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ORBIT EQUIVALENCE ON SELF-SIMILAR GROUPS AND THEIR C^*-ALGEBRAS

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Abstract. Following Matsumoto's definition of continuous orbit equivalence for one-sided subshifts of finite type, we introduce the notion of orbit equivalence to canonically associated dynamical systems, called the limit dynamical systems, of self-similar groups. We show that the limit dynamical systems of two self-similar groups are orbit equivalent if and only if their associated Deaconu groupoids are isomorphic as topological groupoids. We also show that the equivalence class of Cuntz-Pimsner groupoids and the stably isomorphism class of Cuntz-Pimsner algebras of self-similar groups are invariants for orbit equivalence of limit dynamical systems.

1. Introduction

Interdisciplinary study between orbit equivalence of topological dynamics and C^*-algebras has a long and fruitful history. One of the most important and initiating results of this area is the classification of Cantor minimal systems by Giordano, Putnam and Skau [6]. After that, Matsumoto [9] defined continuous orbit equivalence on one-sided subshifts of finite type (SFTs) and studied relations between the orbit structures of one-sided SFTs and their corresponding Cuntz-Krieger algebras.

Recently, Matsumoto's continuous orbit equivalence has been generalized in many directions. In [2, 4], the notion of orbit equivalence of directed graphs was introduced and showed that, under a mild restriction on the graphs, orbit equivalence of directed graphs, their graph groupoids being isomorphic, and the existence of a diagonal preserving $*$-isomorphism between groupoid C^*-algebras are equivalent. Moreover, relations between continuous orbit equivalence on one-sided SFTs and flow equivalence on two-sided SFTs are studied in

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[3,11]. Li [8] generalized the notion of continuous orbit equivalence to topologically free group actions on totally disconnected spaces and their transformation groupoids. Furthermore, Matsumoto [10] developed the notion of asymptotic continuous orbit equivalence on Smale spaces and classified them in terms of their asymptotic equivalence groupoids and associated Ruelle algebras.

In this paper, we present a generalization of continuous orbit equivalence to induced dynamical systems on self-similar groups. For a self-similar group (G, X), the limit dynamical system (T_G, σ) is defined as the quotient of the one-sided full shift (X^ω, σ) by G-action. Then the induced system (T_G, σ) has many similarities and differences when compared to one-sided SFTs. One difference we need to mention is that T_G is a connected space. Boyle and Tomiyama [1] showed that if X and Y are connected spaces and $f\colon X \to X$ and $g\colon Y \to Y$ are homeomorphisms, then (X, f) and (Y, g) are orbit equivalent if and only if they are flip conjugate. In the case of limit dynamical systems of self-similar groups, the space T_G is connected, but the shift map σ on T_G is not a homeomorphism but only a continuous surjective map. Still the connected space property implies that when two limit dynamical systems are orbit equivalent, then they behave like flip conjugacy (Corollary 3.7).

We will show that orbit equivalence on limit dynamical systems is equivalent to the existence of a diagonal preserving isomorphism on the Deaconu groupoids of the limit dynamical systems (Theorem 3.6). This result corresponds to the classification theorems of Matsumoto [9] and Matsumoto-Matui [11] where one-sided subshifts of finite type are continuous orbit equivalent if and only if their associated étale groupoids are isomorphic.

There is a difference between SFTs and self-similar groups. For SFTs or graphs, the groupoid algebras of the aforementioned étale groupoids are isomorphic to Cuntz-Krieger algebras or graph algebras, respectively. However, the corresponding C^*-algebra of a self-similar group (G, X) is the Cuntz-Pimsner algebra $\mathcal{O}_G$, which is isomorphic to the convolution algebra of the Cuntz-Pimsner groupoid C_G [16]. In the case of SFTs or graphs, the Cuntz-Pimsner groupoid, called the extended Weyl groupoid in [2], is isomorphic to the Deaconu groupoid. On self-similar groups, while the Cuntz-Pimsner groupoid C_G is defined on the one-sided full shift X^ω, the Deaconu groupoid D_G is defined on the limit space $T_G = q(X^\omega)$, where $q\colon X^\omega \to T_G$ is the quotient map by the G-action. Because of the quotient structure from X^ω to T_G, the Cuntz-Pimsner groupoid C_G and the Deaconu groupoid D_G are not isomorphic to each other but equivalent in the sense of Muhly, Renault and Williams [13].

This paper is organized as follows. In Section 2, we provide background on self-similar groups and their limit dynamical systems. In Section 3, we define orbit equivalence of limit dynamical systems and show that the Deaconu groupoids and their groupoid algebras with diagonal algebras are complete invariants of orbit equivalence on limit dynamical systems of self-similar groups. In Section 4, we study interrelation between the Deaconu groupoids and the Cuntz-Pimsner groupoids of self-similar groups.

2. Self-similar groups

We review the properties of self-similar groups. Every material in this section is taken from [15, 16].

Suppose that X is a finite set. We denote by X^n the set of words of length n in X with $X^0 = \{\emptyset\}$, and define $X^* = \cup_{n=0}^{\infty} X^n$. The set X^* has a natural structure of a rooted tree: the root is $\emptyset$, the vertices are words in X^*, the edges are of the form u to ux where $u \in X^*$ and $x \in X$. Then the boundary of the tree X^* is identified with the space X^ω of right-infinite paths of the form $x_1 x_2 \cdots$ where $x_i \in X$. The product topology of a discrete set X is given on X^ω. A cylinder set $Z(u)$ for each $u \in X^*$ is

$$Z(u) = \{\xi \in X^\omega \colon \xi = x_0 x_1 \cdots \text{ such that } x_0 \cdots x_{|u|-1} = u\}.$$

Then the collection of all such cylinder sets forms a basis for the product topology on X^ω.

A *self-similar group* (G, X) consists of a finite set X and a faithful action of a group G on X^* such that, for all $g \in G$ and $x \in X$, there exist unique $y \in X$ and $h \in G$ such that

$$g(xu) = yh(u) \text{ for every } u \in X^*.$$

The unique element h is called the *restriction* of g at x and denoted by $g|_x$. The restriction extends to X^* via the inductive formula

$$g|_{xy} = (g|_x)\,|_y$$

so that for every $u, v \in X^*$ we have

$$g(uv) = g(u)g|_u(v).$$

The G-action extends to an action of G on X^ω given by

$$g(x_0 x_1 \cdots) = g(x_0 \cdots x_{n-1}) g|_{x_0 \cdots x_{n-1}}(x_n \cdots).$$

Conditions on self-similar groups

A self-similar group (G, X) is called *contracting* if there is a finite subset N of G satisfying the following: For every $g \in G$, there is $n \geq 0$ such that $g|_v \in N$ for every $v \in X^*$ of length $|v| \geq n$. If the group is contracting, the smallest set N satisfying this condition is called the *nucleus* of the group. We say that (G, X) is *regular* if, for every $g \in G$ and every $w \in X^\omega$, either $g(w) \neq w$ or there is a neighborhood of w such that every point in the neighborhood is fixed by g.

