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ON THE APPROXIMATION BY REGULAR POTENTIALS OF SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS

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Abstract. We prove that wave operators for Schrödinger operators with multi-center local point interactions are scaling limits of the ones for Schrödinger operators with regular potentials. We simultaneously present a proof of the corresponding well known result for the resolvent which substantially simplifies the one by Albeverio et al.

1. Introduction

Let $Y = \{y_1, \ldots, y_N\}$ be the set of N points in $\mathbb{R}^3$ and T_0 be the densely defined non-negative symmetric operator in $\mathcal{H} = L^2(\mathbb{R}^3)$ defined by

$$T_0 = -\Delta|_{C_0^\infty(\mathbb{R}^3 \setminus Y)}.$$

Any of selfadjoint extensions of T_0 is called the Schrödinger operator with point interactions at Y. Among them, we are concerned with the ones with local point interactions $H_{\alpha,Y}$ which are defined by separated boundary conditions at each point y_j parameterized by $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, N$. They can be defined via the resolvent equation (cf. [2]): With $H_0 = -\Delta$ being the free Schrödinger operator and $z \in \mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \Im z > 0\}$,

$$(H_{\alpha,Y} - z^2)^{-1} = (H_0 - z^2)^{-1} + \sum_{j,\ell=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{j\ell}\, \mathcal{G}_z^{y_j} \otimes \overline{\mathcal{G}_z^{y_\ell}}, \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, $\Gamma_{\alpha,Y}(z)$ is an $N \times N$ symmetric matrix whose entries are entire holomorphic functions of $z \in \mathbb{C}$ given by

$$\Gamma_{\alpha,Y}(z) \;:=\; \left(\left(\alpha_j - \frac{iz}{4\pi}\right)\delta_{j\ell} - \mathcal{G}_z(y_j - y_\ell)\hat{\delta}_{j\ell}\right)_{j,\ell=1,\ldots,N}, \tag{2}$$

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where $\delta_{j\ell} = 1$ for $j = \ell$ and $\delta_{j\ell} = 0$ otherwise; $\hat{\delta}_{j\ell} = 1 - \delta_{j\ell}$; $\mathcal{G}_z(x)$ is the convolution kernel of $(H_0 - z^2)^{-1}$:

$$\mathcal{G}_z(x) = \frac{e^{iz|x|}}{4\pi|x|} \quad \text{and} \quad \mathcal{G}_z^y(x) = \frac{e^{iz|x-y|}}{4\pi|x-y|}. \tag{3}$$

Since $(H_{\alpha,Y} - z^2)^{-1} - (H_0 - z^2)^{-1}$ is of rank N by virtue of (1), the wave operators $W^{\pm}_{\alpha,Y}$ defined by the limits

$$W^{\pm}_{\alpha,Y} u = \lim_{t\to\pm\infty} e^{itH_{\alpha,Y}} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{4}$$

exist and are complete in the sense that Image $W^{\pm}_{\alpha,Y} = \mathcal{H}_{ac}$, the absolutely continuous (AC for short) subspace of $\mathcal{H}$ for $H_{\alpha,Y}$. Wave operators are of fundamental importance in scattering theory.

This paper is concerned with the approximation of the wave operators $W^{\pm}_{\alpha,Y}$ by the ones for Schrödinger operators with regular potentials and generalizes a result in [5] for the case $N = 1$, which immediately implies that $W^{\pm}_{\alpha,Y}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$, see remarks below Theorem 1.1. We also give a proof of the corresponding well known result for the resolvent $(H_{\alpha,Y} - z)^{-1}$ which substantially simplifies the one in the seminal monograph [2].

We begin with recalling various properties of $H_{\alpha,Y}$ (see [2]):

- Equation (1) defines a unique selfadjoint operator $H_{\alpha,Y}$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, which is real and local.
- The spectrum of $H_{\alpha,Y}$ consists of the AC part $[0,\infty)$ and at most N non-positive eigenvalues. Positive eigenvalues are absent. We define $\mathcal{E} = \{ik \in i\mathbb{R}^+ \colon -k^2 \in \sigma_p(H_{\alpha,Y})\}$. We simply write $\mathcal{H}_{ac}$ and P_{ac} respectively for the AC subspace $\mathcal{H}_{ac}(H_{\alpha,Y})$ of $\mathcal{H}$ for $H_{\alpha,Y}$ and for the projection $P_{ac}(H_{\alpha,Y})$ onto $\mathcal{H}_{ac}$.
- $H_{\alpha,Y}$ may be approximated by a family of Schrödinger operators with scaled regular potentials

$$\overline{H}_Y(\varepsilon) = -\Delta + \sum_{i=1}^{N} \frac{\lambda_i(\varepsilon)}{\varepsilon^2} V_i\left(\frac{x - y_i}{\varepsilon}\right), \tag{5}$$

  in the sense that for $z \in \mathbb{C}^+$

$$\lim_{\varepsilon\to 0} (\overline{H}_Y(\varepsilon) - z^2)^{-1} u = (H_{\alpha,Y} - z^2)^{-1} u, \quad \forall u \in \mathcal{H}, \tag{6}$$

  where V_j, $j = 1, \ldots, N$ are such that $H_j = -\Delta + V_j(x)$ have threshold resonances at 0 and $\lambda_1(\varepsilon), \ldots, \lambda_N(\varepsilon)$ are smooth real functions of ε such that $\lambda_j(0) = 1$ and $\lambda_j'(0) \neq 0$ (see Theorem 1.1 for more details).

We prove the following theorem (see Section 4 for the definition of the threshold resonance).

Theorem 1.1. *Let Y be the set of N points $Y = \{y_1, \ldots, y_N\}$. Suppose that:*

(1) $V_1, \dots, V_N$ *are real-valued functions such that for some* $p < 3/2$ *and* $q > 3$,

$$\langle x\rangle^2 V_j \in (L^p \cap L^q)(\mathbb{R}^3), \quad j = 1, \dots, N. \tag{7}$$

(2) $\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$ *are real* C^2 *functions of* $\varepsilon \geq 0$ *such that*

$$\lambda_j(0) = 1, \quad \lambda_j'(0) \neq 0, \quad \forall j = 1, \dots, N.$$

(3) $H_j = -\Delta + V_j$, $j = 1, \dots, N$ *admits a threshold resonance at* 0.

Then, the following statements are satisfied:

(a) $\overline{H}_Y(\varepsilon)$ *converges in the strong resolvent sense as in* (6) *as* $\varepsilon \to 0$ *to a Schrödinger operator* $H_{\alpha,Y}$ *with point interactions at* Y *with certain parameters* $\alpha = (\alpha_1, \dots, \alpha_N)$ *to be specified below.*

(b) *Wave operators* $W^{\pm}_{Y,\varepsilon}$ *for the pair* $(\overline{H}_Y(\varepsilon), H_0)$ *defined by the strong limits*

$$W^{\pm}_{Y,\varepsilon} u = \lim_{t\to\pm\infty} e^{it\overline{H}_Y(\varepsilon)} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{8}$$

exist and are complete. $W^{\pm}_{Y,\varepsilon}$ *satisfy*

$$\lim_{\varepsilon\to 0} \|W^{\pm}_{Y,\varepsilon} u - W^{\pm}_{\alpha,Y} u\|_{\mathcal{H}} = 0, \quad u \in \mathcal{H}. \tag{9}$$

Note that Hölder's inequality implies $V_j \in L^r(\mathbb{R}^3)$ for all $1 \leq r \leq q$ under the condition (7).

Remark 1.2. (i) It is known that $W^{\pm}_{Y,\varepsilon}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$ ([14]) and, if $\lambda_j(\varepsilon) = 1$ for all $j = 1, \dots, N$, $\|W^{\pm}_{Y,\varepsilon}\|_{\mathbf{B}(L^p)}$ is independent of $\varepsilon > 0$ and, the proof of Theorem 1.1 shows that Theorem 1.1 holds with $\alpha = 0$. It follows by virtue of (9) that $W_{Y,\varepsilon}$ converges to $W_{\alpha=0,Y}$ weakly in L^p and $W^{\pm}_{\alpha=0,Y}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$. Actually, the latter result is known for general $\alpha = (\alpha_1 \dots, \alpha_N)$ but its proof is long and complicated ([5]). Wave operators satisfy the intertwining property

$$f(H_{\alpha,Y})\mathcal{H}_{ac}(H_{\alpha,Y}) = W^{\pm *}_{\alpha,Y} f(H_0) W^{\pm *}_{\alpha,Y}$$

for Borel functions f on $\mathbb{R}$ and, L^p mapping properties of $f(H_{\alpha,Y})P_{ac}(H_{\alpha,Y})$ are reduced to those for the Fourier multiplier $f(H_0)$ for a certain range of p's.

(ii) If some of $H_j = -\Delta + V_j$ have no threshold resonance, then Theorem 1.1 remains to hold if corresponding points of interactions and parameters (y_j, α_j) are removed from $H_{\alpha,Y}$.

(iii) The first statement is long known (see [2]). We shall present here a simplified proof, providing in particular details of the proof of Lemma 1.2.3 of [2] where [6] is referred to for "a tedious but straightforward calculation" by using a result from [4] and a simple matrix formula.

(iv) The existence and the completeness of wave operators $W^{\pm}_{Y,\varepsilon}$ are well known (cf. [11]).

(v) When $N = 1$ and $\alpha = 0$, (9) is proved in [5]. The theorem is a generalization for general α and $N \geq 2$.

