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A NOTE ON THE EXISTENCE OF HORIZONTAL ENVELOPES IN THE 3D-HEISENBERG GROUP

Yen-Chang Huang

Abstract. By using the support functions on the xy-plane, we show the necessary and sufficient conditions for the existence of envelopes of horizontal lines in the 3D-Heisenberg group. A method to construct horizontal envelopes from the given ones is also derived, and we classify the solutions satisfying the construction.

1. Introduction

Given a family of lines in $\mathbb{R}^2$, the envelope of the family of lines $F(\lambda, x, y)$, depending on the parameter λ, is defined to be a curve which every line in the family contacts exactly at one point. A simple example of finding the envelope of the family of lines $F(\lambda, x, y) = (1-\lambda)x + \lambda y - \lambda(1-\lambda) = 0$ for $\lambda \in [0,1]$ and $(x, y) \in [0,1] \times [0,1]$ can be illustrated as follows: consider the system of differential equations

$$\left\{ \begin{array}{l} F(\lambda, x, y) = 0, \\ \frac{\partial F(\lambda,x,y)}{\partial \lambda} = 0. \end{array} \right. \tag{1}$$

The second equation helps us find λ in terms of x, y, and substitute λ into the first equation to have the envelope

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0. \tag{2}$$

Note that there are in total three variables λ, x, y, two equations in the system of differential equations (1), and hence one gets the solution (2) which is a one-dimensional curve in $\mathbb{R}^2$. Some classical results of the envelope theory has been systematically studied by V. I. Arnold [1, Chapter 9] and Thom [26], and the references therein. In particular, in [2–4] Arnold et al. also develop the theory of Lagrangian and Legendrian mappings by studying the singularities of wave fronts and caustics. See also [27] by Zakalyukin. In [7] Capitanio studies the

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union of the Legendrian lifts of the family curves in the projectivized cotangent bundle of the plane (called *Legendrian graph*) and constructs a Legendrian version of tangential family theory. For the studies of 1-parameter family of surfaces in $\mathbb{R}^3$, we refer the readers to Kock [18] and Izumiya [17].

The applications of the envelope theory have even attracted a great attention to the other scientific fields. For instance, in Economics the Envelope Theorem shows that the optimal production functions of objects can be obtained by the given input and output prices of the objects [10, 21]. Recently the theorem has also been generalized to the functions of multivariables with non-differentiability condition [14]. The other application in Computational Geometry related to the existence of envelops is the complexity of the upper envelopes of n line segments in a given time period [5, 11, 16].

In general, seeking the explicit expression of envelope of a family of lines without any constraints is impossible in the higher dimensional Euclidean spaces $\mathbb{R}^n$ for $n \geq 3$. The reason is that in the higher dimensional spaces, the systems of differential equations similar to (1) are usually over-determined [12, Sec. 26, Chapter 2]. However, it is possible to consider the existence of a kind of envelopes (called *horizontal envelopes*, see Definition 1 later) in the 3-dimensional Heisenberg group $\mathbb{H}_1$. There are a few equivalent ways to introduce the Heisenberg group (for instance, the authors [6] show four equivalent definitions), and in this note, as the by-product of our previous studies, we take the definition same as in [8, 9]. The Heisenberg group $(\mathbb{H}_1, J, \xi)$ is the 3-dimensional space $\mathbb{R}^3$ with the almost complex structure J and the horizontal distribution ξ defined by $\xi = \ker \alpha$, where $\alpha = dz + xdy - ydz$ is the standard contact 1-form (more detail about $\mathbb{H}_1$ will be introduced in the later paragraphs). A horizontal curve in $\mathbb{H}_1$ is a regular curve with tangent vectors on the horizontal distribution ξ. The geodesics in $\mathbb{H}_1$ are the horizontal lines and helices (both are Legendre curves, see [23, Sec. 3] and references therein). It is natural to propose the following question:

Question. *What conditions for given a family of horizontal lines in* $\mathbb{H}_1$ *ensure the existence of a curve such that its tangents are all lines in the family?*

Such a curve is called a *horizontal envelope* for given horizontal lines (see Definition 1), and in this note we will study the necessary and sufficient conditions for the existence of horizontal envelopes in $\mathbb{H}_1$. To the best of our knowledge, there is no literature regarding the horizontal envelope except for the different approach taken by Li-Pei-Takahashi-Yu [19, 25] in 2018 shows that the existence and uniqueness theorems for one-parameter families of spherical Legendre curves by using the curvatures defined on the unit spherical bundle. Some similar topics