For a self-similar group (G, X), let $X \cdot G = \{x \cdot g \colon x \in X, g \in G\}$ be the set of transformations on $X^* \cup X^\omega$ given by

$$x \cdot g \colon u \mapsto xg(u).$$

The group G acts on $X \cdot G$ from left by composition:

$$h \cdot (x \cdot g) = h(x) \cdot (h|_x g).$$

We say that (G, X) is *recurrent* if the left G-action on $X \cdot G$ is transitive, i.e., for all $x \cdot g$ and $y \cdot h$ in $X \cdot G$, there is a $\gamma \in G$ such that $\gamma \cdot (x \cdot g) = y \cdot h$.

The following property was mentioned at [14, p. 235] without proof. So we provide a complete proof.

Lemma 2.1. *A self-similar group (G, X) satisfies the recurrent condition if and only if, for any two words a, b of equal length and every $h \in G$, there is a $g \in G$ such that $g(a) = b$ and $g|_a = h$.*

Proof. For all $x, y \in X$ and $h \in G$, $x \cdot 1$ and $y \cdot h$ are elements of $X \cdot G$. Then recurrent condition implies that there is a $g \in G$ such that

$$g \cdot (x \cdot 1) = g(x) \cdot (g|_x \cdot 1) = g(x) \cdot g|_x = y \cdot h.$$

So we have for every $u \in X^* \cup X^\omega$

$$(g(x) \cdot g|_x)(u) = g(x) g|_x(u) = (y \cdot h)(u) = yh(u).$$

Hence $g(x) = y$ and $g|_x = h$ hold. It is straight to check general case by induction on the length of words.

For the converse, let $x \cdot g$ and $y \cdot h$ be arbitrary elements of $X \cdot G$. Then, for $x, y \in X$ and $hg^{-1} \in G$, there is a $\gamma \in G$ such that $\gamma(x) = y$ and $\gamma|_x = hg^{-1}$. So we have $\gamma|_x g = h$ and

$$\gamma \cdot (x \cdot g) = \gamma(x) \cdot (\gamma|_x g) = y \cdot h.$$

Thus (G, X) satisfies the recurrent condition. □

Standing assumption. In this paper, we assume that our self-similar group (G, X) is a contracting, recurrent, and regular group and that the group G is finitely generated.

Remark 2.2. We only need recurrent and regular conditions for equivalences of dynamical systems and their corresponding groupoids of self-similar groups. The contracting condition is required for the amenability of groupoids and groupoid C^*-algebras. See [15, 16] for dynamical systems and C^*-algerbas of self-similar groups.

Limit dynamical systems of self-similar groups

Suppose that (G, X) is a self-similar group. We consider the space $X^{\mathbb{Z}}$ of bi-infinite paths $\cdots x_{-1}.x_0 x_1 x_2 \cdots$ over X and the shift map $\sigma \colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ given by

$$\cdots x_{-1}.x_0 x_1 x_2 \cdots \mapsto \cdots x_{-1} x_0 . x_1 x_2 \cdots .$$

The direct product topology of the discrete set X is given on $X^{\mathbb{Z}}$. We say that two paths $\cdots x_{-1}.x_0 x_1 x_2 \cdots$ and $\cdots y_{-1}.y_0 y_1 y_2 \cdots$ in $X^{\mathbb{Z}}$ are *asymptotically equivalent* if there is a finite set $I \subset G$ and a sequence $g_n \in I$ such that

$$g_n(x_n x_{n+1} \cdots) = y_n y_{n+1} \cdots$$

for every $n \in \mathbb{Z}$. The quotient of $X^{\mathbb{Z}}$ by the asymptotic equivalence relation is called the *limit solenoid* of (G, X) and is denoted S_G. The shift map on $X^{\mathbb{Z}}$ is transferred to an induced homeomorphism on S_G, which we will denote by σ when there is no confusion.

Let $q\colon X^{\mathbb{Z}} \to S_G$ be the quotient map by the asymptotic equivalence relation and $\pi\colon X^{\mathbb{Z}} \to X^{\omega}$ the canonical projection map. We restrict the asymptotic equivalence relation on $X^{\mathbb{Z}}$ to X^{ω} so that $x_0x_1x_2\cdots$ and $y_0y_1y_2\cdots$ in X^{ω} are *asymptotically equivalent* if there is a $g \in G$ such that

$$g(x_0x_1\cdots) = y_0y_1\cdots .$$

The quotient of X^{ω} by the asymptotic equivalence is called the *limit space* of (G, X) and denoted as T_G. Then the canonical projection $\pi\colon X^{\mathbb{Z}} \to X^{\omega}$ and the shift map $\sigma\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ induce a natural projection map $S_G \to T_G$ and a shift map $T_G \to T_G$

$$\cdots\xi_{-1}.\xi_0\xi_1\cdots \mapsto \xi_0\xi_1\cdots \text{ and } \xi_0\xi_1\cdots \mapsto \xi_1\xi_2\cdots, \text{ respectively.}$$

We denote these induced projection, quotient and shift maps as π, q and σ, respectively, when there is no confusion. The restricted dynamical system (T_G, σ) is called the *limit dynamical system* of (G, X). Then it is easy to check that the projection maps on $X^{\mathbb{Z}}$ and S_G, quotient maps on $X^{\mathbb{Z}}$ and X^{ω}, and shift maps on $X^{\mathbb{Z}}$, X^{ω}, S_G and T_G are commuting with each other.

Remark 2.3. (1) In [15, 16], Nekrashevych used the shift map as

$$\cdots x_{-2}x_{-1}.x_0\cdots \mapsto \cdots x_{-2}.x_{-1}x_0\cdots$$

so that the limit space is given as the quotient of left-hand-sided full shift.

(2) The limit solenoids and limit dynamical system are the quotients of two-sided and one-sided full shifts on X, respectively, by the asymptotic equivalence relation.