(vi) The matrix $\Gamma_{\alpha,Y}(k)$ is non-singular for all $k \in (0,\infty)$ by virtue of the selfadjointness of $H_{\alpha,Y}$ and H_0. Indeed, if it occurred that $\det \Gamma_{\alpha,Y}(k_0) = 0$ for some $0 < k_0$, then the selfadjointness of $H_{\alpha,Y}$ and H_0 implied that $\Gamma_{\alpha,Y}(k)^{-1}$ had a simple pole at k_0 and

$$\begin{aligned}(10)\qquad & 2k_0 \mathrm{Res}_{z=k_0} (\Gamma_{\alpha,Y}(z)^{-1})_{j\ell} (\mathcal{G}_z^{y_j}, v)(u, \mathcal{G}_z^{y_\ell}) \\ &= \lim_{z=k_0+i\varepsilon, \varepsilon\downarrow 0} (z^2 - k_0^2) \sum_{j,\ell=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{j\ell} (\mathcal{G}_z^{y_j}, v)(u, \mathcal{G}_z^{y_\ell}) \neq 0\end{aligned}$$

for some $u, v \in C_0^\infty(\mathbb{R}^3)$. However, the absence of positive eigenvalues of $H_{\alpha,Y}$ (see [2, pp. 116–117]) and the Lebesgue dominated convergence theorem imply for all $u, v \in C_0^\infty(\mathbb{R}^3)$ that

$$\begin{aligned}&\lim_{z=k_0+i\varepsilon, \varepsilon\downarrow 0} (z^2 - k_0^2)((H_{\alpha,Y} - z^2)^{-1} u, v) \\ &= \lim_{z=k_0+i\varepsilon, \varepsilon\downarrow 0} \int_{\mathbb{R}} \frac{2ik_0\varepsilon - \varepsilon^2}{\mu - (k_0 + i\varepsilon)^2} (E(d\mu)u, v) = (E(\{k_0^2\})u, v) = 0\end{aligned}$$

and the likewise for $(z^2 - k_0^2)((H_0 - z^2)^{-1} u, v)$, where $E(d\mu)$ is the spectral projection for $H_{\alpha,Y}$, which contradict (10).

For more about point interactions we refer to the monograph [2] or the introduction of [5] and jump into the proof of Theorem 1.1 immediately. We prove (9) only for $W_{Y,\varepsilon}^{+}$ as $\overline{H}_Y(\varepsilon)$ and $H_{\alpha,Y}$ are real operators and the complex conjugation $\mathcal{C}$ changes the direction of the time which implies $W_{Y,\varepsilon}^{-} = \mathcal{C} W_{Y,\varepsilon}^{+} \mathcal{C}^{-1}$.

We write $\mathcal{H}$ for $L^2(\mathbb{R}^3)$, (u, v) for the inner product and $\|u\|$ the norm. $u \otimes v$ and $|u\rangle\langle v|$ indiscriminately denote the one dimentional operator

$$(u \otimes v) f(x) = |u\rangle\langle v|f\rangle(x) = \int_{\mathbb{R}^3} u(x)\overline{v(y)} f(y) dy.$$

Integral operators T and their integral kernels $T(x, y)$ are identified. Thus we often say that operator $T(x, y)$ satisfies such and such properties and etc. $\mathbf{B}_2(\mathcal{H})$ is the space of Hilbert-Schmidt operators in $\mathcal{H}$ and

$$\|T\|_{HS} = \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |T(x, y)|^2 dx dy \right)^{1/2}$$

is the norm of $\mathbf{B}_2(\mathcal{H})$. $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $a \le_{|\,\cdot\,|} b$ means $|a| \le |b|$. For subsets D_1 and D_2 of the complex plane $\mathbb{C}$, $D_1 \Subset D_2$ means $\overline{D_1}$ is a compact subset of the interior of D_2.

2. Scaling

For $\varepsilon > 0$, we let

$$(U_\varepsilon f)(x) = \varepsilon^{-3/2} f(x/\varepsilon).$$

This is unitary in $\mathcal{H}$ and $H_0 = \varepsilon^2 U_\varepsilon^* H_0 U_\varepsilon$. We define $H(\varepsilon)$ by

$$(11)\qquad H(\varepsilon) = \varepsilon^2 U_\varepsilon^* \overline{H}_Y(\varepsilon) U_\varepsilon, \quad (\overline{H}_Y(\varepsilon) - z^2)^{-1} = \varepsilon^2 U_\varepsilon (H(\varepsilon) - \varepsilon^2 z^2)^{-1} U_\varepsilon^*.$$

Then, $H(\varepsilon)$ is written as

$$H(\varepsilon) = -\Delta + \sum_{i=1}^{N} \lambda_i(\varepsilon) V_i \left(x - \frac{y_i}{\varepsilon}\right) \equiv -\Delta + V(\varepsilon)$$

and $W^{\pm}_{Y,\varepsilon}$ are transformed as

$$W^{\pm}_{Y,\varepsilon} = \lim_{t\to\pm\infty} U_\varepsilon e^{itH(\varepsilon)/\varepsilon^2} e^{-itH_0/\varepsilon^2} U^*_\varepsilon = U_\varepsilon W^{\pm}_Y(\varepsilon) U^*_\varepsilon, \tag{12}$$

$$W^{\pm}_Y(\varepsilon) = \lim_{t\to\pm\infty} U_\varepsilon e^{itH(\varepsilon)} e^{-itH_0} U^*_\varepsilon\,. \tag{13}$$

We write the translation operator by $\varepsilon^{-1} y_j$ by

$$\tau_{j,\varepsilon} f(x) = f\left(x + \frac{y_j}{\varepsilon}\right), \quad j = 1, \dots, N.$$

When $\varepsilon = 1$, we simply denote $\tau_j = \tau_{j,1}$, $j = 1, \dots, N$. Then,

$$V_j\left(x - \frac{y_j}{\varepsilon}\right) = \tau^*_{j,\varepsilon} V_j(x) \tau_{j,\varepsilon}.$$

3. Stationary representation

The following lemma is obvious and well known:

Lemma 3.1. *The subspace $\mathcal{D}_* = \{u \in L^2 \colon \hat{u} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})\}$ is a dense linear subspace of $L^2(\mathbb{R}^3)$.*

It is obvious that $\|W^+_{Y,\varepsilon} u\| = \|W^+_{\alpha,Y} u\| = \|u\|$ for every $u \in \mathcal{H}$ and, for proving (9) it suffices to show that

$$\lim_{\varepsilon\to 0} (W^+_{Y,\varepsilon} u, v) = (W^+_{\alpha,Y} u, v), \quad u, v \in \mathcal{D}_*. \tag{14}$$

We express $W^+_{Y,\varepsilon}$ and $W^+_{\alpha,Y}$ via stationary formulae. We recall from [5] the following representation formula for $W^+_{\alpha,Y}$.

Lemma 3.2. *Let $u, v \in \mathcal{D}_*$ and let $\Omega_{j\ell} u$ be defined for $j, \ell \in \{1, \dots, N\}$ by*

$$\frac{1}{\pi i} \int_0^\infty \left(\int_{\mathbb{R}^3} (\Gamma_{\alpha,Y}(-k)^{-1})_{j\ell}\, \mathcal{G}_{-k}(x) \big(\mathcal{G}_k(y) - \mathcal{G}_{-k}(y)\big) u(y) dy \right) k dk. \tag{15}$$

Then

$$\langle W^+_{\alpha,Y} u, v\rangle \;=\; \langle u, v\rangle + \sum_{j,\ell=1}^{N} \langle \tau_j^* \Omega_{j\ell} \tau_\ell u, v\rangle. \tag{16}$$

Note that for $u \in \mathcal{D}_*$ the inner integral in (15) *produces a smooth function of $k \in \mathbb{R}$ which vanishes outside the compact set $\{|\xi| \colon \xi \in \operatorname{supp} \hat{u}\}$.*

For describing the formula for $W^+_{Y,\varepsilon}$ corresponding to (15) and (16), we introduce some notation. $\mathcal{H}^{(N)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ is the N-fold direct sum of $\mathcal{H}$.

Likewise $T^{(N)} = T \oplus \cdots \oplus T$ for an operator T on $\mathcal{H}$. For $i = 1, \dots, N$ we decompose $V_i(x)$ as the product:

$$V_i(x) = a_i(x)b_i(x), \quad a_i(x) = |V_i(x)|^{1/2}, \quad b_i(x) = |V_i(x)|^{1/2}\text{sign}(V_i(x)),$$

where $\operatorname{sign} a = \pm 1$ if $\pm a > 0$ and $\operatorname{sign} a = 0$ if $a = 0$. We use matrix notation for operators on $\mathcal{H}^{(N)}$. Thus, we define

$$A = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_N \end{pmatrix}, \; B = \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_N \end{pmatrix}, \; \Lambda(\varepsilon) = \begin{pmatrix} \lambda_1(\varepsilon) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N(\varepsilon) \end{pmatrix}.$$

Since a_j, b_j and $\lambda_j(\varepsilon)$, $j = 1, \dots, N$ are real valued, multiplications with A, B and $\Lambda(\varepsilon)$ are selfadjoint operators on $\mathcal{H}^{(N)}$. We also define the operator τ_ε by

$$\tau_\varepsilon \colon \mathcal{H} \ni f \mapsto \tau_\varepsilon f = \begin{pmatrix} \tau_{1,\varepsilon} f \\ \vdots \\ \tau_{N,\varepsilon} f \end{pmatrix} \in \mathcal{H}^{(N)}$$

so that

$$V(\varepsilon) = \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j\left(x - \frac{y_j}{\varepsilon}\right) = \tau_\varepsilon^* A \Lambda(\varepsilon) B \tau_\varepsilon.$$

We write for the case $\varepsilon = 1$ simply as $\tau = \tau_1$ as previously. For $z \in \mathbb{C}$, $G_0(z)$ is the integral operator defined by

$$G_0(z)u(y) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{iz|x-y|}}{|x-y|} u(y) dy.$$

It is a holomorphic function of $z \in \mathbb{C}^+$ with values in $\mathbf{B}(\mathcal{H})$ and

$$G_0(z) = (H_0 - z^2)^{-1} \quad \text{for} \quad z \in \mathbb{C}$$

and, it can be extended to various subsets of $\mathbb{C}^+$ when considered as a function with values in a space of operators between suitable function spaces. We also write

$$G_\varepsilon(z) = (H(\varepsilon) - z^2)^{-1} \text{ for } z \in \mathbb{C}^+ \setminus \{z \colon z^2 \in \sigma_p(H(\varepsilon))\}.$$

Lemma 3.3. *Let $V_1, \dots, V_N$ satisfy the assumption* (7) *and $z \in \overline{\mathbb{C}}^+$. Then*:

(1) $a_i, b_j \in L^2(\mathbb{R}^3)$, $i, j = 1, \dots, N$.