We recall some terminologies for our purpose. For more details about the Heisenberg groups, we refer the readers to [8, 9, 20, 22, 23]. The 3-dimensional Heisenberg group $\mathbb{H}_1$ is the Lie group $(\mathbb{R}^3, \star)$, where the group operation $\star$ is defined, for any point (x, y, z), $(x', y', z') \in \mathbb{R}^3$, by

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + yx' - xy').$$

For $p \in \mathbb{H}_1$, the *left translation* by p is the diffeomorphism $L_p(q) := p \star q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$\mathring{e}_1(p) := \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \mathring{e}_2(p) := \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ T(p) := (0, 0, 1).$$

The *horizontal distribution* (or *contact plane* ξ_p at any point $p \in \mathbb{H}_1$) is the smooth plane distribution generated by $\mathring{e}_1(p)$ and $\mathring{e}_2(p)$. We shall consider on $\mathbb{H}_1$ the (left invariant) Riemannian metric $g := \langle \cdot, \cdot \rangle$ so that $\{\mathring{e}_1, \mathring{e}_2, T\}$ is an orthonormal basis in the Lie algebra of $\mathbb{H}_1$. The endomorphism $J : \mathbb{H}_1 \to \mathbb{H}_1$ is defined such that $J(\mathring{e}_1) = \mathring{e}_2$, $J(\mathring{e}_2) = -\mathring{e}_1$, $J(T) = 0$ and $J^2 = -1$.

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{H}_1$ is called *horizontal* (or *Legendrian*) if its tangent at any point on the curve is on the contact plane. More precisely, if we write the curve in coordinates $\gamma := (x, y, z)$ with the tangent vector $\gamma' = (x', y', z') = x'\mathring{e}_1(\gamma) + y'\mathring{e}_2(\gamma) + T(z' - x'y + xy')$, then the curve γ is horizontal if and only if

$$z' - x'y + xy' = 0, \tag{3}$$

where the prime $'$ denotes the derivative with respect to the parameter of the curve. The velocity γ' has the natural decomposition

$$\gamma' = \gamma'_\xi + \gamma'_T,$$

where γ'_ξ (resp. γ'_T) is the orthogonal projection of γ' on ξ along T (resp. on T along ξ) with respect to the metric g. Recall that a *horizontally regular curve* is a parametrized curve $\gamma(u)$ such that $\gamma'_\xi(u) \neq 0$ for all $u \in I$ (Definition 1.1 [9]). Also, in Proposition 4.1 [9], we show that any horizontally regular curve can be uniquely reparametrized by *horizontal arc-length* s, up to a constant, such that $|\gamma'_\xi(s)| = 1$ for all s, and called the curve being *with horizontal unit-speed*. Moreover, two geometric quantities for horizontally regular curves parametrized by horizontal arc-length, the *p-curvature* $k(s)$ and the *contact normality* $\tau(s)$, are defined by

$$k(s) := \langle \frac{d\gamma'(s)}{ds}, J\gamma'(s) \rangle,$$
$$\tau(s) := \langle \gamma'(s), T \rangle,$$

which are invariant under pseudo-hermitian transformations of horizontally regular curves [9, Section 4]. Note that $k(s)$ is analogous to the curvature of the curve in the Euclidean space $\mathbb{R}^3$, while $\tau(s)$ measures how far the curve is from being horizontal. When the curve $\gamma(u)$ is parametrized by arbitrary parameter u (not necessarily the horizontal arc-length s), the p-curvature is given by

$$k(u) := \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}(u). \tag{4}$$

It is clear that a curve $\gamma(s)$ is horizontal if and only if $\gamma'(s) = \gamma'_{\xi}(s)$ for all s. One of the examples for horizonal curves is the horizontal lines which can be characterized by the following proposition.

Proposition 1.1. *Any horizontal line ℓ in $\mathbb{H}_1$ can be uniquely determined by three parameters (p, θ, t) for any $p \geq 0$, $\theta \in [0, 2\pi]$, $t \in \mathbb{R}$. Any point on the line can be represented in coordinates*

$$\ell : \begin{cases} x = p\cos\theta - s\ \sin\theta, \\ y = p\sin\theta + s\ \cos\theta, \\ z = t - sp, \end{cases} \tag{5}$$

for all $s \in \mathbb{R}$.