(3) The existence of $g \in G$ satisfying $g(x_0x_1\cdots) = y_0y_1\cdots$ is equivalent to the existence of a $g_n = g|_{x_0\cdots x_{n-1}} \in G$ such that

$$g(x_0x_1\cdots) = y_0\cdots y_{n-1}g_n(x_n\cdots)$$

for every $n \in \mathbb{N}$.

(4) The limit solenoid S_G and limit space T_G are compact, connected and Hausdorff spaces [16, Propoistion 2.4].

(5) The limit solenoid (S_G, σ) is a mixing Smale space [16, Propoistion 6.10].

3. Orbit equivalence of limit dynamical systems

We generalize Matsumoto's definition of continuous orbit equivalence for one-sided SFTs to limit dynamical systems of self-similar groups. See [9, 11] for more details.

Definition 3.1. Suppose that (G, X) and (H, Y) are self-similar groups and that (T_G, σ) and (T_H, σ) are their corresponding limit dynamical systems, respectively. The limit dynamical systems (T_G, σ) and (T_H, σ) are said to be *orbit equivalent* if there are a homeomorphism $h\colon T_G \to T_H$ and nonnegative integers k_1, l_1, k_2, l_2 such that

$$\sigma^{k_1} \circ h \circ \sigma(\xi) = \sigma^{l_1} \circ h(\xi) \text{ and } \sigma^{k_2} \circ h^{-1} \circ \sigma(\eta) = \sigma^{l_2} \circ h^{-1}(\eta)$$

for every $\xi \in T_G$ and $\eta \in T_H$.

Remark 3.2. In Matsumoto's definition for one-sided SFTs, $k_1, l_1\colon T_G \to \mathbb{N} \cup \{0\}$ and $k_2, l_2\colon T_H \to \mathbb{N} \cup \{0\}$ are continuous maps. In the case of limit dynamical systems, T_G and T_H are connected spaces by Remark 2.3 so that k_i and l_i become constant maps.

Lemma 3.3. *For every natural number n, we have*

$$\sigma^{k_1 n} \circ h \circ \sigma^n(\xi) = \sigma^{l_1 n} \circ h(\xi) \text{ and } \sigma^{k_2 n} \circ h^{-1} \circ \sigma^n(\eta) = \sigma^{l_2 n} \circ h^{-1}(\eta).$$

Proof. If $\sigma^{k_1 n} \circ h \circ \sigma^n(\xi) = \sigma^{l_1 n} \circ h(\xi)$ holds for some $n \in \mathbb{N}$, then we have

$$\begin{aligned} \sigma^{k_1 n} \circ h \circ \sigma^{n+1}(\xi) &= \sigma^{k_1 n} \circ h \circ \sigma^n \circ \sigma(\xi) \\ &= \sigma^{l_1 n} \circ h \circ \sigma(\xi) \end{aligned}$$

so that

$$\begin{aligned} \sigma^{k_1(n+1)} \circ h \circ \sigma^{n+1}(\xi) &= \sigma^{k_1} \circ \sigma^{k_1 n} \circ h \circ \sigma^{n+1}(\xi) \\ &= \sigma^{k_1} \circ \sigma^{l_1 n} \circ h \circ \sigma(\xi) \\ &= \sigma^{l_1 n} \circ \sigma^{k_1} \circ h \circ \sigma(\xi) \\ &= \sigma^{l_1 n} \circ \sigma^{l_1} \circ h(\xi) \\ &= \sigma^{l_1(n+1)} \circ h(\xi). \end{aligned}$$

The case of h^{-1} is the same. Thus the above equations are true by induction. $\square$

Proposition 3.4. *The limit dynamical systems (T_G, σ) and (T_H, σ) are orbit equivalent if and only if the natural numbers given in the definition satisfy $k_1 - l_1 = k_2 - l_2 = 1$ or -1.*

Proof. Suppose that (T_G, σ) and (T_H, σ) are orbit equivalent. We remark that (T_G, σ) has non-eventually periodic elements by [20, Lemma 4.7]. Let η be a non-eventually periodic element in T_G. Then Lemma 3.3 implies

$$\begin{aligned} \sigma^{k_1 l_2} \circ h \circ \sigma^{l_2} \circ h^{-1}(\eta) &= (\sigma^{k_1 l_2} \circ h \circ \sigma^{l_2})(h^{-1}(\eta)) \\ &= \sigma^{l_1 l_2} \circ h \circ h^{-1}(\eta) = \sigma^{l_1 l_2}(\eta) \\ &= (\sigma^{k_1 l_2} \circ h)(\sigma^{l_2} \circ h^{-1}(\eta)) \\ &= \sigma^{k_1 l_2} \circ h \circ \sigma^{k_2} \circ h^{-1} \circ \sigma(\eta) \end{aligned}$$

and

$$\begin{aligned} \sigma^{k_1 k_2} \circ \sigma^{l_1 l_2}(\eta) &= \sigma^{k_1 k_2} \circ \sigma^{k_1 l_2} \circ h \circ \sigma^{k_2} \circ h^{-1} \circ \sigma(\eta) \\ &= \sigma^{k_1 l_2} \circ \sigma^{k_1 k_2} \circ h \circ \sigma^{k_2} \circ h^{-1} \circ \sigma(\eta) \end{aligned}$$

$$
\begin{aligned}
&= \sigma^{k_1 l_2} \circ (\sigma^{k_1 k_2} \circ h \circ \sigma^{k_2})(h^{-1} \circ \sigma(\eta)) \\
&= \sigma^{k_1 l_2} \circ (\sigma^{l_1 k_2} \circ h)(h^{-1} \circ \sigma(\eta)) \\
&= \sigma^{k_1 l_2 + k_2 l_1 + 1}(\eta).
\end{aligned}
$$

So we have $k_1k_2 + l_1l_2 = k_1l_2 + k_2l_1 + 1$ and

$$k_1k_2 + l_1l_2 - k_1l_2 - k_2l_1 = 1 = (k_1 - l_1)(k_2 - l_2).$$

Since k_i and l_i are natural numbers, we conclude that $k_1-l_1 = k_2-l_2 = 1$ or -1. The converse is trivial. □

Suppose that (G, X) is a self-similar group and that (T_G, σ) is its limit dynamical system. The *Deaconu groupoid* of (T_G, σ) is

$$D_G = \{(\xi, m - n, \eta) \colon \xi, \eta \in T_G, m, n \in \mathbb{N}, \sigma^m(\xi) = \sigma^n(\eta)\}.$$

A pair $\{(\xi, m - n, \eta), (\chi, k - l, \zeta)\} \in D_G^{(2)}$ is composable if $\eta = \chi$, and multiplication and inverse are given by

$$(\xi, m-n, \eta)(\eta, k-l, \zeta) = (\xi, m-n+k-l, \zeta) \text{ and } (\xi, m-n, \eta)^{-1} = (\eta, n-m, \xi).$$