(2) $a_i G_0(z) b_j \in \mathbf{B}_2(\mathcal{H})$, $1 \le i, j \le N$.

Proof. (1) We have $a_i, b_j \in L^2(\mathbb{R}^3)$ for $V_j \in L^1(\mathbb{R}^3)$ as was remarked below Theorem 1.1.

(2) We also have $|a_j|^2 = |b_j|^2 = |V_j| \in L^{3/2}(\mathbb{R}^3)$ and $|x|^{-2} \in L^{3/2,\infty}(\mathbb{R}^3)$. It follows by the generalized Young inequality that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|a_i(x)|^2 |b_j(y)|^2}{|x-y|^2} dxdy \le C \|V_i\|_{L^{3/2}} \|V_j\|_{L^{3/2}}.$$

Hence, $a_i G_0(z) b_j$ is of Hilbert-Schmidt type in $L^2(\mathbb{R}^3)$. ☐

Using this notation, we have from (16) that

$$(W^+_{\alpha,Y}u,v) = (u,v) + \left\langle (\Omega_{j\ell})\tau^* u, \tau^* v\right\rangle_{\mathcal{H}^{(N)}}. \tag{17}$$

The resolvent equation for $H(\varepsilon)$ may be written as

$$G_\varepsilon(z) - G_0(z) = -G_0(z)\tau_\varepsilon^* A\Lambda(\varepsilon)B\tau_\varepsilon G_\varepsilon(z)$$

and the standard argument (see e.g. [13]) yields

$$G_\varepsilon(z) = G_0(z) - G_0(z)\tau_\varepsilon^* A(1+\Lambda(\varepsilon)B\tau_\varepsilon G_0(z)\tau_\varepsilon^* A)^{-1}\Lambda(\varepsilon)B\tau_\varepsilon G_0(z). \tag{18}$$

Note that $\tau_\varepsilon R_0(z)\tau_\varepsilon^* \neq R_0(z)$ in general unless $N = 1$.

Under the assumption (7) on $V_1, \dots, V_N$ the first two statements of the following lemma follow from the limiting absorption principle for the free Schrödinger operator ([1], [7], [12]) and the last from the absence of positive eigenvalues for $H(\varepsilon)$ ([10]). In what follows we often write k for z when we want emphasize that k can also be real.

Lemma 3.4. *Suppose that $V_1, \dots, V_N$ satisfy the assumption of* Theorem 1.1. *Let $0 < \varepsilon \le 1$. Then*:

(1) *For $u \in \mathcal{D}_*$, $\lim_{\delta\downarrow 0}\sup_{k\in\mathbb{R}} \|A\tau_\varepsilon G_0(k+i\delta)u - A\tau_\varepsilon G_0(k)u\|_{\mathcal{H}^{(N)}} = 0$.*

(2) $\lim_{\delta\downarrow 0}\sup_{k\in\mathbb{R}} \|\Lambda(\varepsilon)A\tau_\varepsilon(G_0(k+i\delta) - G_0(k))\tau_\varepsilon^* A\|_{\mathbf{B}(\mathcal{H}^{(N)})} = 0$.

(3) *Define for $k \in \overline{\mathbb{C}}^+ = \{k \in \Im k \ge 0\}$,*

$$M_\varepsilon(k) = \Lambda(\varepsilon)B\tau_\varepsilon G_0(k)\tau_\varepsilon^* A. \tag{19}$$

Then, $M_\varepsilon(k)$ is a compact operator on $\mathcal{H}^{(N)}$ and $1+M_\varepsilon(k)$ is invertible for all $k \neq 0$. $(1+M_\varepsilon(k))^{-1}$ is a locally Hölder continuous function of $\overline{\mathbb{C}}^+ \setminus \{0\}$ with values in $\mathbf{B}(\mathcal{H}^{(N)})$.

Statements (1) *and* (2) *remain to hold when A is replaced by B.*

The well known stationary formula for wave operators ([12]) and the resolvent equation (18) yield

$$\begin{aligned}(20)\quad &(W^+_Y(\varepsilon)u,v) - (u,v)\\ &= -\frac{1}{\pi i}\int_0^\infty \big((1+M_\varepsilon(-k))^{-1}\Lambda(\varepsilon)B\tau_\varepsilon\{G_0(k)-G_0(-k)\}u, A\tau_\varepsilon G_0(k)v\big)\,kdk.\end{aligned}$$

For obtaining the corresponding formula for $W^+_{Y,\varepsilon}$, we scale back (20) by using the identity (12) and (13). Then

$$\tau_\varepsilon U_\varepsilon^* = U_\varepsilon^* \tau,$$

and change of variable k to εk produce the first statement of the following lemma. Recall $\tau = \tau_{\varepsilon=1}$. The second formula is proven in parallel with the first by using (11).

Lemma 3.5. (1) *For $u, v \in \mathcal{D}^*$, we have*

$$(W^+_{Y,\varepsilon}u,v) = (u,v) - \frac{\varepsilon^2}{\pi i}\int_0^\infty kdk\,\big((1+M_\varepsilon(-\varepsilon k))^{-1}\Lambda(\varepsilon) \tag{21}$$

$$\times\, B\{G_0(k\varepsilon)-G_0(-k\varepsilon)\}^{(N)}U_\varepsilon^*\tau u, AG_0(k\varepsilon)^{(N)}U_\varepsilon^*\tau v\Big).$$

(2) *For $k\in\mathbb{C}^+$ with sufficiently large $\Im k$,*

$$(\overline{H}_Y(\varepsilon)-k^2)^{-1}=G_0(k)-\varepsilon^2\tau^*U_\varepsilon G_0(k\varepsilon)^{(N)}A(1+M_\varepsilon(\varepsilon k))^{-1} \times \Lambda(\varepsilon)BG_0(k\varepsilon)^{(N)}U_\varepsilon^*\tau, \tag{22}$$

where $G_0(\pm k\varepsilon)^{(N)}=G_0(\pm k\varepsilon)\oplus\cdots\oplus G_0(\pm k\varepsilon)$ is the N-fold direct sum of $G_0(\pm k\varepsilon)$.

Notice that for $u\in\mathcal{D}_*$, $\{G_0(k\varepsilon)-G_0(-k\varepsilon)\}^{(N)}U_\varepsilon^*\tau u\neq 0$ only for $R^{-1}<k<R$ for some $R>0$ and the integral on the right of (21) is only over $[R^{-1},R]\subset(0,\infty)$ uniformly for $0<\varepsilon<1$. Indeed, if $u\in\mathcal{D}_*$ and $\hat{u}(\xi)=0$ unless $R^{-1}\le|\xi|\le R$ for some $R>1$, then, since the translation τ does not change the support of $\hat{u}(\xi/\varepsilon)$, we have

$$\mathcal{F}(U_\varepsilon^*\tau u)(\xi)=\varepsilon^{-\frac{3}{2}}\mathcal{F}(\tau u)\left(\frac{\xi}{\varepsilon}\right)=0$$

unless $R^{-1}\varepsilon\le|\xi|\le R\varepsilon$ and

$$\{G_0(k\varepsilon)-G_0(-k\varepsilon)\}U_\varepsilon^*\tau u=2i\pi\delta(\xi^2-k^2\varepsilon^2)\mathcal{F}(U_\varepsilon^*\tau u)(\xi)=0$$

for $k>R$ or $k<R^{-1}$.

4. Limits as $\varepsilon\to 0$

We study the small $\varepsilon>0$ behavior of the right hand sides of (21) and (22). For (21), the argument above shows that we need only consider the integral over a compact set $K\equiv[R^{-1},R]\subset\mathbb{R}$ *which will be fixed in this section.* Splitting $\varepsilon^2=\varepsilon\cdot\varepsilon^{1/2}\cdot\varepsilon^{1/2}$ in front of the second term on the right, we place one $\varepsilon^{1/2}$ each in front of $BG_0(\pm k\varepsilon)^{(N)}U_\varepsilon^*$ and $AG_0(\pm k\varepsilon)^{(N)}U^*$ or $U_\varepsilon G_0(k\varepsilon)^{(N)}A$ and the remaining ε in front of $(1+M_\varepsilon(\pm\varepsilon k))^{-1}$. We begin with the following lemma. Recall the definition (3) of $\mathcal{G}_k$.