When $s = 0$, denote by

$$Q' := (x, y, z) = (p\cos\theta, p\sin\theta, t). \tag{6}$$

Observe that ℓ is an oriented line through the point Q' with the directional vector $-\sin\theta\ \mathring{e}(Q') + \cos\theta\ \mathring{e}(Q')$. The value p actually is the value of the support function $p(\theta)$ for the projection $\pi(\ell)$ of ℓ on the xy-plane (see the proof of Proposition 1.1 in next section).

Inspired by the envelopes in the plane, with the assistance of contact planes in $\mathbb{H}_1$ we introduce the horizontal envelopes tangent to a family of horizontal lines.

Definition. Given a family of horizontal lines in $\mathbb{H}_1$, a *horizontal envelope* is a horizontal curve γ such that γ contacts with exactly one line in the family at one point.

Back to Proposition 1.1, p actually is the distance of the projection $\pi(\ell)$ of ℓ onto the xy-plane to the origin, and θ is the angle from the x-axis to the line perpendicular to the projection (see Fig. 3 next section). To obtain the horizontal envelope γ, it is natural to consider p as a function of θ, namely, the support function $p = p(\theta)$ for the projection $\pi(\gamma)$ of the curve on the xy-plane. Moreover, by (6) we know that the value t dominates the height of the point Q'. As long as θ is fixed, the projection of Q' onto the xy-plane is fixed. Thus, we may consider t as a function of θ. Under these circumstances, the family of horizontal lines is only controlled by one parameter θ, and so the following is our main theorem.

Theorem 1.2. *Let $p = p(\theta) \geq 0$ and $t = t(\theta)$ be C^1-functions defined on $\theta \in [0, 2\pi]$ satisfying*

$$t' = (p')^2 - p^2. \tag{7}$$

There exists a horizontally regular curve γ parametrized by horizontal arc-length such that the curve is the horizontal envelope of the family of horizontal lines determined by θ (and hence p and t). In coordinates, the envelope $\gamma = (x, y, z)$ can be represented by

$$\gamma : \begin{cases} x(\theta) = p(\theta)\cos\theta - p'(\theta)\sin\theta, \\ y(\theta) = p(\theta)\sin\theta + p'(\theta)\cos\theta, \\ z(\theta) = t(\theta) - p'(\theta)p(\theta). \end{cases} \tag{8}$$

Moreover, if p is a C^2-function, the p-curvature and the contact normality of γ are given by $k = \frac{1}{p+p''}$ and $\tau \equiv 0$.

Since the functions $p(\theta)$ and $t(\theta)$ uniquely determine a family of horizontal lines by Proposition 1.1, we say that the horizontal envelope γ in Theorem 1.2 is generated by the family of horizontal lines $(p(\theta), \theta, t(\theta))$.

Remark 1.3. A geodesic in $\mathbb{H}_1$ is a horizontally regular curve with minimal length with respect to the Carnot-Carathéodory distance. For two given points in $\mathbb{H}_1$ one can find, by Chow's connectivity Theorem ([15, p. 95]), a horizontal curve joining these points. Note that when $p \equiv c$, a constant function, by (5), (7), and (8), the horizontal envelope γ generated by the family of horizontal lines $\{(c, \theta, t(\theta)),\ \theta \in [0, 2\pi]\}$ is a (helix) geodesic with radius c; the same result occurs if t is also a constant function. In particular, when $p \equiv 0$, the horizontal envelope is the line parallel to the xy-plane and through the z-axis.

Example 1.4. Let $p(\theta) = \sin\theta\cos\theta$ and $t(\theta) = \frac{3\theta}{8} + \frac{5}{32}\sin(4\theta)$, $\theta \in [0, 2\pi]$. The horizontal envelope $\gamma(s)$ generated by $p(\theta)$ and $t(\theta)$ from different viewpoints in $\mathbb{H}_1$ are shown in Fig. 1. The dashed curve is the projection of γ onto the xy-plane.