With these operations, D_G is a groupoid. For $(\xi, m - n, \eta) \in D_G$, the domain and range are given by

$$d(\xi, m - n, \eta) = (\xi, 0, \xi) \text{ and } r(\xi, m - n, \eta) = (\eta, 0, \eta).$$

The unit space of D_G denoted by $D_G^{(0)}$ is identified with T_G via the diagonal map, and the isotropy group bundle is given by $I = \{(\xi, m, \xi) \in D_G\}$. For open sets U, V of $T_{(G,E)}$ and $k, l \geq 0$, let

$$Z(U, k, l, V) = \{(\xi, k - l, \eta) : \xi \in U, \eta \in V, \sigma^k x = \sigma^l y\}.$$

Then the collection of these sets is a basis for a second countable locally compact Hausdorff topology on D_G, and the counting measure is a Haar system of D_G [5]. If (G, X) is a self-similar group satisfying the Standing Assumption, then the Deaconu groupoid D_G is an étale, amenable, topologically principal, locally compact, Hausdorff groupoid.

Proposition 3.5. *For every $(\xi, m - n, \eta) \in D_G$, $\sigma^k(\xi) = \sigma^l(\eta)$ holds for all $k, l \in \mathbb{N}$ such that $k - l = m - n$.*

Proof. If $k > m$, then $k - m = l - n$ implies

$$\sigma^k(\xi) = \sigma^{k-m} \circ \sigma^m(\xi) = \sigma^{l-n} \circ \sigma^n(\eta) = \sigma^l(\eta).$$

If $k < m$, choose any $x = x_0x_1 \cdots \in q^{-1}(\xi)$ and $y = y_0y_1 \cdots \in q^{-1}(\eta)$ where $q \colon X^\omega \to T_G$ is the quotient map. As the shift maps on X^ω and T_G, respectively, and the quotient map are commuting to each other, we have

$$\sigma^m(x) = x_m x_{m+1} \cdots \in q^{-1}(\sigma^m(\xi)) \text{ and } \sigma^n(y) = y_n y_{n+1} \cdots \in q^{-1}(\sigma^n(\eta)).$$

Thus $\sigma^m(\xi) = \sigma^n(\eta)$ implies that $\sigma^m(x) = x_m x_{m+1} \cdots$ and $\sigma^n(y) = y_n y_{n+1} \cdots$ are asymptotically equivalent so that there is an $h \in G$ such that

$$h(x_m x_{m+1} \cdots) = y_n y_{n+1} \cdots .$$

Because $m - k = n - l$, we have $|x_k \cdots x_{m-1}| = |y_l \cdots y_{n-1}|$. Then Lemma 2.1 implies that there is a $g \in G$ such that

$$g(x_k \cdots x_{m-1}) = y_l \cdots y_{n-1} \text{ and } g|_{x_k \cdots x_{m-1}} = h.$$

Hence

$$\begin{aligned} g(\sigma^k(x)) &= g(x_k \cdots x_{m-1} x_m \cdots) \\ &= g(x_k \cdots x_{m-1}) g|_{x_k \cdots x_{m-1}}(x_m \cdots) \\ &= y_l \cdots y_{n-1} h(x_m \cdots) \\ &= y_l \cdots y_{n-1} y_n \cdots \\ &= \sigma^l(y) \end{aligned}$$

implies that $\sigma^k(x)$ is asymptotically equivalent to $\sigma^l(y)$. So we have

$$q(\sigma^k(x)) = \sigma^k \circ q(x) = \sigma^k(\xi) = \sigma^l(\eta) = \sigma^l \circ q(y) = q(\sigma^l(y)).$$

Therefore $\sigma^k(\xi) = \sigma^l(\eta)$ holds for all $k, l \in \mathbb{N}$ such that $k - l = m - n$. $\square$

Theorem 3.6 ([11, Theorem 2.3])**.** *Let (T_G, σ) and (T_H, σ) be the limit dynamical systems of self-similar groups (G, X) and (H, Y), respectively. Then the following assertions are equivalent*:

(1) *(T_G, σ) and (T_H, σ) are orbit equivalent.*
(2) *The Deaconu groupoids D_G and D_H are isomorphic.*
(3) *There is a C^*-isomorphism $\Psi\colon C^*(D_G) \to C^*(D_H)$ with $\Psi(C(T_G)) = C(T_H)$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that (T_G, σ) and (T_H, σ) are orbit equivalent and that $h\colon T_G \to T_H$ is the corresponding homeomorphism. For every $(\xi, m - n, \eta) \in D_G$, $\sigma^m(\xi) = \sigma^n(\eta)$ and Lemma 3.3 imply that

$$\sigma^{ml_1} \circ h(\xi) = \sigma^{mk_1} \circ h \circ \sigma^m(\xi) = \sigma^{mk_1} \circ h \circ \sigma^n(\eta)$$

and

$$\begin{aligned} \sigma^{nk_1} \circ \sigma^{ml_1} \circ h(\xi) &= \sigma^{nk_1} \circ \sigma^{mk_1} \circ h \circ \sigma^n(\eta) \\ &= \sigma^{mk_1} \circ \sigma^{nk_1} \circ h \circ \sigma^n(\eta) \\ &= \sigma^{mk_1} \circ \sigma^{nl_1} \circ h(\eta). \end{aligned}$$

So $(h(\xi), (m-n)(l_1 - k_1), h(\eta)) \in D_H$. Here, we recall that $l_1 - k_1 = 1$ or -1 by Proposition 3.4. Define $\tilde{h}\colon D_G \to D_H$ by

$$(\xi, m - n, \eta) \mapsto (h(\xi), (m-n)(l_1 - k_1), h(\eta)).$$

Then it is trivial that $\tilde{h}$ is a continuous groupoid isomorphism.