Lemma 4.1. *Suppose $a\in L^2(\mathbb{R}^3)$. Then, following statements are satisfied:*

(1) *Let $u\in\mathcal{D}_*$. Then, uniformly in $k\in K$, we have*

$$\lim_{\varepsilon\to 0}\|\varepsilon^{\frac{1}{2}}aG_0(\pm k\varepsilon)U_\varepsilon^*u-|a\rangle\langle\mathcal{G}_{\pm k},u\rangle\|_{L^2}=0. \tag{23}$$

(2) *Let $u\in L^2(\mathbb{R}^3)$. Then, uniformly on compacts of $k\in\mathbb{C}^+$, we have*

$$\|\varepsilon^{\frac{1}{2}}aG_0(k\varepsilon)U_\varepsilon^*u\|_{L^2}\le C(\Im k)^{-1/2}\|a\|_{L^2}\|u\|_{L^2} \tag{24}$$

and the convergence (23) *with k in place of $\pm k$.*

(3) *Let $u\in L^2(\mathbb{R}^3)$. Then, uniformly on compacts of $k\in\mathbb{C}^+$, we have*

$$\lim_{\varepsilon\to 0}\|\varepsilon^{\frac{1}{2}}U_\varepsilon G_0(k\varepsilon)au-|\mathcal{G}_k\rangle\langle a,u\rangle\|_{L^2}=0. \tag{25}$$

Proof. (1) We prove the + case only. The proof for the − case is similar. We have $u \in \mathcal{S}(\mathbb{R}^3)$ and

$$\varepsilon^{\frac{1}{2}} G_0(k\varepsilon)U_\varepsilon^* u(x) = \frac{1}{4\pi}\varepsilon^2 \int_{\mathbb{R}^3} \frac{e^{ik\varepsilon|x-y|}}{|x-y|} u(\varepsilon y)dy = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{ik|y|}}{|y|} u(y+\varepsilon x)dy.$$

It is then obvious for any $R > 0$ and a compact $K \subset \mathbb{R}$ that

$$\lim_{\varepsilon\to 0} \sup_{|x|\le R, k\in K} |\varepsilon^{\frac{1}{2}} G_0(k\varepsilon)U_\varepsilon^* u(x) - \langle \mathcal{G}_k, u\rangle| = 0. \tag{26}$$

Moreover, Hölder's inequality in Lorentz spaces implies that

$$|\langle \mathcal{G}_k, u\rangle| + \|\varepsilon^{\frac{1}{2}} G_0(k\varepsilon)U_\varepsilon^* u\|_\infty \le \|(4\pi|x|)^{-1}\|_{3,\infty}\|u\|_{\frac{3}{2},1}. \tag{27}$$

It follows from (26) that for any $R > 0$

$$\lim_{\varepsilon\to 0} \sup_{k\in K} \|\varepsilon^{\frac{1}{2}} a G_0(k\varepsilon)U_\varepsilon^* u - a\langle \mathcal{G}_k, u\rangle\|_{L^2(|x|\le R)} = 0 \tag{28}$$

and, from (27) that

$$\begin{aligned} &\|\varepsilon^{\frac{1}{2}} a G_0(k\varepsilon)U_\varepsilon^* u - a\langle \mathcal{G}_k, u\rangle\|_{L^2(|x|\ge R)} \\ &\le 2\|a\|_{L^2(|x|\ge R)}\|(4\pi|x|)^{-1}\|_{3,\infty}\|u\|_{\frac{3}{2},1} \to 0. \end{aligned} \tag{29}$$

Combining (26) and (29), we obtain (23) for $u \in \mathcal{D}_*$. (Since $\mathcal{D}_*$ is dense in $L^{3,1}(\mathbb{R}^3)$, (23) actually holds for $u \in L^{\frac{3}{2},1}(\mathbb{R}^3)$.)

(2) We have

$$\|aG_0(k\varepsilon)\|_{HS}^2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|a(x)|^2 e^{-2\Im k\varepsilon|x-y|}}{16|x-y|^2} dxdy \le C(\Im k\varepsilon)^{-1}\|a\|_{L^2}^2.$$

This implies (24) as U_ε^* is unitary in $L^2(\mathbb{R}^3)$ and it suffices to prove the strong convergence in L^2 for $u \in C_0^\infty(\mathbb{R}^3)$. This, however, follows as in the case (1).

(3) We have

$$\varepsilon^{\frac{1}{2}}(U_\varepsilon G_0(k\varepsilon)au)(x) = \int_{\mathbb{R}^3} \frac{e^{ik|x-\varepsilon y|}}{4\pi|x-\varepsilon y|} a(y)u(y)dy$$

and Minkowski's inequality implies

$$\|\varepsilon^{\frac{1}{2}} U_\varepsilon G_0(k\varepsilon)au - |\mathcal{G}_k\rangle\langle a, u\rangle\| \le \int_{\mathbb{R}^3} \|\mathcal{G}_k(\cdot - \varepsilon y) - \mathcal{G}_k\|_{L^2(\mathbb{R}^3)}|a(y)u(y)|dy. \tag{30}$$

Plancherel's and Lebesgue's dominated convergence theorems imply that for a compact subset $\tilde{K}$ of $\mathbb{C}^+$

$$\begin{aligned} \sup_{k\in\tilde{K}} \|\mathcal{G}_k(\cdot + \varepsilon y) - \mathcal{G}_k\| &= \sup_{k\in\tilde{K}} \|(\mathcal{F}^{-1}\mathcal{G}_k)(\xi)(e^{\varepsilon y\xi} - 1)\|_{L^2(\mathbb{R}^3_\xi)} \\ &= \left(\int_{\mathbb{R}^3} \sup_{k\in\tilde{K}} |(|\xi|^2 - k^2)^{-1}(e^{i\varepsilon y\xi} - 1)|^2 d\xi\right)^{\frac{1}{2}} \\ &\le C\left(\int_{\mathbb{R}^3} \langle\xi\rangle^{-4}|(e^{i\varepsilon y\xi} - 1)|^2 d\xi\right)^{\frac{1}{2}} \end{aligned}$$

is uniformly bounded for $y \in \mathbb{R}^3$ and converges to 0 as $\varepsilon \to 0$. Thus, (25) follows from (30) by applying Lebesgue's dominated convergence theorem. $\square$

We next study $\varepsilon(1+M_\varepsilon(\varepsilon k))^{-1}$ for $\varepsilon \to 0$ and $k \in \overline{\mathbb{C}}^+ \setminus \{0\}$. We decompose $M_\varepsilon(k) = \Lambda(\varepsilon)B\tau_\varepsilon G_0(\varepsilon k)\tau_\varepsilon^* A$ into the diagonal and the off-diagonal parts:

$$M_\varepsilon(k) = D_\varepsilon(\varepsilon k) + \varepsilon E_\varepsilon(\varepsilon k), \tag{31}$$

where the diagonal part is given by

$$D_\varepsilon(\varepsilon k) = \begin{pmatrix} \lambda_1(\varepsilon)b_1 G_0(\varepsilon k)a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N(\varepsilon)b_N G_0(\varepsilon k)a_N \end{pmatrix} \tag{32}$$

and, the off diagonal part $\varepsilon E_\varepsilon(\varepsilon k) = \left(\lambda_i(\varepsilon)b_i\tau_{i,\varepsilon}G_0(\varepsilon k)\tau_{j,\varepsilon}^* a_j\hat{\delta}_{ij}\right)$ by

$$\varepsilon E_\varepsilon(\varepsilon k) = \varepsilon\left(\lambda_i(\varepsilon)\frac{b_i(x)e^{ik|\varepsilon(x-y)+y_i-y_j|}a_j(y)}{4\pi|\varepsilon(x-y)+y_i-y_j|}\hat{\delta}_{ij}\right)_{ij}. \tag{33}$$

We study $E_\varepsilon(\varepsilon k)$ first. Define constant matrix $\hat{\mathcal{G}}(k)$ by

$$\hat{\mathcal{G}}_{ij}(k) = \mathcal{G}_{ij}(k)\hat{\delta}_{ij}, \quad \mathcal{G}_{ij}(k) = \frac{1}{4\pi}\frac{e^{ik|y_i-y_j|}}{|y_i-y_j|}, \quad i \neq j.$$

Lemma 4.2. *Assume* (7) *and let* $\Omega \subset \overline{\mathbb{C}}^+$ *be compact. We have uniformly for* $k \in \Omega$ *that*

$$\lim_{\varepsilon\to 0}\|E_\varepsilon(\pm\varepsilon k) - |B\rangle\hat{\mathcal{G}}(\pm k)\langle A|\|_{\mathbf{B}(\mathcal{H}^{(N)})} = 0. \tag{34}$$

$|B\rangle\hat{\mathcal{G}}(\pm k)\langle A|$ *is an operator of rank at most* N *on* $\mathcal{H}^{(N)}$:

$$|B\rangle\hat{\mathcal{G}}(\pm k)\langle A| \equiv \left(b_i(x)\mathcal{G}_{ij}(\pm k)a_j(y)\hat{\delta}_{ij}\right).$$

Proof. We prove the + case only. The − case may be proved similarly. Let $k \in K$. Then,

$$\left|\frac{e^{ik|\varepsilon(x-y)+y_i-y_j|}}{|\varepsilon(x-y)+y_i-y_j|} - \frac{e^{ik|y_i-y_j|}}{|y_i-y_j|}\right|$$

$$\leq \frac{|k||\varepsilon(x-y)|}{|\varepsilon(x-y)+y_i-y_j|} + \frac{|\varepsilon(x-y)|}{|\varepsilon(x-y)+y_i-y_j||y_i-y_j|} \tag{35}$$

$$\leq \frac{C|x-y|}{|(x-y)+(y_i-y_j)/\varepsilon|} \tag{36}$$

for a constant $C > 0$ and we may estimate as

$$\begin{aligned}\|(E_{\varepsilon,ij}(\varepsilon k) - \lambda_i(\varepsilon)b_i\mathcal{G}_{ij}(k)a_j)u\|_{L^2} &\leq C\left\|\int_{\mathbb{R}^3}\frac{|b_i(x)|x-y|a_j(y)u(y)|}{|(x-y)+(y_i-y_j)/\varepsilon|}dy\right\| \\ &\leq C\left\|\int_{\mathbb{R}^3}\frac{|\langle x\rangle b_i(x)\langle y\rangle a_j(y)u(y)|}{|(x-y)+(y_i-y_j)/\varepsilon|}dy\right\|\end{aligned}$$

$$= C \left\| \int_{\mathbb{R}^3} \frac{|\tau_{i,\varepsilon}(\langle x\rangle b_i)(x)\tau_{j,\varepsilon}(\langle y\rangle a_j u)(y)|}{|x-y|} dy \right\|.$$

Since the convolution with the Newton potential $|x|^{-1}$ maps $L^{\frac{6}{5}}(\mathbb{R}^3)$ to $L^6(\mathbb{R}^3)$ by virtue of Hardy-Littlewood-Sobolev's inequality, Hölder's inequality implies that the right hand side is bounded by

$$\begin{aligned} (37) \qquad & C\|\langle x\rangle b_i\|_{L^3}\|\langle y\rangle a_j u\|_{L^{6/5}} \\ & \le C\|\langle x\rangle b_i\|_{L^3}\|\langle x\rangle a_j\|_{L^3}\|u\|_{L^2} = C\|\langle x\rangle^2 V_i\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\langle x\rangle^2 V_j\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|u\|_{L^2}. \end{aligned}$$