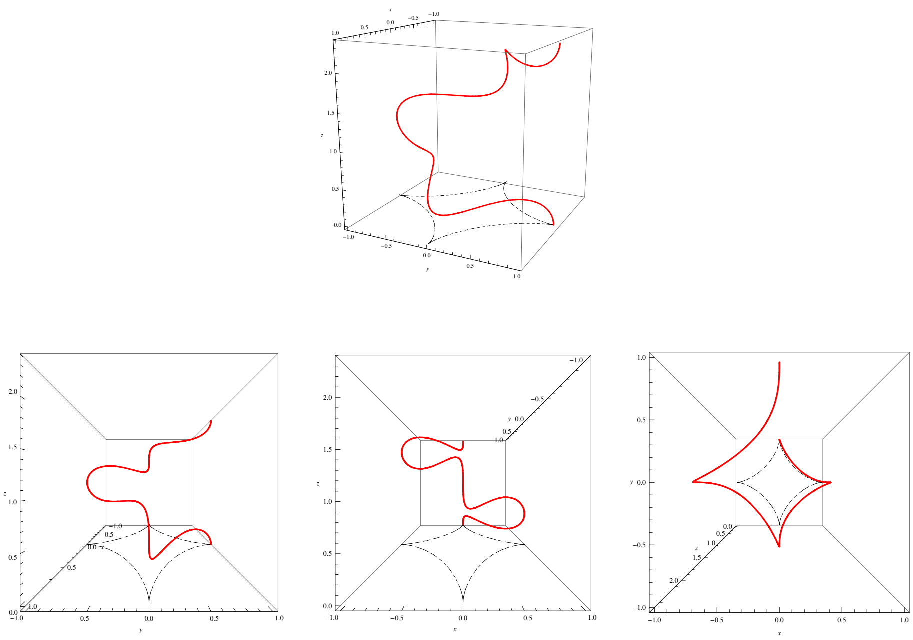
We also point out that the construction of horizontal envelopes introduced in Theorem 1.2 can be achieved by the concept of the space of contact elements in contact topology. First, the contact form $\alpha = dz + xdy - ydz$ used in the paper and $\alpha_1 = dz + xdy$ are contactomorphic via the diffeomorphism

$$f : (\mathbb{H}_1, \ker\alpha) \to (\mathbb{H}_1, \ker\alpha_1)$$
$$(x, y, z) \mapsto (2x, y, z - xy)$$

and both contact structures are diffeomorphic to $\ker\alpha_2$, where $\alpha_2 = dz - ydx$ (see Example 2.1.3 in [13] for more detail). Recall that a *contact element* for a smooth n-dimensional manifold B is a hyperplane in a tangent space to B. *The space of contact elements* of B is the collection

$$\mathcal{B} = \{(b, V) \mid b \in B \text{ and contact element } V \in T_bB\}.$$

By identifying the space $\mathcal{B}$ with the projective cotangent bundle $\mathbb{P}T^*B$ of dimension $2n - 1$, $\mathcal{B}$ forms a contact manifold with a natural contact structure, but without a globally defined contact form (see [13, page 6]). In particular, when $B = \mathbb{R}^2$ and identify the real projective space $\mathbb{R}P^1$ with $\mathbb{R}/2\pi\mathbb{Z}$ with



the yz-plane the xz-plane the xy-plane

FIGURE 1. The horizontal envelope shown in Example 1 with the projections to the specific planes.

coordinate θ, we have the contact manifold $\mathcal{B} \cong \mathbb{P}T^*B \cong \mathbb{R}^2 \times \mathbb{S}^1$ with the natural contact structure $\ker\Big(\sin(\theta)dx - \cos(\theta)dy\Big)$. Thus, the restriction

$$\{(b, \ell) \,|\, b \in \mathbb{R}^2, \text{ oriented non-vertical lines } \ell \in T_b\mathbb{R}^2\} \subset \mathbb{R}^2 \times \mathbb{S}^1$$

and $(\mathbb{H}_1, \ker\alpha_2)$ are isomorphic via the map $g(x, y, \theta) = (x, \tan\theta, y)$.