(2) $\Longrightarrow$ (1). Suppose that $\tilde{h}\colon D_G \to D_H$ is a continuous isomorphism. When we define $h = \tilde{h}|_{D_G^{(0)}} : T_G \to T_H$, h is obviously a homeomorphism. Since $\tilde{h}$ is a groupoid isomorphism,

$$\begin{aligned} d(\tilde{h}(\xi, m, \eta)) &= \tilde{h}(r(\xi, m, \eta)) = \tilde{h}(\xi, 0, \xi) = h(\xi) \text{ and} \\ r(\tilde{h}(\xi, m, \eta)) &= \tilde{h}(d(\xi, m, \eta)) = \tilde{h}(\eta, 0, \eta) = h(\eta) \end{aligned}$$

imply that $\tilde{h}(\xi, m, \eta) = (h(\xi), \tilde{h}|_{\mathbb{Z}}(m), h(\eta))$. Then we observe

$$\begin{aligned} \tilde{h}((\xi, m, \eta)(\eta, n, \zeta)) &= \tilde{h}(\xi, m+n, \zeta) = (h(\xi), \tilde{h}|_{\mathbb{Z}}(m+n), h(\zeta)) \\ &= \tilde{h}(\xi, m, \eta)\tilde{h}(\eta, n, \zeta) \\ &= (h(\xi), \tilde{h}|_{\mathbb{Z}}(m), h(\eta))(h(\eta), \tilde{h}|_{\mathbb{Z}}(n), h(\zeta)) \\ &= (h(\xi), \tilde{h}|_{\mathbb{Z}}(m) + \tilde{h}|_{\mathbb{Z}}(n), h(\zeta)) \end{aligned}$$

and for every $m, n \in \mathbb{Z}$,

$$\tilde{h}|_{\mathbb{Z}}(m+n) = \tilde{h}|_{\mathbb{Z}}(m) + \tilde{h}|_{\mathbb{Z}}(n).$$

Hence $\tilde{h}|_{\mathbb{Z}}$ is a group homomorphism on $\mathbb{Z}$. Because $\tilde{h}$ is surjective, $\tilde{h}|_{\mathbb{Z}}$ is also a surjective homomorphism, i.e., $\tilde{h}|_{\mathbb{Z}}$ is an automorphism on $\mathbb{Z}$ so that $\tilde{h}|_{\mathbb{Z}}(m) = m$ or $-m$ for every $m \in \mathbb{Z}$.

For every $\xi \in T_G$, consider $(\xi, 1, \sigma(\xi)) \in D_G$ and

$$\tilde{h}(\xi, 1, \sigma(\xi)) = (h(\xi), \tilde{h}|_{\mathbb{Z}}(1), h(\sigma(\xi))) \in D_H.$$

Then Proposition 3.5 implies that

$$\begin{aligned} \sigma \circ h(\xi) = h \circ \sigma(\xi) \qquad &\text{if } \tilde{h}|_{\mathbb{Z}}(1) = 1, \\ h(\xi) = \sigma \circ h \circ \sigma(\xi) \ &\text{if } \tilde{h}|_{\mathbb{Z}}(1) = -1. \end{aligned}$$

By Proposition 3.4, we have the same relations for $\tilde{h}^{-1}$. Therefore (T_G, σ) and (T_H, σ) are orbit equivalent.

(2) $\Longleftrightarrow$ (3) follows from [12, Theorem 5.1] as D_G and D_H are topologically principal. $\square$

We obtain the following from the proof of the above theorem.

Corollary 3.7 ([9, Theorem 5.6])**.** *Let (T_G, σ) and (T_H, σ) be the limit dynamical systems of self-similar groups (G, X) and (H, Y), respectively. Then (T_G, σ_G) and (T_H, σ_H) are orbit equivalent if and only if there is a homeomorphism $h\colon T_G \to T_H$ such that either $h \circ \sigma_G = \sigma_H \circ h$ or $\sigma_H \circ h \circ \sigma_G = h$.*

Remark 3.8. Let X and Y be compact connected Hausdorff spaces and $f\colon X \to X$ and $g\colon Y \to Y$ homeomorphisms. In [1], Boyle and Tomiyama showed that orbit equivalence between (X, f) and (Y, g) implies flip conjugacy between them, i.e., $h \circ f = g \circ h$ or $h \circ f = g^{-1} \circ h$ for some homeomorphism $h\colon X \to Y$. The above Corollary is a generalization of Boyle and Tomiyama's result to epimorphism case.

4. Groupoids on self-similar groups

Cuntz-Pimsner algebras and groupoids

Suppose that (G, X) is a self-similar group. The Cuntz-Pimsner algebra $\mathcal{O}_G$ of a self-similar group (G, X) is the universal C^*-algebra generated by the sets G and $\{s_x : x \in X\}$ satisfying the following relations ([16, Definition 3.1]):

(1) All relations of G,
(2) $s_x^* s_x = 1$ for every $x \in X$ and $\sum_{x \in X} s_x s_x^* = 1$, and
(3) for all $g \in G$ and $x \in X$, $g \cdot s_x = s_{g(x)} \cdot g|_x$.

We denote by $\langle G, X \rangle$ the inverse semigroup generated by the elements s_x, s_x^* and G in $\mathcal{O}_G$. For $u = u_1 \cdots u_n \in X^n$, we let $s_u = s_{u_1} \cdots s_{u_n}$ and $s_u^* = s_{u_n}^* \cdots s_{u_1}^*$. Likewise, we also let $s_\emptyset$ be the identity. Then it is easy to observe that every element of $\langle G, X \rangle$ is uniquely written in the form $s_u g s_v^*$ for some $u, v \in X^*$ and $g \in G$ [16, Proposition 3.2].

The inverse semigroup $\langle G, X \rangle$ acts faithfully on X^ω by the partial homeomorphism

$$s_u g s_v^*(vw) = ug(w)$$

with domain vX^ω and range uX^ω. The groupoid of germs of $\langle G, X \rangle$ is denoted by C_G and called the *Cuntz-Pimsner groupoid* of (G, X). It is a well-known fact that the groupoid C_G is an étale, amenable, topologically principal, locally compact and Hausdorff groupoid. The étale, topologically principal, locally compact and Hausdorff properties come from the definition of groupoids of germs. Amenable is from [16, Theorem 5.6]. We refer to [17, 19] for the definition and properties of groupoids of germs and groupoid algebras.

Theorem 4.1 ([16, Theorem 5.1])**.** *The Cuntz-Pimsner algebra $\mathcal{O}_G$ is isomorphic to the convolution C^*-algebra of C_G.*

Proposition 4.2. *In the Cuntz-Pimsner groupoid C_G, $[s_u g s_v^*, vw] = [s_a h s_b^*, bc]$ if and only if $vw = bc$, $ug(w) = ah(c)$ and $|v| - |b| = |u| - |a|$ hold.*

Proof. By definition of the groupoids of germs, $[s_u g s_v^*, vw] = [s_a h s_b^*, bc]$ if and only if $vw = bc$ and $s_u g s_v^* = s_a h s_b^*$ on a neighborhood of vw.

