $(1 \le m \le n+1)$, $\vec{\eta}_m = (\eta_0, \eta_1, \ldots, \eta_m) \in \mathbb{R}^{m+1}$

$$\begin{aligned}J_3(h,\vec{\eta}_m) = \exp\Bigg\{\sum_{j=1}^{m} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})}\Bigg[\eta_j - \eta_{j-1} - \alpha(t_j)\\ + \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})]\Bigg]\Bigg\}.\end{aligned}$$

Proof. Let $\varphi_1 = \frac{1}{\varphi_a(\mathbb{R})}\varphi_a$. By Theorems 2.2 and 4.1, we have for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}&GE[F|X_{n+1}](\vec{\eta}_{n+1})\\ &= \frac{1}{\varphi(\mathbb{R})}\int_{C[0,T]} F(x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1}))dw_{\alpha,\beta;\varphi}(x)\\ &= \frac{J_1(h)}{\varphi_a(\mathbb{R})}\int_{C[0,T]} F(x_0 + x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1} - x_0))J_2(h,x)\\ &\quad dw_{\alpha,\beta;\varphi_a}(x)\\ &= J_1(h)\int_{C[0,T]} F(x_0 + x - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1} - x_0))J_2(h,x\\ &\quad - P_\beta^{n+1}(x) + P_\beta^{n+1}(\vec{\eta}_{n+1} - x_0))J_2(h, P_\beta^{n+1}(x) - P_\beta^{n+1}(\vec{\eta}_{n+1} - x_0))\\ &\quad dw_{\alpha,\beta;\varphi_1}(x)\end{aligned}$$

since $\varphi_a(\mathbb{R}) = \varphi(\mathbb{R}+a) = \varphi(\mathbb{R})$. Since φ_1 is a probability measure, $x - P_\beta^{n+1}(x)$ and $P_\beta^{n+1}(x)$ are independent with respect to $w_{\alpha,\beta;\varphi_1}$ by Theorem 2.1. Hence we have

$$GE[F|X_{n+1}](\vec{\eta}_{n+1}) = J_1(h)GE_{w_{\varphi_a}}[F(\cdot + x_0)J_2(h,\cdot)|X_{n+1}](\vec{\eta}_{n+1} - \vec{x}_0)J_2(h,$$

$$- P_\beta^{n+1}(\vec{\eta}_{n+1} - x_0)) \int_{C[0,T]} J_2(h, P_\beta^{n+1}(x)) dw_{\alpha,\beta;\varphi_1}(x)$$

since $P_\beta^{n+1}(x_0) = P_\beta^{n+1}(\vec{x}_0)$. Note that for $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1})$

$$\int_0^T h(t) dP_\beta^{n+1}(\vec{\eta}_{n+1} - x_0)(t) = \sum_{j=1}^{n+1} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} [\eta_j - \eta_{j-1} - x_0(t_j) + x_0(t_{j-1})]$$

and by Theorem 1.1, we have

$$\begin{aligned}
&\int_{C[0,T]} J_2(h, P_\beta^{n+1}(x)) dw_{\alpha,\beta;\varphi_1}(x) \\
&= \int_{C[0,T]} \exp\bigg\{ - \sum_{j=1}^{n+1} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} [x(t_j) - x(t_{j-1})] \bigg\} dw_{\alpha,\beta;\varphi_1}(x) \\
&= \exp\bigg\{ \sum_{j=1}^{n+1} \bigg[\frac{[x_0(t_j) - x_0(t_{j-1})]^2}{2[\beta(t_j) - \beta(t_{j-1})]} - \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} [\alpha(t_j) - \alpha(t_{j-1})] \bigg] \bigg\}.
\end{aligned}$$

Now, we have the desired result by a simple calculation. □

Theorem 4.4. *Under the assumptions as in Theorem 4.1, we have for m_{X_n} a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$GE[F|X_n](\vec{\eta}_n) = J_1(h) J_3(h, \vec{\eta}_n) GE_{w_{\varphi_a}}[F(\cdot + x_0) J_2(h, \cdot)|X_n](\vec{\eta}_n - \vec{x}_0),$$

where $\vec{x}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n))$ and J_3 is as given in Theorem 4.3.

Proof. Let $\vec{y}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n), x_0(t_{n+1}))$. By Theorems 2.3 and 4.3, we have for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$GE[F|X_n](\vec{\eta}_n) = J_1(h) \int_{\mathbb{R}} J_3(h, \vec{\eta}_{n+1}) GE_{w_{\varphi_a}}[F(\cdot + x_0) J_2(h, \cdot)|X_{n+1}](\vec{\eta}_{n+1} - \vec{y}_0) W_T(\eta_n, \eta_{n+1}) dm_L(\eta_{n+1}),$$

where $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1})$. By the change of variable theorem with a simple calculation, we have

$$\begin{aligned}
&GE[F|X_n](\vec{\eta}_n) \\
&= J_1(h) \int_{\mathbb{R}} J_3(h, \vec{\eta}_n) GE_{w_{\varphi_a}}[F(\cdot + x_0) J_2(h, \cdot)|X_{n+1}](\vec{\eta}_n - \vec{x}_0, \eta_{n+1} \\
&\quad - x_0(t_{n+1})) W_T(\eta_n - x_0(t_n), \eta_{n+1} - x_0(t_{n+1})) dm_L(\eta_{n+1}) \\
&= J_1(h) J_3(h, \vec{\eta}_n) \int_{\mathbb{R}} GE_{w_{\varphi_a}}[F(\cdot + x_0) J_2(h, \cdot)|X_{n+1}](\vec{\eta}_n - \vec{x}_0, \eta_{n+1}) \\
&\quad \times W_T(\eta_n - x_0(t_n), \eta_{n+1}) dm_L(\eta_{n+1}).
\end{aligned}$$

Now, the theorem follows from Theorem 2.3 as desired. □

Finally, we have the next example which is useful in the Feynman integration theory.

Example 4.5. Let the assumptions be as in Theorem 4.1. Let $A(\lambda, x) = \exp\{\lambda \int_0^T h(t)dx(t)\}$ for $\lambda \in \mathbb{R}$ and for $x \in C[0,T]$. Letting $a = 0$ and $F \equiv 1$, we have the followings:

(a) By Theorem 4.1, we have

$$\int_{C[0,T]} A(\lambda, x) dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R}) \exp\bigg\{ \frac{\lambda^2}{2} \int_0^T [h(t)]^2 d\beta(t) + \lambda \int_0^T h(t) d\alpha(t) \bigg\}$$

which can also be obtained by applications of [4, Theorem 3.4] and [4, Corollary 3.7].

(b) By Theorem 4.3, we have for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \dots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$

$$\begin{aligned} &GE[A(1,\cdot)|X_{n+1}](\vec{\eta}_{n+1}) \\ = \ &\exp\bigg\{ \frac{1}{2} \int_0^T [h(t)]^2 d\beta(t) + \int_0^T h(t) d\alpha(t) + \sum_{j=1}^{n+1} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} \\ &\times \bigg[\eta_j - \eta_{j-1} - \alpha(t_j) + \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})] \bigg] \bigg\}. \end{aligned}$$

(c) By Theorem 4.4, we have for m_{X_n} a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &GE[A(1,\cdot)|X_n](\vec{\eta}_n) \\ = \ &\exp\bigg\{ \frac{1}{2} \int_0^T [h(t)]^2 d\beta(t) + \int_0^T h(t) d\alpha(t) + \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} \\ &\times \bigg[\eta_j - \eta_{j-1} - \alpha(t_j) + \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})] \bigg] \bigg\}. \end{aligned}$$

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