If $\gamma(t) = (x(t), y(t))$ is a (regular) plane curve parametrized by arc-length, then we can find $\theta(t)$ such that the tangent vector equals $(x'(t), y'(t)) = (\cos\theta(t), \sin\theta(t))$. Thus, $\tilde{\gamma}(t) = (x(t), y(t), \theta(t))$ in $\mathbb{R}^2 \times \mathbb{S}^1$ is the horizontal lift of γ. To complete the construction of horizontal envelopes, now one can note the following. Suppose L_t is a family of lines in $\mathbb{R}^2$ and E is the corresponding envelope of L_t. One lifts the family L_t (E resp.) to the horizontal lines $\tilde{L}_t$ (horizontal lift $\tilde{E}$ resp.) in $\mathbb{R}^2 \times \mathbb{S}^1$ and one sees that $\tilde{E}$ is the horizontal envelope of $\tilde{L}_t$. The condition (7) can be reproduced by using the composition of the functions f, g mentioned above and we leave the derivation to the reader.

Our second theorem shows that the converse statement of Theorem 1.2 also holds for horizontal curves with "jumping" ends. The key observation for the proof is that the tangent lines of a horizontal curve are all horizontal lines and they can be represented by parameters (p,θ,t) by Proposition 1.1.

Theorem 1.5. *Let $\gamma:\theta\in[0,2\pi]\mapsto(x(\theta),y(\theta),z(\theta))\in\mathbb{H}_1$ be a horizonal curve with finite length. Suppose $x(0)=x(2\pi)$, $y(0)=y(2\pi)$, and $z(0)\neq z(2\pi)$. Then the set of its tangent lines is uniquely determined by $p=p(\theta)\geq 0$ and $t=t(\theta)$ satisfying (7).*

We introduce a method to construct horizontal envelopes from the given ones.

Corollary 1.6. *Let $p_i=p_i(\theta)\geq 0, t_i=t_i(\theta)$ be C^1-functions defined on $[0,2\pi]$ satisfying (7) for $i=1,2$. Suppose $\gamma_i=(x_i,y_i,z_i)$ is a horizontal envelope generated by the family of horizontal lines (p_i,θ,t_i) for $i=1,2$. Denote by $p=p_1+p_2$ and $t=t_1+t_2$. The curve $\gamma=(x,y,z)$ is a horizontal envelope generated by a family of horizontal lines $(p(\theta),\theta,t(\theta))$ if and only if*

$$p_1p_2=p_1'p_2'. \tag{9}$$

By Corollary 1.6 and Remark 1.3, we know that if at least one of γ_1 or γ_2 is a (helix) geodesic with nonzero radius, γ can not be a horizontal envelope.

Now we seek the classification of functions p_1, p_2 satisfying the condition $p_1p_2=p_1'p_2'$ in Corollary 1.6. Actually, for any subinterval $[a,b]\subset[0,2\pi]$ such that $p_i\neq 0$ and $p_i'\neq 0$ for $i=1,2$, the condition (9) is equivalent to that $p_1(\theta)=p_1(a)\exp\left(\int_a^\theta \frac{p_2(\alpha)}{p_2'(\alpha)}d\alpha\right)$ for any $\theta\in[a,b]$. In addition, if $p_i(\theta_i)=0$ for some $\theta_i\in[0,2\pi]$, we may move the horizontal envelope γ by a left translation such that $p_i>0$ in $[0,2\pi]$. Without loss of generality we may assume that $p_i>0$ on $[0,2\pi]$ and obtain the corollary of classification.

Corollary 1.7. *Let $p_i(\theta)>0$ $(i=1,2)$ be the C^2-functions defined on $[0,2\pi]$ satisfying the condition $p_1'p_2'=p_1p_2$. Suppose $p_i'\neq 0$ and $p_i''\neq 0$ in any subinterval $[a,b]\subset[0,2\pi]$. We have the following results in $[a,b]$:*

(1) *If $p_1'>0, p_2'>0$, $p_2''>0$, then $p_1''>0$.*
(2) *If $p_1'<0, p_2'<0$, $p_2''<0$, then $p_1''>0$.*
(3) *We do not have to consider the assumptions for $p_1'<0, p_2'<0$, $p_2''>0$ and $p_1'>0, p_2'>0$, $p_2''<0$.*

Finally, we emphasis that unlike the differential system (1) mentioned in the first paragraph, the horizontal envelope in $\mathbb{H}_1$, in general, does not have an analytic expression similar to the one in (2). Indeed, an alternative expression of a horizontal line can be obtained by the intersection of two planes in $\mathbb{H}_1$

$$F_1(x,y,z,\theta):=\cos\theta x+\sin\theta y-p=0, \tag{10}$$

$$F_2(x,y,z,\theta):=-p\sin\theta x+p\cos\theta y+z-t=0, \tag{11}$$

where the set of points such that $F_1(x,y,z,\theta)=0$ is a vertical plane passing through the line $p=x\cos\theta+y\sin\theta$, and the set of $F_2(x,y,z,\theta)=0$ is the contact plane spanned by $\mathring{e}_1(Q')$ and $\mathring{e}_2(Q')$ through the point $Q'=$