For $v = v_0 \cdots v_n$ and $b = b_0 \cdots b_m$ with $vw = bc$, we have

$$\begin{cases} b = v b_{n+1} \cdots b_m \text{ and } w = b_{n+1} \cdots b_m c & \text{if } n < m, \\ v = b v_{m+1} \cdots v_n \text{ and } c = v_{m+1} \cdots v_n w & \text{if } n > m. \end{cases}$$

If $vw = bc$, $|v| < |b|$ and $s_u g s_v^* = s_a h s_b^*$ on a neighborhood of vw, then

$$\begin{aligned} ug(w) = s_u g s_v^*(vw) &= s_u g s_v^*(v b_{n+1} \cdots b_m c) \\ &= s_u g(b_{n+1} \cdots b_m c) \\ &= s_u g(b_{n+1} \cdots b_m) g_{b_{n+1} \cdots b_m}(c) \\ &= ug(b_{n+1} \cdots b_m) g|_{b_{n+1} \cdots b_m}(c) \\ &= s_a h s_b^*(bc) \end{aligned}$$

$$= ah(c)$$

implies $ug(w) = ah(c)$ and $|a| = |u| + |b| - |v|$.

Conversely, if $vw = bc$, $|v| < |b|$, $ug(w) = ah(c)$ and $|v| - |b| = |u| - |a|$ hold, then $ug(w) = ah(c)$ and $|a| = |b| - |v| + |u|$ mean $u = a_0 \cdots a_{|u|-1}$ and

$$g(w) = g(b_{n+1} \cdots b_m c) = g(b_{n+1} \cdots b_m) g|_{b_{n+1} \cdots b_m}(c) = a_{|u|} a_{|u|+1} \cdots a_{|a|-1} h(c).$$

So we have

$$ug(b_{n+1} \cdots b_m) = a \text{ and } g|_{b_{n+1} \cdots b_m}(c) = h(c).$$

On the second equality above, the regular condition on (G, X) implies that there is a neigborhood U of c such that $g|_{b_{n+1} \cdots b_m}(\beta) = h(\beta)$ for every $\beta \in U$. It is trivial that $bU = \{b\beta \in X^\omega : \beta \in U\}$ is a neighborhood of $bc = vw$. Now we show $s_u g s_v^* = s_a h s_b^*$ on bU. For every $b\beta = v b_{n+1} \cdots b_m \beta \in bU$, we have

$$\begin{aligned} s_u g s_v^*(b\beta) &= s_u g s_v^*(v b_{n+1} \cdots b_m \beta) \\ &= s_u g(b_{n+1} \cdots b_m \beta) \\ &= s_u g(b_{n+1} \cdots b_m) g|_{b_{n+1} \cdots b_m}(\beta) \\ &= ug(b_{n+1} \cdots b_m) h(\beta) \\ &= ah(\beta) \\ &= s_a h s_b^*(b\beta), \end{aligned}$$

and $s_u g s_v^* = s_a h s_b^*$ on bU. The case for $|v| > |b|$ can be obtained by the same argument. □

Computation gives us the following lemma.

Corollary 4.3. *If $[s_u g s_v^*, vw]$ and $[s_a h s_b^*, bc]$ are composable in C_G, then*

$$[s_u g s_v^*, vw] \cdot [s_a h s_b^*, bc] = \begin{cases} [s_{ug(\alpha)} g|_\alpha h s_b^*, bc] & \text{if } a = v\alpha, \\ [s_u g (h^{-1}|_\beta)^{-1} s_{bh^{-1}(\beta)}^*, bc] & \text{if } v = a\beta. \end{cases}$$

Relations between C_G and D_G

To study properties of C_G and D_G more conveniently, we construct another groupoid between them.

Lemma 4.4. *The quotient map $q \colon X^\omega \to T_G$ by the asymptotic equivalence relation is an open map.*

Proof. Let $\pi' \colon X^{\mathbb{Z}} \to X^\omega$ and $\pi \colon S_G \to T_G$ be canonical projection maps and $q' \colon X^{\mathbb{Z}} \to S_G$ the quotient map. Then it is trivial that the projection maps are open maps and $q \circ \pi' = \pi \circ q'$. Since q' is also an open map by [20, Proposition 3.4], for every open set U in X^ω, $q(U) = \pi \circ q' \circ \pi^{-1}(U)$ shows that q is an open map. □

Because the quotient map q is an open map, the following holds.

Proposition 4.5 ([7, Lemma 5.1]). *The Deaconu groupoid D_G is equivalent to*

$$D_G^q = \{(x, (q(x), m-n, q(y)), y) \colon x, y \in X^\omega, (q(x), m-n, q(y)) \in D_G\}$$

in the sense of Muhly, Renault and Williams.

Proposition 4.6. *The Cuntz-Pimsner groupoid C_G is isomorphic to D_G^q as topological groupoids.*

Proof. For $(x, (q(x), m-n, q(y)), y) \in D_G^q$, $(q(x), m-n, q(y)) \in D_G$ means that

$$\sigma^m \circ q(x) = \sigma^n \circ q(y) = q \circ \sigma^m(x) = q \circ \sigma^n(y)$$

so that there is a $g \in G$ such that $g(\sigma^m(x)) = \sigma^n(y)$. Thus we have

$$g(\sigma^m(x)) = g s^*_{x_0 \cdots x_{m-1}}(x) = \sigma^n(y)$$

and

$$y = y_0 \cdots y_{n-1} \sigma^n(y) = s_{y_0 \cdots y_{n-1}} g s^*_{x_0 \cdots x_{m-1}}(x).$$

Define $f \colon D_G^q \to C_G$ by

$$(x, (q(x), m-n, q(y)), y) \mapsto [s_{y_0 \cdots y_{n-1}} g s^*_{x_0 \cdots x_{m-1}}, x].$$

Then it is routine to check that f is a continuous groupoid isomorphism. □

Remark 4.7. A group element g satisfying $g(\sigma^m(x)) = \sigma^n(y)$ is not unique. But, if $g, h \in G$ are such that $g(\sigma^m(x)) = h(\sigma^m(x))$, then the regular condition implies $g = h$ on a neighborhood of $\sigma^m(x)$ so that

$$[s_{y_0 \cdots y_{n-1}} g s^*_{x_0 \cdots x_{m-1}}, x] = [s_{y_0 \cdots y_{n-1}} h s^*_{x_0 \cdots x_{m-1}}, x].$$

Now the following four corollaries are trivial from Propositions 4.5, 4.6 and [13].