Let $B_R(0) = \{x\colon |x| \le R\}$ for an $R > 0$. Then, for $\varepsilon > 0$ such that $4R\varepsilon < \min|y_i - y_j|$, we have

$$(35) \le 4C\varepsilon, \quad \forall x, y \in B_R(0).$$

Thus, if $V_j \in C_0^\infty(\mathbb{R}^3)$, $j = 1, \dots, N$ are supported by $B_R(0)$, then

$$\|E_\varepsilon(\varepsilon k) - \Lambda(\varepsilon)B\hat{\mathcal{G}}(k)A\|_{\mathbf{B}(\mathcal{H}^{(N)})} \le 4C\varepsilon \sum_{j=1}^N \|V_j\|_{L^1} \xrightarrow{\varepsilon\to 0} 0.$$

Since $C_0^\infty(\mathbb{R}^3)$ is a dense subspace of the Banach space $(\langle x\rangle^{-2}L^{3/2}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^3)$, (37) implies $\|E_\varepsilon(\varepsilon k) - \Lambda(\varepsilon)B\hat{\mathcal{G}}(k)A\|_{\mathbf{B}(\mathcal{H}^{(N)})} \to 0$ as $\varepsilon \to 0$ for general V_j's which satisfies the assumption (7). The lemma follows because $\Lambda(\varepsilon)$ converges to the identity matrix. $\square$

We have shown in Lemma 3.3 that $b_iG_0(k\varepsilon)a_j$ is of Hilbert-Schmidt type for $k \in \overline{\mathbb{C}}^+$ and it is well known that $1 + \lambda_j(\varepsilon)b_jG_0(k\varepsilon)a_j$ is an isomorphism of $\mathcal{H}$ unless $k^2\varepsilon^2$ is an eigenvalue of $H_j(\varepsilon) = -\Delta + \lambda_j(\varepsilon)V_j$ (see [7]). Hence, the absence of positive eigenvalues for $H_j(\varepsilon)$ (see e.g. [10]) implies that $1 + \lambda_j(\varepsilon)b_jG_0(k\varepsilon)a_j$ is an isomorphism in $\mathcal{H}$ for all $k \in \overline{\mathbb{C}}^+ \setminus (\varepsilon^{-1}i\mathcal{E}_j(\varepsilon) \cup \{0\})$ where $\mathcal{E}_j(\varepsilon) = \{k > 0\colon -k^2 \in \sigma_p(H_j(\varepsilon))\}$. Thus, if we fix a compact set $\Omega \subset \overline{\mathbb{C}}^+ \setminus \{0\}$. $1 + D_\varepsilon(\varepsilon k)$ is invertible in $\mathbf{B}(\mathcal{H}^{(N)})$ for small $\varepsilon > 0$ and $k \in \Omega$ and

$$1 + M_\varepsilon(\varepsilon k) = (1 + D_\varepsilon(\varepsilon k))(1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}E_\varepsilon(\varepsilon k)).$$

It follows that

$$(38) \qquad (1 + M_\varepsilon(\varepsilon k))^{-1} = (1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}E_\varepsilon(\varepsilon k))^{-1}(1 + D_\varepsilon(\varepsilon k))^{-1}$$

and we need study the right hand side of (38) as $\varepsilon \to 0$.

We begin by studying $\varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}$ and, since $1 + D_\varepsilon(\varepsilon k)$ is diagonal, we may do it component-wise. We first study the case $N = 1$.

4.1. Threshold analysis for the case $N = 1$

When $N = 1$, we have $M_\varepsilon(\varepsilon k) = D_\varepsilon(\varepsilon k)$.

Lemma 4.3. *Let $N = 1$, $a = a_1$ and etc. and, let Ω be compact in $\overline{\mathbb{C}}^+ \setminus \{0\}$. Then, for any $0 < \rho < \rho_0$, $\rho_0 = (3-p)/2p > 1/2$, we have following expansions in Ω in the space of Hilbert-Schmidt operators $\mathbf{B}_2(\mathcal{H})$:*

$$bG_0(k\varepsilon)a = bD_0a + ik\varepsilon bD_1a + O((k\varepsilon)^{1+\rho}), \tag{39}$$

$$M_\varepsilon(\varepsilon k) = bD_0a + \varepsilon\big(\lambda'(0)bD_0a + ikbD_1a\big) + O(\varepsilon^{1+\rho}), \tag{40}$$

$$D_0 = \frac{1}{4\pi|x-y|}, \quad D_1 = \frac{1}{4\pi}, \tag{41}$$

where $O((k\varepsilon)^{1+\rho})$ and $O(\varepsilon^{1+\rho})$ are $\mathbf{B}_2(\mathcal{H})$-valued functions of (k, ε) such that

$$\|O((k\varepsilon)^{1+\rho})\|_{HS} \le C|k\varepsilon|^{1+\rho}, \quad \|O(\varepsilon^{1+\rho})\|_{HS} \le C|\varepsilon|^{1+\rho}, \quad 0 < \varepsilon < 1,\ k \in \Omega.$$

Proof. Since $\Im k \ge 0$ for $k \in \Omega$, Taylor's formula and the interpolation imply that for any $0 \le \rho \le 1$ there exists a constant $C_\rho > 0$ such that

$$|e^{ik\varepsilon|x-y|} - (1 + ik\varepsilon|x-y|)\,| \le C_\rho|\varepsilon k|^{1+\rho}|x-y|^{1+\rho}.$$

Hence

$$\left|D_\varepsilon(\varepsilon k)(x,y) - \frac{b(x)a(y)}{4\pi|x-y|} - ik\varepsilon\frac{b(x)a(y)}{4\pi}\right| \le C_\rho|k|^{1+\rho}\varepsilon^{1+\rho}|x-y|^\rho|b(x)a(y)|.$$

We have shown in Lemma 3.3 that $D_\varepsilon(\varepsilon k)$ and bD_0a are Hilbert-Schmidt operators and bD_1a is evidently so as $a, b \in L^2(\mathbb{R}^3)$ (see the remark below Theorem 1.1). As $\langle x\rangle b(x), \langle y\rangle a(y) \in L^{2p}(\mathbb{R}^3)$, we have $\langle x\rangle^\rho a(x), \langle x\rangle^\rho a(y) \in L^2(\mathbb{R}^3)$ for $\rho < \rho_0$, and

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{2\rho}|b(x)a(y)|^2 dxdy \le C\|\langle x\rangle^\rho b(x)\|_{L^2}^2\|\langle y\rangle^\rho a(y)\|_{L^2}^2.$$

This prove estimate (39). (40) follows from (39) and Taylor's expansion of $\lambda(\varepsilon)$. This completes the proof of the lemma. ☐

We define

$$Q_0 = 1 + bD_0a, \quad Q_1 = \lambda'(0)bD_0a + ikbD_1a, \quad bD_1a = (4\pi)^{-1}|b\rangle\langle a|. \tag{42}$$

Regular case.

Definition. $H = -\Delta + V(x)$ is said to be of regular type at 0 if Q_0 is invertible in $\mathcal{H}$. It is of exceptional type if otherwise.

Lemma 4.4. *Suppose $N = 1$ and that $H = -\Delta + V(x)$ is of regular type at 0. Let Ω be a compact subset of $\overline{\mathbb{C}}^+$. Then*

$$\lim_{\varepsilon\to 0}\sup_{k\in\Omega}\|\varepsilon(1 + M_\varepsilon(\varepsilon k))^{-1}\|_{\mathbf{B}(\mathcal{H})} = 0. \tag{43}$$

Proof. Since $Q_0 = 1 + bD_0a$ is invertible, (40) implies the same for $1 + M_\varepsilon(\varepsilon k)$ for $k \in \Omega$ and small $\varepsilon > 0$ and,

$$\lim_{\varepsilon\to 0}\sup_{k\in\Omega}\|(1 + M_\varepsilon(\varepsilon k))^{-1} - Q_0^{-1}\|_{\mathbf{B}(\mathcal{H})} = 0.$$

(43) follows evidently. ☐

An application of Lemma 3.4, Lemma 4.1 and Lemma 4.4 to (21) and (22) immediately produces the following proposition for the case $N = 1$.

Proposition 4.5. *Suppose $H = -\Delta + V$ is of regular type at 0. Then:*

(1) *As $\varepsilon \to 0$, $W_{Y,\varepsilon}^{+}$ converges strongly to the identity operator.*
(2) *Let $\Omega_0 \subset \overline{\mathbb{C}}^{+}$ be compact. Then, $a(\overline{H}_Y(\varepsilon) - k^2)^{-1}b - aG_0(k)b \to 0$ in the norm of $\mathbf{B}(\mathcal{H})$ as $\varepsilon \to 0$ uniformly with respect to $k \in \Omega_0$.*
(3) *Let $\Omega_1 \Subset \mathbb{C}^{+}$. Then, $\lim_{\varepsilon\to 0} \sup_{k\in\Omega_1} \|(\overline{H}_Y(\varepsilon) - k^2)^{-1} - G_0(k)\|_{\mathbf{B}(\mathcal{H})} = 0$.*

Exceptional case. Suppose next that Q_0 is *not* invertible and define

$$\mathcal{M} =: \operatorname{Ker} Q_0, \quad \mathcal{N} = \operatorname{Ker} Q_0^*, \quad Q_0^* = 1 + aD_0 b.$$

By virtue of the Riesz-Schauder theorem $\dim \mathcal{M} = \dim \mathcal{N}$ are finite and $\mathcal{M}$ and $\mathcal{N}$ are dual spaces of each other with respect to the inner product of $\mathcal{H}$. Let S be the Riesz projection onto $\mathcal{M}$.