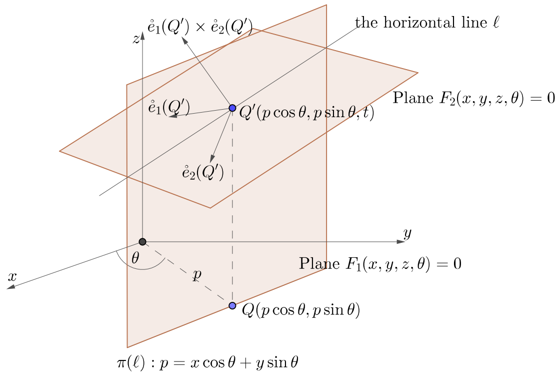


FIGURE 2. Alternative expression of horizontal line ℓ in $\mathbb{H}_1$

$(p\cos\theta, p\sin\theta, t)$ (see Fig. 2). By taking the derivatives, respectively, of F_1 and F_2 with respect to θ, we have

$$\frac{\partial F_1}{\partial \theta} = -x\sin\theta + y\cos\theta - p' = 0, \tag{12}$$

$$\frac{\partial F_2}{\partial \theta} = (-p'\sin\theta - p\cos\theta)x + (p'\cos\theta - p\sin\theta)y - t' = 0. \tag{13}$$

Using (10), (12), and substituting p, p', into (13), the condition $\frac{\partial F_2}{\partial \theta} = 0$ is equivalent to (7). Therefore, it may be only for seldom special cases that one can eliminate the parameter θ in (10), (11), and (12), to find the exact expression for the horizontal envelope.

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2. Proofs of theorems and corollaries

In this section we shall give the proofs of Theorems 1.2 and 1.5. First we prove Proposition 1.1 which plays the essential role in the notes.

Proof of Proposition 1.1. Although the proof has been shown in [9] Proposition 8.2, we describe the proof again for completely understanding the representations of horizontal lines in $\mathbb{H}_1$. Suppose $\pi(\ell)$ is the projection of horizontal line $\ell \in \mathbb{H}_1$ onto the xy-plane and the function $p = p(\theta)$ is the distance from the origin to $\pi(\ell)$ with angle θ from the positive direction of the x-axis (see Fig. 3).

Let the point $Q = (p\cos\theta, p\sin\theta, 0)$ be the intersection of the line through the origin perpendicular to $\pi(\ell)$. We choose the unit directional vector $(-\sin\theta,$

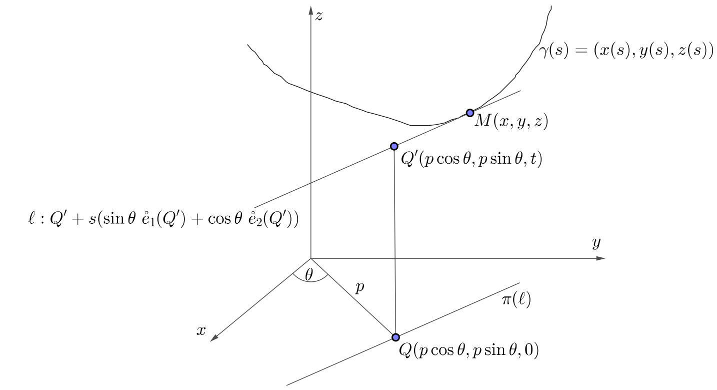

FIGURE 3. The horizontal line ℓ in $\mathbb{H}_1$

$\cos\theta, 0)$ along $\pi(\ell)$, and so any point $(x, y, 0)$ on $\pi(\ell)$ can be parametrized by horizontal arc-length s

$$\begin{cases} x = p\cos\theta - s\ \sin\theta, \\ y = p\sin\theta + s\ \cos\theta. \end{cases} \tag{14}$$