Corollary 4.8. (1) *The Cuntz-Pimsner groupoid C_G and the Deaconu groupoids D_G are equivalent in the sense of Muhly, Renault and Williams.*

(2) *The Cuntz-Pimsner algebras $\mathcal{O}_G$ and the groupoid algebras $C^*(D_G)$ are stably isomorphic.*

Remark 4.9. The Cuntz-Pimsner algebra $\mathcal{O}_G$ is stably isomorphic to the unstable Ruelle algebra of the limit solenoid (S_G, σ) [16, Theorem 6.11]. Since the unstable equivalence on S_G is determined by asymptotic equivalence on right-tail of bi-infinite sequences [16, Proposition 6.8] and T_G is the restriction of S_G on the right-hand-side, the Deaconu groupoid D_G is equivalent to the unstable Ruelle groupoid of (S_G, σ) [18].

Corollary 4.10. *Suppose that (G, X) and (H, Y) are self-similar groups.*

1. *The Cuntz-Pimsner groupoids C_G and C_H are equivalent if and only if the Deaconu groupoids D_G and D_H are equivalent in the sense of Muhly, Renault and Williams.*

(2) *The Cuntz-Pimsner algebras* $\mathcal{O}_G$ *and* $\mathcal{O}_H$ *are stably isomorphic if and only if the groupoid algebras* $C^*(D_G)$ *and* $C^*(D_H)$ *are stably isomorphic.*

Corollary 4.11. *The induced quotient map* $\tilde{q}\colon C_G \to D_G$ *defined by*

$$\begin{aligned}[s_u g s_v^*, vw] \simeq &(vw, (q(vw), |v| - |u|, q(ug(w))), ug(w)) \\ &\mapsto (q(vw), |v| - |u|, q(ug(w)))\end{aligned}$$

is a continuous groupoid epimorphism.

Corollary 4.12. *For every* $(q(ab), |a| - |c|, q(cd)) \in D_G$,

$$\begin{aligned}&\tilde{q}^{-1}(q(ab), |a| - |c|, q(cd)) \\ = &\left\{ (\alpha\beta, (q(\alpha\beta), |\alpha| - |\gamma|, q(\gamma\delta)), \gamma\delta) \colon \begin{array}{l} \alpha\beta \in q^{-1}(q(ab)), \gamma\delta \in q^{-1}(q(cd)), \\ |a| - |c| = |\alpha| - |\gamma| \end{array} \right\} \\ = &\left\{ [s_\gamma g s_\alpha^*, \alpha\beta] \colon \begin{array}{l} g \in G, \alpha\beta \in q^{-1}(q(ab)), \gamma g(\beta) \in q^{-1}(q(cd)), \\ |a| - |c| = |\alpha| - |\gamma| \end{array} \right\}.\end{aligned}$$

Theorem 4.13. *If the Cuntz-Pimsner groupoids* C_G *and* C_H *are isomorphic as topological groupoids, then the Deaconu groupoids* D_G *and* D_H *are isomorphic as topological groupoids.*

Proof. We consider D_G^q and D_H^q instead of C_G and C_H, respectively. Suppose that $\phi\colon D_G^q \to D_H^q$ is a groupoid isomorphism such that ϕ and ϕ^{-1} are continuous. Then

$$\phi({D_G^q}^{(0)}) = {D_H^q}^{(0)}$$

implies that there is a homeomorphism $f\colon X^\omega \to Y^\omega$ such that $f = \phi|_{{D_G^q}^{(0)}}$.

Since the groupoid isomorphism ϕ satisfies

$$d \circ \phi = \phi \circ d \text{ and } r \circ \phi = \phi \circ r,$$

where d and r are domain and source maps, respectively, of groupoids, we have

$$\begin{aligned}d \circ \phi(x, (q(x), m-n, q(y)), y) &= \phi \circ d(x, (q(x), m-n, q(y)), y) \\ &= \phi(x, (q(x), 0, q(x)), x) \\ &= (f(x), (q \circ f(x), 0, q \circ f(x)), f(x)) \text{ and} \\ r \circ \phi(x, (q(x), m-n, q(y)), y) &= (f(y), (q \circ f(y), 0, q \circ f(y)), f(y))\end{aligned}$$

so that ϕ is given as one of the followings:

$$\begin{aligned}\phi(x, (q(x), m-n, q(y)), y) &= (f(x), (q \circ f(x), m-n, q \circ f(y)), f(y)) \text{ or} \\ \phi(x, (q(x), m-n, q(y)), y) &= (f(x), (q \circ f(x), n-m, q \circ f(y)), f(y)).\end{aligned}$$

We only prove the first case, and leave the second one to the readers.

When $\phi(x, (q(x), m-n, q(y)), y) = (f(x), (q \circ f(x), m-n, q \circ f(y)), f(y))$, we define $\psi\colon D_G \to D_H$ by

$$(q(x), m-n, q(y)) \mapsto (q \circ f(x), m-n, q \circ f(y)).$$

First check that ϕ is well-defined: Let $(q(x), m-n, q(y)) = (q(a), m-n, q(b))$ in D_G, and show

$$(q \circ f(x), m-n, q \circ f(y)) = (q \circ f(a), m-n, q \circ f(b)).$$

Since $(x, (q(x), 0, q(a)), a)$ and $(y, (q(y), 0, q(b)), b)$ are elements of D_G^q,

$$\phi(x, (q(x), 0, q(a)), a) = (f(x), (q \circ f(x), 0, q \circ f(a)), f(a)) \text{ and}$$
$$\phi(y, (q(y), 0, q(b)), b) = (f(y), (q \circ f(y), 0, q \circ f(b)), f(b))$$

are elements of D_H^q so that

$$(q \circ f(x), 0, q \circ f(a)) \text{ and } (q \circ f(y), 0, q \circ f(b))$$

are included in D_H. Then Proposition 3.5 induces

$$q \circ f(x) = q \circ f(a) \text{ and } q \circ f(y) = q \circ f(b).$$

Thus ϕ is a well-defined map.

For $(q(x), m-n, q(y))$ and $(q(y), k-l, q(z))$ in D_G,

$$(q(x), m-n, q(y)) \cdot (q(y), k-l, q(z)) = (q(x), m+k-n-l, q(z))$$

and

$$\begin{aligned} &\psi(q(x), m-n, q(y)) \cdot \psi(q(y), k-l, q(z)) \\ &= (q \circ f(x), m-n, q \circ f(y)) \cdot (q \circ f(y), k-l, q \circ f(z)) \\ &= (q \circ f(x), m-n+k-l, q \circ f(z)) \\ &= \psi(q(x), m+k-n-l, q(z)) \end{aligned}$$

show that ψ is a groupoid homomorphism.