Lemma 4.6. (1) *aD_0a is an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$ and bD_0b from $\mathcal{N}$ onto $\mathcal{M}$. They are inverses of each other.*
(2) *$(a\varphi, D_0a\varphi)$ is an inner product on $\mathcal{M}$ and $(b\psi, D_0b\psi)$ on $\mathcal{N}$.*
(3) *For an orthonormal basis $\{\varphi_1, \dots, \varphi_n\}$ of $\mathcal{M}$ with respect to the inner product $(a\varphi, D_0a\varphi)$, define $\psi_j = aD_0a\varphi_j$, $j = 1, \dots, n$. Then:*
   (a) *$\{\psi_1, \dots, \psi_n\}$ is an orthonormal basis of $\mathcal{N}$ with respect to $(b\psi, D_0b\psi)$.*
   (b) *$\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ are dual basis of $\mathcal{M}$ and $\mathcal{N}$ respectively.*
   (c) *$Sf = \langle f, \psi_1\rangle\varphi_1 + \dots + \langle f, \psi_n\rangle\varphi_n$, $f \in \mathcal{H}$.*

Proof. (1) Let $\varphi \in \mathcal{M}$. Then, $\varphi = -bD_0a\varphi$ and $aD_0a\varphi = -aD_0b \cdot aD_0a\varphi$. Hence $aD_0a\varphi \in \mathcal{N}$. Likewise bD_0b maps $\mathcal{N}$ into $\mathcal{M}$. We have

$$bD_0b \cdot aD_0a\varphi = (bD_0a)^2\varphi = \varphi, \quad \varphi \in \mathcal{M},$$
$$aD_0a \cdot bD_0b\psi = (aD_0b)^2\psi = \psi, \quad \psi \in \mathcal{N}$$

and aD_0a and bD_0b are inverses of each other.

(2) Let $\varphi \in \mathcal{M}$. Then $a\varphi \in L^1 \cap L^\sigma$ for some $\sigma > 3/2$ (see the proof of Lemma 4.8 below) and $\widehat{a\varphi} \in L^\infty \cap L^\rho$ for some $\rho < 3$ by Hausdorff-Young's inequality. It follows that

$$(a\varphi, D_0a\varphi) = \int_{\mathbb{R}^3} \frac{|\widehat{a\varphi}(\xi)|^2}{|\xi|^2} d\xi \geq 0$$

and $(a\varphi, D_0a\varphi) = 0$ implies $a\varphi = 0$ hence, $\varphi = -bD_0a\varphi = 0$. Thus, $(a\varphi, D_0a\varphi)$ is an inner product of $\mathcal{M}$. The proof for $(b\psi, D_0b\psi)$ is similar.

(3) We have for any $j, k = 1, \dots, n$ that

$$(b\psi_j, D_0b\psi_k) = (baD_0a\varphi_j, D_0baD_0a\varphi_k) = (-a\varphi_j, -D_0a\varphi_k) = \delta_{jk}$$

and $\{\psi_1, \dots, \psi_n\}$ is orthonormal with respect to the inner product $(b\psi, D_0 b\psi)$. Since $n = \dim \mathcal{N}$, it is a basis of $\mathcal{N}$.

$$(\varphi_j, \psi_k) = (\varphi_j, aD_0 a\varphi_k) = (a\varphi_j, D_0 a\varphi_k) = \delta_{jk}, \quad j, k = 1, \dots, n.$$

Hence $\{\varphi_j\}$ and $\{\psi_k\}$ are dual basis of each other. Because of this, (c) is a well known fact for Riesz projections to eigen-spaces of compact operators ([9]). This completes the proof of the lemma. □

The following lemma should be known for a long time. We give a proof for readers' convenience.

Lemma 4.7. *Let* $1 < \gamma \le 2$ *and* $\sigma < 3/2 < \rho$. *Then, the integral operator*

$$(\mathcal{Q}_\gamma u)(x) = \int_{\mathbb{R}^3} \frac{\langle y \rangle^{-\gamma} u(y)}{|x-y|} dy \tag{44}$$

is bounded from $(L^\sigma \cap L^\rho)(\mathbb{R}^3)$ *to the space* $C_*(\mathbb{R}^3)$ *of bounded continuous functions on* $\mathbb{R}^3$ *which converge to* 0 *as* $|x| \to 0$:

$$\|\mathcal{Q}_\gamma u\|_{L^\infty} \le C \|u\|_{(L^\sigma \cap L^\rho)(\mathbb{R}^3)}. \tag{45}$$

For $R \ge 1$, *there exists a constant* C *independent of* u *such that for* $|x| \ge R$

$$\left| (Q_\gamma u)(x) - \frac{C(u)}{|x|} \right| \le C \frac{\|u\|_{L^\sigma \cap L^\rho}}{\langle x \rangle^\gamma}, \quad C(u) = \int_{\mathbb{R}^3} \langle y \rangle^{-\gamma} u(y) dy. \tag{46}$$

Proof. We omit the index γ in the proof. Since $|x|^{-1} \in L^{3,\infty}(\mathbb{R}^3)$, it is obvious that $\mathcal{Q}u(x)$ is a bounded continuous function and that (45) is satisfied. Thus, it suffices to prove (46) for $|x| \ge 100$. Let K_x be the unit cube with center x. Combining the two integrals on the left hand side of (46), we write it as

$$\begin{aligned}(Q_\gamma u)(x) - \frac{C(u)}{|x|} &= \frac{1}{|x|} \left(\int_{K_x} + \int_{\mathbb{R}^3 \setminus K_x} \right) \frac{(2yx - y^2) \langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy \\ &\equiv I_0(x) + I_1(x).\end{aligned}$$

When $|x-y| \le 1$ and $|x| \ge 100$, $|x|, \langle x \rangle, |y|$ and $|x-y|$ are comparable in the sense that $0 < C_1 \le |x|/\langle x \rangle \le C_2 < \infty$ and etc. and we may estimate the integral over K_x as follows:

$$|I_0(x)| \le \frac{C}{|x| \langle x \rangle^{\gamma - 1}} \int_{K_x} \frac{|u(y)|}{|x-y|} dy \le \frac{C}{\langle x \rangle^\gamma} \|u\|_{L^\rho(K_x)}. \tag{47}$$

We estimate the integral $I_1(x)$ by splitting it as $I_1(x) = I_{10}(x) + I_{11}(x)$:

$$I_{10}(x) = \frac{-1}{|x|} \int_{\mathbb{R}^3 \setminus K_x} \frac{y^2 \langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy,$$

$$I_{11}(x) = \frac{1}{|x|} \int_{\mathbb{R}^3 \setminus K_x} \frac{2yx \langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy.$$

Since $|x-y|+|x| \geq C\langle x\rangle^{\gamma-1}\langle y\rangle^{2-\gamma}$ for $|x| \geq 100$, Hölder's inequality implies

$$|I_{10}(x)| \leq \frac{C}{|x|\langle x\rangle^{\gamma-1}} \int_{\mathbb{R}^3\setminus K_x} \frac{|u(y)|}{|x-y|} dy \leq \frac{C}{\langle x\rangle^{\gamma}} \|u\|_{L^{\rho}(\mathbb{R}^3)}. \tag{48}$$

Let σ' be the dual exponent of σ. Then, $\sigma' > 3$ and via Hölder's inequality

$$|I_{11}(x)| \leq C \left(\int_{\mathbb{R}^3} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \|u\|_{L^{\sigma}(\mathbb{R}^3)}. \tag{49}$$

If $|x| < 100|y|$, then $\langle y\rangle^{\gamma-1}(\langle x\rangle + \langle y\rangle) \geq C\langle x\rangle^{\gamma}$ and

$$\left(\int_{|x|<100|y|} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \leq \frac{C}{\langle x\rangle^{\gamma}} \|\langle x\rangle^{-1}\|_{L^{\sigma'}}. \tag{50}$$

When $|x| > 100|y|$, we may estimate for $1 < \gamma \leq 2$ as

$$\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(|x|+|y|)} \leq \frac{C}{\langle x-y\rangle\langle x\rangle^{\gamma}}.$$

It follows that

$$\left(\int_{|x|>100|y|} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \leq \frac{C}{\langle x\rangle^{\gamma}} \|\langle x\rangle^{-1}\|_{L^{\sigma'}}. \tag{51}$$

Estimates (50) and (51) imply

$$|I_{11}(x)| \leq \frac{C}{\langle x\rangle^{\gamma}} \|u\|_{L^{\sigma}}. \tag{52}$$

Combining (52) with (48), we obtain (46). □

Lemma 4.8. (1) *The following is a continuous functional on* $\mathcal{N}$:

$$\mathcal{N} \ni \varphi \mapsto L(\varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} a(x)\varphi(x)dx = \frac{1}{4\pi}\langle a, \varphi\rangle \in \mathbb{C}.$$

(2) *For* $\varphi \in \mathcal{N}$, *let* $u = D_0(a\varphi)$. *Then,*

(a) u *is a sum* $u = u_1 + u_2$ *of* $u_1 \in C^{\infty}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$ *and* $u_2 \in (W^{\frac{3}{2}+\varepsilon,2} \cap W^{2,\frac{3}{2}+\varepsilon})(\mathbb{R}^3)$ *for some* $\varepsilon > 0$. *It satisfies*

$$(-\Delta + V)u(x) = 0. \tag{53}$$

(b) u *is bounded continuous and satisfies*

$$u(x) = \frac{L(\varphi)}{|x|} + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{54}$$

(c) u *is an eigenfunction of* H *with eigenvalue* 0 *if and only if* $L(\varphi) = 0$ *and it is a threshold resonance of* H *otherwise.*

(3) *The space of zero eigenfunctions in* $\mathcal{N}$ *has codimension at most one.*

Proof. (1) Since $a \in L^2$, $|L(\varphi)| \le (4\pi)^{-1}\|a\|_{L^2}\|\varphi\|_{L^2}$.