Denote the lift of the point Q on ℓ by Q'. We may assume $Q' = (p\cos\theta, p\sin\theta, t)$ for some $t \in \mathbb{R}$. Since ℓ is horizontal, the tangent line of ℓ must be on the contact plane ξ. Thus, the parametric equations for ℓ can be represented by

$$\ell : (x, y, z) = (p\cos\theta, p\sin\theta, t) + s(A\mathring{e}_1(Q') + B\mathring{e}_2(Q')) \tag{15}$$

for some constants A, B to be determined. Expand (15) by using the definitions of $\mathring{e}_1$ and $\mathring{e}_2$, and compare the coefficients with (14), one gets $A = -\sin\theta$ and $B = \cos\theta$. Thus, the parametric equations for ℓ are obtained as shown in (5). By (5), the parameters p, θ, and t, uniquely determine the horizontal line ℓ. □

Proposition 2.1. *Given any horizontal curve $\gamma(s) = (x(s), y(s), z(s))$ parametrized by horizontal arc-length s. Suppose the horizontal line ℓ determined by (p, θ, t) intersects γ at the unique point $M(x, y, z)$ on the contact plane ξ_M (see Fig. 3). If $p = p(\theta)$ is a function of θ, then the intersection $M(x, y, z)$ can be uniquely represented by*

$$\begin{cases} x = p\cos\theta - p'\sin\theta, \\ y = p\sin\theta + p'\cos\theta, \\ z = t - p'p, \end{cases} \tag{16}$$

where p' denotes the partial derivative of the function p with respect to θ.

Proof. We first solve the projection $(x, y, 0)$ of the intersection M onto the xy-plane in terms of p and θ, and then the z-component of M. By (14), any point (x, y) on the projection of ℓ satisfies

$$p = x\cos\theta + y\sin\theta. \tag{17}$$

Since $p = p(\theta)$, take the derivative on both sides to get

$$p' = -x\sin\theta + y\cos\theta. \tag{18}$$

Use (17) and (18) we obtain the first two components of the intersection M on the xy-plane, namely,

$$\begin{aligned} x &= p\cos\theta - p'\sin\theta, \\ y &= p\sin\theta + p'\cos\theta. \end{aligned}$$

To determine the third component of M, z, by using (14), (18), we have $s = y\cos\theta - x\sin\theta = p'$. Finally, (5) implies that $z = t - p'p$. The unique intersection point M immediately implies the uniqueness of the expression (16) and the result follows. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. According to the assumptions for the functions p and t, the curve $\gamma(\theta) = (x(\theta), y(\theta), z(\theta))$ defined by (8) is well-defined. By [9, Proposition 4.1], since any horizontally regular curve can be reparametrized by its horizontal arc-length, it suffices to show that the curve $\gamma(\theta)$ is horizontal. Indeed, by Proposition 2.1, a straight-forward calculation shows that

$$\begin{aligned} & z' - x'y + y'x \\ = & \left(t' - p''p - (p')^2\right) - \left(p'\cos\theta - p\sin\theta - p''\sin\theta - p'\cos\theta\right)\left(p\sin\theta + p'\cos\theta\right) \\ & + \left(p'\sin\theta + p\cos\theta + p''\cos\theta - p'\sin\theta\right)\left(p\cos\theta - p'\sin\theta\right) \\ = & \; t' - (p')^2 + p^2. \end{aligned}$$

Thus, $z' - x'y + y'x = 0$ if and only if the functions p and t satisfy (7). Therefore the curve defined by (8) is horizontal by (3).

To derive the p-curvature for the curve γ, substitute (8) into (4) and the result follows. It is also clear that $\tau \equiv 0$ since γ is horizontal. □

The horizontal length of the horizontal envelope can be represented by the function $p(\theta)$. Actually, by (8) we have the horizontal length

$$L(\gamma) := \int_0^{2\pi} \left[\left(x'(\theta)\right)^2 + \left(y'(\theta)\right)^2 \right]^{1/2} d\theta = \int_0^{2\pi} |p + p''| d\theta.$$

Compare the p-curvature k in Theorem 1.2 and the function on the right-hand side of the integral, we conclude that the length of the horizontal envelope γ is the integral of the radius of curvature for the projection $\pi(\gamma)$

$$L(\gamma) = \int_0^{2\pi} \frac{1}{|k(\theta)|} d\theta.$$

Next we show Theorem 1.5.