If $\psi(q(x), m-n, q(y)) = \psi(q(u), m-n, q(v))$, then

$$q \circ f(x) = q \circ f(u) \text{ and } q \circ f(y) = q \circ f(v).$$

So we have

$$(q \circ f(x), 0, q \circ f(u)) \text{ and } (q \circ f(y), 0, q \circ f(v)) \in D_H,$$

which imply $(q(x), 0, q(u))$ and $(q(y), 0, q(v))$ are elements of D_G. Hence Proposition 3.5 implies $q(x) = q(u)$ and $q(y) = q(v)$. Thus ψ is a monomorphism.

Because f and $\tilde{q}$ are onto maps, it is not difficult to see that ψ is also a continuous onto map. The inverse of ψ is defined by

$$(q(u), m-n, q(v)) \mapsto (q \circ f^{-1}(u), m-n, q \circ f^{-1}(v)).$$

It is trivial that ψ^{-1} is a continuous groupoid isomorphism. Therefore ψ is a groupoid isomorphism such that ψ and ψ^{-1} are continuous. □

Remark that the inverse semigroups $\langle G, X\rangle$ and $\langle H, Y\rangle$ of the self-similar groups (G, X) and (H, Y), respectively, are isomorphic if there is a homeomorphism $f\colon X^\omega \to Y^\omega$ such that $f \circ \langle G, X\rangle \circ f^{-1} = \{f \circ \alpha \circ f^{-1} \colon \alpha \in \langle G, X\rangle\} = \langle H, Y\rangle$.

Theorem 4.14. *Suppose that (G, X) and (H, Y) are self-similar groups. Then the following are equivalent*:

(1) $\langle G, X\rangle$ *is isomorphic to* $\langle H, Y\rangle$.
(2) C_G *is isomorphic to* C_H *as topological groupoids.*
(3) *There is a $*$-isomorphism* $\Phi\colon \mathcal{O}_G \to \mathcal{O}_H$ *with* $\Phi(C(X^\omega)) = C(Y^\omega)$.

Proof. (1) $\Longrightarrow$ (2). If $\langle G, X\rangle$ is isomorphic to $\langle H, Y\rangle$ with a homeomorphism $f\colon X^\omega \to Y^\omega$, we define

$$\phi\colon C_G \to C_H \text{ by } [\alpha, x] \mapsto [f \circ \alpha \circ f^{-1}, f(x)].$$

If $[\alpha, x] = [\beta, x]$, then there is a neighborhood U of x such that $\alpha|_U = \beta|_U$. Since f is a homeomorphism, $f(U)$ is also a neighborhood of $f(x)$ so that

$$f \circ \alpha f^{-1}|_{f(U)} = f \circ \alpha|_U = f \circ \beta|_U = f \circ \beta \circ f^{-1}|_{f(U)}.$$

Hence ϕ is well-defined. It is routine to check that ϕ is a groupoid isomorphism. To show that ϕ is a continuous map, choose a base element (γ, V) of germ topology on C_H. Then V is an open set in Y^ω, and $\phi^{-1}(\gamma, V) = (f^{-1} \circ \gamma \circ f, f^{-1}(V))$ is a base element of C_G. So ϕ is continuous. By the same argument, ϕ^{-1} is also continuous. Thus C_G is isomorphic to C_H as topological groupoids.

(2) $\Longrightarrow$ (1). If $\phi\colon C_G \to C_H$ is a continuous isomorphism, we identify X^ω with the unit space $C_G^{(0)}$ and define

$$f = \phi|_{X^\omega}.$$

Then it is trivial that f is a homeomorphism. We show $f^{-1} \circ \langle H, Y\rangle \circ f = \langle G, X\rangle$.

For every $s_a g s_b^* \in \langle G, X\rangle$ and $bx \in X^\omega$ with $s_a g s_b^*(bx) = ay$, we have

$$(q(bx), |b| - |a|, q(ay)) \in D_G$$

so that, by Proposition 3.5,

$$\sigma^{|b|} \circ q(bx) = q(x) = q(y) = \sigma^{|a|} \circ q(ay).$$

Because of $q(x) = q(y)$, x is asymptotically equivalent to y, and there is a $g \in G$ such that $g(x) = y$. Moreover

$$\phi(q(bx), |b| - |a|, q(ay)) = (q \circ f(bx), |b| - |a|, q \circ f(ay)) \in D_H$$

implies

$$\sigma^{|b|} \circ q \circ f(bx) = q \circ \sigma^{|b|} \circ f(bx) = q \circ \sigma^{|a|} \circ f(ay) = \sigma^{|a|} \circ q \circ f(ay),$$

and there is an $h \in H$ such that

$$h(\sigma^{|b|} \circ f(bx)) = \sigma^{|a|} \circ f(ay).$$

Let $u \in X^{|a|}$ and $v \in X^{|b|}$ be the prefixes of $f(ay)$ and $f(bx)$, respectively, so that

$$u \cdot \sigma^{|a|} \circ f(ay) = f(ay) \text{ and } v \cdot \sigma^{|b|} \circ f(bx) = f(bx).$$

Then

$$f(ay) = u \cdot (\sigma^{|a|} \circ f(ay))$$

$$
\begin{aligned}
&= u \cdot h(\sigma^{|b|} \circ f(bx)) \\
&= s_u h s_v^*(v \cdot \sigma^{|b|} \circ f(bx)) \\
&= s_u h s_v^*(f(bx)) \\
&= s_u h s_v^* \circ f(bx)
\end{aligned}
$$

implies

$$ay = f^{-1} \circ s_u h s_v^* \circ f(bx) = s_a g s_b^*(bx).$$

Therefore $f^{-1} \circ \langle H, Y \rangle \circ f = \langle G, X \rangle$ holds, and $\langle G, X \rangle$ is isomorphic to $\langle H, Y \rangle$.

(2) $\Longleftrightarrow$ (3) follows from [12, Theorem 5.1]. □

Corollary 4.15. *If there is a $*$-isomorphism $\Phi\colon \mathcal{O}_G \to \mathcal{O}_H$ with $\Phi(C(X^\omega)) = C(Y^\omega)$, then (T_G, σ) and (T_H, σ) are orbit equivalent. Conversely, if (T_G, σ) and (T_H, σ) are orbit equivalent, then the Cuntz-Pimsner algebras $\mathcal{O}_G$ and $\mathcal{O}_H$ are stably isomorphic.*

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