(2a) Assumption (7) implies $a(x) = \langle x\rangle^{-1}\tilde a(x)$ with $\tilde a \in (L^{2p}\cap L^{2q})(\mathbb{R}^3)$ and $1 \le 2p < 3$ and $2q > 6$. It follows by Hölder's inequality that $\tilde a\varphi \in L^{\frac65-\varepsilon}\cap L^{\frac32+\varepsilon}$ for an $\varepsilon > 0$. Using the the Fourier multiplier $\chi(D)$ by $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$,

$$\chi(D)u = \frac{1}{(2\pi)^{\frac32}}\int_{\mathbb{R}^3} e^{ix\xi}\chi(\xi)\hat u(\xi)d\xi,$$

we decompose u:

$$u = u_1 + u_2, \quad u_1 = \chi(D)D_0(a\varphi), \ \ u_2 = \{(1-\chi(D))(1-\Delta)D_0\}(1-\Delta)^{-1}(a\varphi).$$

Since $a\varphi \in L^1(\mathbb{R}^3)$ it is obvious that

$$u_1(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\xi}\chi(\xi)\frac{\widehat{a\varphi}(\xi)}{|\xi|^2}d\xi \in C^\infty(\mathbb{R}^3), \qquad \lim_{|x|\to\infty}\partial^\alpha u_1(x) = 0$$

for all α. Since $(1-\chi(\xi))(1+|\xi|^2)|\xi|^{-2}$ is a symbol of Hörmander class S_0, the multiplier $(1-\chi(D))(1-\Delta)D_0$ is bounded in any Sobolev space $W^{k,p}(\mathbb{R}^3)$ for $1 < p < \infty$ by Mikhlin's theorem and,

$$(1-\Delta)^{-1}(a\varphi) \in W^{2,\frac32+\varepsilon}(\mathbb{R}^3)\cap W^{\frac32+\varepsilon,2}(\mathbb{R}^3)$$

for an $\varepsilon > 0$ by the Sobolev embedding theorem. It follows that

$$u_2 \in W^{2,\frac32+\varepsilon}(\mathbb{R}^3)\cap W^{\frac32+\varepsilon,2}(\mathbb{R}^3),$$

in particular, u is bounded and Hölder continuous. If $(1 + bD_0 a)\varphi = 0$, then

$$a(1 + bD_0a)\varphi = (1 + VD_0)a\varphi = (-\Delta + V)D_0a\varphi = 0$$

and $(-\Delta + V)u(x) = 0$.

(2b) We just proved that u is bounded and Hölder continuous. We use the notation in the proof of Lemma 4.7. We have $a\varphi = -VD_0(a\varphi)$ and

$$D_0(a\varphi)(x) = \frac{1}{4\pi}\left(\int_{K_x} + \int_{\mathbb{R}^3\setminus K_x}\right)\frac{\langle y\rangle^{-1}\tilde a(y)\varphi(y)dy}{|x-y|} = I_1(x) + I_2(x).$$

Since $\langle y\rangle$ is comparable with $\langle x\rangle$ when $|x-y| < 1$,

$$|I_1(x)| \le C\langle x\rangle^{-1}\|\tilde a\varphi\|_{L^{\frac32+\varepsilon}}\||x|^{-1}\|_{L^\tau(K_x)}, \quad \tau = \tfrac{3+2\varepsilon}{1+2\varepsilon} < 3.$$

For estimating the integral over $\mathbb{R}^3\setminus K_x$, we use that $\tilde a\varphi \in L^{\frac65-\varepsilon}$ for some $0 < \varepsilon < 1/5$. Let $\delta = (6-5\varepsilon)/(1-5\varepsilon)$. Then, $\delta > 6$ and Hölder's inequality implies

$$|I_2(x)| \le C\|\tilde a\varphi\|_{L^{\frac65-\varepsilon}}\left(\int_{\mathbb{R}^3}\frac{dy}{\langle x-y\rangle^\delta\langle y\rangle^\delta}\right)^{\frac1\delta} \le \frac{C\|\tilde a\varphi\|_{L^{\frac65-\varepsilon}}}{\langle x\rangle}.$$

Hence, $a\varphi = -VD_0(a\varphi) \in \langle x\rangle^{-3}(L^p\cap L^q)(\mathbb{R}^3)$ and Lemma 4.7 with $\gamma = 2$ implies statement (2b).

Statements (2a) and (2b) obviously implies (2c). (3) follows from (1) and (2c). □

We distinguish following three cases:

Case (a): $\mathcal{N} \cap \mathrm{Ker}\,(L) = \{0\}$. Then, Lemma 4.8 implies $\dim \mathcal{N} = 1$, H has no zero eigenvalue and has only threshold resonances $\{u = D_0(a\varphi) \colon \varphi \in \mathcal{N}\}$.

Case (b): $\mathcal{N} = \mathrm{Ker}\,(L)$. Then, $\{u = D_0(a\varphi) \colon \varphi \in \mathcal{N}\}$ consists only of eigenfunctions of H with eigenvalue 0.

Case (c): $\{0\} \subsetneqq \mathcal{N} \cap \mathrm{Ker}\,(L) \subsetneqq \mathcal{N}$. In this case H has both zero eigenvalue and threshold resonances.

In case (c), we take an orthonormal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of $\mathcal{N}$ such that $\varphi_2, \ldots, \varphi_n \in \mathrm{Ker}\,(L)$ and $\varphi_1 \in \mathrm{Ker}\,(L)^\perp$ such that $L(\varphi_1) > 0$ which uniquely determines φ_1.

We study $\varepsilon(1 + M_\varepsilon(\varepsilon k))^{-1}$, $M_\varepsilon(\varepsilon k) = \lambda_0(\varepsilon) b G_0(\varepsilon k) a$ as $\varepsilon \to 0$ by applying the following Lemma 4.9 due to Jensen and Nenciu ([8]). We consider the case (c) only. The modification for the cases (a) and (b) should be obvious.

Lemma 4.9. *Let $\mathcal{A}$ be a closed operator in a Hilbert space $\mathcal{H}$ and S a projection. Suppose $\mathcal{A} + S$ has a bounded inverse. Then, $\mathcal{A}$ has a bounded inverse if and only if*

$$\mathcal{B} = S - S(\mathcal{A} + S)^{-1} S$$

has a bounded inverse in $S\mathcal{H}$ and, in this case,

$$\mathcal{A}^{-1} = (\mathcal{A} + S)^{-1} + (\mathcal{A} + S)^{-1} S \mathcal{B}^{-1} S (\mathcal{A} + S)^{-1}. \tag{55}$$

We recall (40) and (42). We apply Lemma 4.9 to

$$\mathcal{A} = 1 + M_\varepsilon(\varepsilon k) \equiv 1 + \lambda(\varepsilon) b G_0(\varepsilon k) a. \tag{56}$$

We take as S the Riesz projection onto the kernel $\mathcal{M}$ of $Q_0 = 1 + bD_0 a$. Since $bD_0 a$ is compact, $Q_0 + S$ is invertible. Hence, by virtue of (40), $\mathcal{A} + S$ is also invertible for small $\varepsilon > 0$ and the Neumann expansion formula yields,

$$\begin{aligned}
(\mathcal{A} + S)^{-1} &= (Q_0 + \varepsilon Q_1 + O(\varepsilon^2) + S)^{-1} \\
&= \Big(1 + \varepsilon (Q_0 + S)^{-1} Q_1 + O(\varepsilon^2)\Big)^{-1} (Q_0 + S)^{-1} \\
&= (Q_0 + S)^{-1} - \varepsilon (Q_0 + S)^{-1} Q_1 (Q_0 + S)^{-1} + O(\varepsilon^2).
\end{aligned} \tag{57}$$

Since $S(Q_0 + S)^{-1} = (Q_0 + S)^{-1} S = S$, the operator $\mathcal{B}$ of Lemma 4.9 corresponding to $\mathcal{A}$ of (56) becomes

$$\mathcal{B} = \varepsilon S Q_1 S + O(\varepsilon^2), \quad \sup_{k \in \Omega} \|O(\varepsilon^2)\|_{\mathbf{B}(\mathcal{H})} \le C\varepsilon^2, \tag{58}$$

where $\Omega \Subset \overline{\mathbb{C}}^+ \setminus \{0\}$. Take the dual basis $(\{\varphi_j\}, \{\psi_j\})$ of $(\mathcal{M}, \mathcal{N})$ defined in Lemma 4.6. Then, $bD_0 a\varphi = -\varphi$ for $\varphi \in \mathcal{M}$, $(a, \varphi_j) = 0$ for $2 \le j \le n$ and

$(\psi_j, b) = (aD_0 a\varphi_j, b) = -(\varphi_j, a)$ imply

$$SQ_1S = S(\lambda'(0)bD_0a + ikbD_1a)S = -\lambda'(0)S - \frac{ik}{4\pi}|(a,\varphi_1)|^2(\varphi_1 \otimes \psi_1).$$

It follows from (58) that uniformly with respect to $k \in \Omega$ we have

$$\left\| \varepsilon\mathcal{B}^{-1} + \left(\lambda'(0) + i\frac{k|(a,\varphi_1)|^2}{4\pi}\right)^{-1} \varphi_1 \otimes \psi_1 + \lambda'(0)^{-1}\sum_{j=2}^{n} \varphi_j \otimes \psi_j \right\| \le C\varepsilon. \tag{59}$$

Then, since $\|(\mathcal{A}+S)^{-1}\|_{\mathbf{B}(\mathcal{H})}$ is bounded as $\varepsilon \to 0$ and $k \in \Omega$ and

$$\lim_{\varepsilon\to 0} \sup_{k\in\Omega}(\|S(\mathcal{A}+S)^{-1} - S\|_{\mathbf{B}(\mathcal{H})} + \|(\mathcal{A}+S)^{-1}S - S\|_{\mathbf{B}(\mathcal{H})} = 0,$$

(55), (57) and (59) imply the first statement of the following proposition.