Proof of Theorem 1.5. Suppose that the horizontal line ℓ represented by (p, θ, t) is tangent to γ at $M(x, y, z)$. Since $M \in \ell$, by (5), we can solve the point Q' representing the horizontal line in terms of x, y, z. Let

$$\begin{aligned} x &= p\cos\theta - s\ \sin\theta, \\ y &= p\sin\theta + s\ \cos\theta, \\ z &= t - sp. \end{aligned} \tag{19}$$

One can solve

$$s = -x\sin\theta + y\cos\theta. \tag{20}$$

Substitute s into the third equation in (19) to get

$$t = z + (-x\sin\theta + y\cos\theta)p. \tag{21}$$

The first two equations in (19) imply that

$$p = x\cos\theta + y\sin\theta, \tag{22}$$

which means that the distance p is a smooth function of θ defined on $[0, 2\pi]$ if ℓ intersects γ at exactly one point. Similarly, (21) implies that t is a smooth function of θ. Finally, use (20), (21), (22) it is easy to check that the horizontal line $\ell(p, \theta, t)$ satisfies the condition (7) for any θ. □

Proof of Corollary 1.6. By Theorem 1.2, it suffices to show that the identity $t' = (p')^2 - p^2$ holds. By assumption we have

$$\begin{aligned} t' - (p')^2 + p^2 &= t_1' + t_2' - (p_1' + p_2')^2 + (p_1 + p_2)^2 \\ &= -2p_1'p_2' + 2p_1p_2. \end{aligned}$$

Thus, $t' = (p')^2 - p^2$ if and only if the identity $p_1'p_2' = p_1p_2$ holds. □

Proof of Corollary 1.7. (1) The condition $p_1'p_2' = p_1p_2$ is equivalent to that $p_1(\theta) = p_1(a)\exp\left(\int_a^\theta \frac{p_2}{p_2'}d\alpha\right)$. Take the derivative twice we have $p_1''(\theta) = p_1(a)\left(\frac{(p_2')^2 - p_2p_2'' + (p_2)^2}{(p_2')^2}\right) \cdot \exp\left(\int_a^\theta \frac{p_2}{p_2'}d\alpha\right)$. Thus, the sign of p_1'' only depends on the sign of the numerator

$$(p_2')^2 - p_2p_2'' + (p_2)^2. \tag{23}$$

We claim that $(p_2')^2 - p_2p_2'' > 0$ under the assumptions, and so (23) is positive. Indeed, if $p_2, p_2' > 0$, then $0 < \left(\log \frac{p_2}{p_2'}\right)' = (\log p_2)' - (\log p_2')' = \frac{p_2'}{p_2} - \frac{p_2''}{p_2'}$, and the result follows.

(2) Use the similar method as (1).

(3) If $p_2' < 0$ and $p_2'' > 0$, then $0 < (\log(-p_2'))' = \frac{p_2''}{p_2'}$ which contradicts with the assumptions for the signs of p_2' and p_2''. The similar argument for the assumptions $p_1' > 0$, $p_2' > 0$, $p_2'' < 0$. □

Finally we point out that the construction in Corollary 1.6 can not obtain a closed horizontal envelope γ. Indeed, take the derivative with respect to θ in (8) and use (7), we have

$$z' = t' - (p')^2 - pp'' = -p^2 - pp'',$$

which is equivalent to

$$z(\theta) = -\int_0^\theta p^2(\alpha) + p(\alpha)p''(\alpha)d\alpha + z(0) \quad \text{for any } \theta \in [0, 2\pi]. \tag{24}$$

If the curve is closed, say $z(0) = z(2\pi)$ and $p(0) = p(2\pi)$, by using integration by parts in (24) one gets

$$\int_0^{2\pi} p^2(\alpha) - \left(p'(\alpha)\right)^2 d\alpha = 0. \tag{25}$$

However, according to Santaló [24] (equation (1.8) in I.1.2) we know that

$$F = \frac{1}{2}\int_0^{2\pi} p^2(\alpha) - (p'(\alpha))^2 d\alpha,$$

where F is the enclosed area of the projection $\pi(\gamma)$ of the curve γ on the xy-plane. Therefore, (25) is equivalent to that γ must be a vertical line segment which contradicts with closeness of the curve.

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Yen-Chang Huang
Department of Applied Mathematics
National University of Tainan
Tainan City 70005, Taiwan
Email address: ychuang@mail.nutn.edu.tw