Proposition 4.10. *Let $N = 1$ and the assumption* (7) *be satisfied. Suppose that H is of exceptional type at* 0 *of the case* (c)*. Then, with the notation of* Lemma 4.6*, uniformly with respect to $k \in \Omega$ in the operator norm of $\mathcal{H}$ we have that*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1} \\ &= -\left(\lambda'(0) + i\frac{k|(a,\varphi_1)|^2}{4\pi}\right)^{-1} \varphi_1 \otimes \psi_1 - \lambda'(0)^{-1}\sum_{j=2}^{n} \varphi_j \otimes \psi_j \equiv \mathcal{L} \end{aligned} \tag{60}$$

and that

$$\langle a|\,(60)\,|b\rangle = -\left(\alpha - \frac{ik}{4\pi}\right)^{-1}, \quad \alpha = -\frac{\lambda'(0)}{|(a,\varphi_1)|^2}. \tag{61}$$

The same result holds for other cases with the following changes: *For the case* (a) *replace φ_1 and ψ_1 by φ and ψ respectively which are normalized as φ_1 and ψ_1 and, for the case* (b) *set $\varphi_1 = \psi_1 = 0$.*

4.2. Proof of Theorem 1.1

Let $\mathcal{L}_j$, $j = 1, \dots, N$ be the $\mathcal{L}$ of (60) corresponding to $H_j(\varepsilon) = -\Delta + \lambda_j(\varepsilon)V_j$. Then, applying Proposition 4.10 to $H_j(\varepsilon)$, we have

$$\lim_{\varepsilon\to 0} \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1} = \oplus_{j=1}^{N}\mathcal{L}_j \equiv \tilde{\mathcal{L}}. \tag{62}$$

It follows by combining Lemma 4.2 and (62) that

$$\lim_{\varepsilon\to 0} \big(1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))\big)^{-1} E_\varepsilon(\varepsilon k) = 1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|. \tag{63}$$

We apply the following lemma due to Deift ([4]) to the right of (63).

Lemma 4.11. *Suppose that $1 + \langle A|\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)$ is invertible in $\mathbf{B}(\mathbb{C}^N)$. Then, $1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|$ is also invertible in $\mathbf{B}(\mathcal{H}^{(N)})$ and*

$$\langle A|(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|)^{-1} = (1 + \langle A|\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k))^{-1}\langle A|. \tag{64}$$

Proof. Since $a_1, \dots, a_N \in L^2(\mathbb{R}^3)$, $|A\rangle\colon \mathbb{C}^N \to \mathcal{H}^{(N)}$ and $\langle A|\colon \mathcal{H}^{(N)} \to \mathbb{C}^N$ are both bounded operators. Then, the lemma is an immediate consequence of Theorem 2 of [4]. □

For the next lemma we use the following simple lemma for matrices. Let

$$\mathcal{A} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}$$

be matrices decomposed into blocks.

Lemma 4.12. *Suppose V and $1 + VZ$ are invertible. Then,*

$$\left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1}$$

exists and

$$\left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (V^{-1} + Z)^{-1} \end{pmatrix}. \tag{65}$$

Proof. It is elementary to see

$$\begin{aligned} \left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1} &= \begin{pmatrix} 1 & 0 \\ VY & 1 + VZ \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -(1 + VZ)^{-1} VY & (1 + VZ)^{-1} \end{pmatrix} \end{aligned} \tag{66}$$

and the left side of (65) is equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & (1 + VZ)^{-1} V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (V^{-1} + Z)^{-1} \end{pmatrix}$$

which proves the lemma. □

Lemma 4.13. *Let $k \in \Omega$. Then, $1 + \langle A|\tilde{\mathcal{L}}|B\rangle \hat{\mathcal{G}}(k)$ is invertibe in $\mathbb{C}^N$. If $H_1, \dots, H_N$ are arranged in such a way that $H_1, \dots, H_{n_1}$ have no resonances and $H_{n_1+1}, \dots, H_N$ do and, $N = n_1 + n_2$, then*

$$(1 + \langle A|\tilde{\mathcal{L}}|B\rangle \hat{\mathcal{G}}(k))^{-1} \langle A|\tilde{\mathcal{L}}|B\rangle = \begin{pmatrix} \mathbb{O}_{n_1 n_1} & \mathbb{O}_{n_1 n_2} \\ \mathbb{O}_{n_2 n_1} & -\tilde{\Gamma}(k)^{-1} \end{pmatrix}, \tag{67}$$

where $\mathbb{O}_{n_1 n_1}$ is the zero matrix of size $n_1 \times n_1$ and etc. and

$$\tilde{\Gamma}(k) = \left(\left(\alpha_j - \frac{ik}{4\pi}\right)\delta_{j,\ell} - \mathcal{G}_k(y_j - y_\ell)\hat{\delta}_{j\ell}\right)_{j,\ell = n_1+1, \dots, N}. \tag{68}$$

Proof. We let φ_{j1} be the resonance of H_j, $j = n_1 + 1, \dots, N$, corresponding to φ_1 of the previous section and define

$$\alpha_j = -\frac{\lambda'(0)}{|(a_j, \varphi_{j1})|^2}. \tag{69}$$

Then, Proposition 4.10 implies that,

$$\langle A|\tilde{\mathcal{L}}|B\rangle = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -\left(\alpha_{n_2+1} - \frac{ik}{4\pi}\right)^{-1} & & \\ & & & & \ddots & \\ & & & & & -\left(\alpha_{n_1+n_2} - \frac{ik}{4\pi}\right)^{-1} \end{pmatrix}$$

and we obtain (67) by applying Lemma 4.12 to the left of (67) with

$$V = \begin{pmatrix} -\left(\alpha_{n_2+1} - \frac{ik}{4\pi}\right)^{-1} & & \\ & \ddots & \\ & & -\left(\alpha_{n_1+n_2} - \frac{ik}{4\pi}\right)^{-1} \end{pmatrix}$$

and with

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \hat{\mathcal{G}}(k).$$

□

Lemma 4.11 and Lemma 4.13 imply that the following limit exists in $\mathbf{B}(\mathcal{H})$ and

$$\lim_{\varepsilon\to 0}\left(1+\varepsilon(1+D_\varepsilon(\varepsilon k))^{-1}E_\varepsilon(\varepsilon k)\right)^{-1} = \left(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|\right)^{-1}$$

and hence so does

$$\lim_{\varepsilon\to 0}\varepsilon\left(1+M_\varepsilon(\varepsilon k)\right)^{-1} = \left(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|\right)^{-1}\tilde{\mathcal{L}}. \tag{70}$$

Completion of the proof of Theorem 1.1. By the assumption of the theorem, we may assume $n_1 = 0$ in Lemma 4.13. Abusing notation, we write

$$\hat{\mathcal{G}}_k^{(N)}u = (\hat{\mathcal{G}}_k u)^{(N)}, \quad \hat{\mathcal{G}}_k u = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{ik|x|}u(x)}{|x|}dy.$$

We first prove (9) for the + case. We let $u, v \in \mathcal{D}_*$ and $R > 0$. Then, (23) and (70) imply that

$$\varepsilon^2((1+M_\varepsilon(-\varepsilon k))^{-1}\Lambda(\varepsilon)B(G_0(k\varepsilon)-G_0(-k\varepsilon))^{(N)}U_\varepsilon u, AG_0(k\varepsilon)^{(N)}U_\varepsilon v) \tag{71}$$

converges as $\varepsilon \to 0$ to

$$(\langle A|(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(-k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle\langle(\mathcal{G}_k^{(N)} - \mathcal{G}_{-k}^{(N)})u, \mathcal{G}_k^{(N)}v) \tag{72}$$

uniformly with respect to $k \in [R^{-1}, R]$. Here we have

$$\begin{aligned}\langle A|(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(-k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle &= (1+\langle A|\mathcal{L}|B\rangle\hat{\mathcal{G}}(-k))^{-1}\langle A|\mathcal{L}|B\rangle \\ &= -\tilde{\Gamma}(-k)^{-1}\end{aligned} \tag{73}$$

by virtue of (64) and (67). Thus, (71) converges as $\varepsilon \to 0$ to

$$-(\Gamma_{\alpha,Y}(-k)^{-1}\left(\hat{\mathcal{G}}_k - \hat{\mathcal{G}}_{-k}\right)^{(N)}u, \hat{\mathcal{G}}_k^{(N)}v)$$

uniformly on $[R^{-1}, R]$. Thus, replacing u and v respectively by τu and τv, we obtain $W^+_{Y,\varepsilon} \to W^+_{\alpha,Y}$ strongly as $\varepsilon \to 0$ in view of (15) and (21).

By virtue of (1) and (22), for proving the convergence (6) of the resolvent, it suffices to show that as $\varepsilon \to 0$ in the strong topology of $\mathbf{B}(\mathcal{H})$

$$\begin{aligned}(74)\qquad & \varepsilon^2 U_\varepsilon G_0(k\varepsilon)^{(N)} A(1+M_\varepsilon(\varepsilon k))^{-1}\Lambda(\varepsilon)\varepsilon B G_0(k\varepsilon)^{(N)} U_\varepsilon \\ & \to\ -|\hat{\mathcal{G}}^{(N)}_k\rangle \Gamma_{\alpha,Y}(k)^{-1}\langle \hat{\mathcal{G}}^{(N)}_k|\end{aligned}$$

for every $k \in \mathbb{C}^+ \setminus \mathcal{E}$. However, (23), (25) and (70) imply that for $k \in \mathbb{C}^+ \setminus \mathcal{E}$ the first line of (74) converges strongly in $\mathbf{B}(\mathcal{H})$ as $\varepsilon \to 0$ to

$$(75)\qquad |\mathcal{G}^{(N)}_k\rangle\langle A|(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle\langle\mathcal{G}^{(N)}_k|.$$

This is equal to the second line by virtue of (73) with k in place of $-k$. This completes the proof of the theorem.

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