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CONDITIONAL EXPECTATION OF PETTIS INTEGRABLE UNBOUNDED RANDOM SETS

Mohamed El Harami

Abstract. In this paper we established new results of existence of conditional expectation for closed convex and unbounded Pettis integrable random sets without assuming the Radon Nikodym property of the Banach space. As application, new versions of multivalued Lévy's martingale convergence theorem are proved by using the Slice and the linear topologies.

1. Introduction

There are many papers in the literature showed that the conditional expectation of Bochner integrable vector valued random variables (resp. multi-valued random sets) defined in a probability space is always exists (see for example, [6, 8, 11, 19, 23]). This theory of conditional expectation is the basic foundation of the study of conditional expectation convergence theorems, martingales convergence theorems, and strong law of large number theorems. So this study has been developed extensively and applied to the mathematical economics and optimal control theory. Therefore it is natural to ask, does the conditional expectation of Pettis integratable random set exists? With no additional conditions imposed, the answer of this question is negative, see counter example 6-4-1, which is taken from Talagrand [25]. But if the Banach space E possesses the weak Radon-Nikodym property ($WRNP$), [21] has given a necessary and sufficient condition of existence of conditional expectation for Pettis integrable random variables. In the multi-valued case a same result has been proved by [28] for convex and weakly compact valued Pettis integrable random sets. For more information on this study, some results can be found in [3, 15]. In the last ten years, without the $WRNP$ property of the Banach space and under the condition $E^{\mathcal{B}}(|X|) < +\infty$, [1] have proved this existence result for convex and weakly compact valued Pettis integrable random sets. This result has been

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extended by [13] in case where $(\Omega, \mathcal{F}, \mu)$ is only a σ-finite measure space. Recently, [2] established an extension of this result to the case where the random sets are only with closed convex and bounded values. In this paper we are interested to the same question of existence of this operator, but for random sets are only with closed convex and unbounded values, which is more general than the case where the random sets are with convex weakly compact (resp. closed convex and bounded) values. The technique used in this section to prove this existence result based on decomposable sets. As application of the existence of $E^{\mathcal{B}}(X)$ in the second section, we extend the Lévy's convergence theorem to the closed convex valued Pettis integrable random sets treating by the Slice and the linear topologies, compare with the works in [1,9,10,14,24,26] dealing with the Mosco topology and closed convex and bounded (resp. convex and weakly compact) random sets.

2. Elementary material

Throughout this paper, $(\Omega, \mathcal{F}, \mu)$ is a complete probability measure space, E a real Banach space, E' the topological dual of E, and B' the closed unit ball of E'. Denote by $cl(E)$ (resp. $cc(E)$) (resp. $ccb(E)$) (resp. $cwk(E)$) the family of all nonempty closed (resp. closed convex) (resp. closed convex bounded) (resp. convex and weakly compact) subsets of E. For a subset $C \in 2^E \setminus \emptyset$, $\overline{C}$ (resp. $\overline{co}(C)$) the norm-closure (resp. the close convex hull) of C. A subset of $cc(E)$ will be called w-ball-compact if its intersection with any closed ball is weakly compact, the family of all w-ball compact is denoted by

$$R_w = \{C \in cc(E), C \cap \overline{B}(x_0, r) \text{ is weakly compact } \forall\, r > 0\},$$

where $B(x_0, r)$ is the open ball with the center x_0 and radius r. The support function and radius of a subset C are defined as follows:

$$\delta^*(x', C) = \sup_{x \in C} \langle x', x \rangle, \quad |C| = \sup_{x \in C} \|x\|.$$

The topology determined by convergence of support functional is denoted by T_{scalar}, a sequence (C_n) is T_{scalar} convergent to some subset C if

$$\lim_n \delta^*(x', C_n) = \delta^*(x', C) \quad \forall x' \in E'.$$

The distance functional is the mapping: $d : E \times 2^E \setminus \emptyset \longrightarrow \mathbf{R}^+$ such that

$$d(x, C) = \inf_{a \in C} \|x - a\|.$$

The topology determined by convergence of distance functionals is called the Wijsman topology and is denoted by $T_{wijsman}$. For $(C_n) \subset cc(E)$ and $C \in cc(E)$, let

$$s - liC_n = \{x, x = s - \lim_k x_k, x_k \in C_k, k \geq 1\},$$

$$w - lsC_n = \{x, x = w - \lim_k x_k, x_k \in C_{n_k}, k \geq 1\}.$$

s and w are respectively the norm topology and the weak topology of E. We say that (C_n) is Mosco convergent to C (denoted $C = Mosco - \lim_n C_n$) if

$$s - liC_n = w - lsC_n = C.$$

This hold if and only if we have $w - lsC_n \subset C \subset s - liC_n$ (see [20], p. 188). The Slice topology on $\mathcal{P}(E)$ is the initial topology T_{Slice} determined by the family of gap functionals

$$\{D(C,\cdot),\ C \text{ is non empty slice of a ball}\},$$

where $D(B,C) = \inf\{\|b - c\|, b \in B, c \in C\}$, and a slice of a ball is an intersection of $B(x_0,r) \cap \{x, \langle x', x\rangle \leq \alpha\}$ (provided it is not empty). The Slice topology is generally stronger than the Mosco and the Wijsman topologies. It coincides with the Mosco topology if the space E is reflexive (see Beer [5]). We have the following characterization of T_{Slice}:

Lemma 2.1. *Let $(C_n, C)_{n\geq 1}$ be a sequence in $cc(E)$. Then $C = T_{Slice} - \lim_n C_n$ if and only if*

(i) $\lim_n d(x, C_n) = d(x, C) \quad \forall x \in E$.

(ii) $\forall x' \in E'$, *and whenever $r > 0$ such that $C \cap \overline{B}(0,r) \neq \emptyset$ we have*

$$\lim_n \delta^*(x', C_n \cap \overline{B}(0,r)) = \delta^*(x', C \cap \overline{B}(0,r)).$$

Proof. See Theorem 5.4 in Beer [5]. □

The linear topology T_B introduced by Beer [5] is the upper bound of the following topologies:

(1-) the one of simple convergence of distance functions on E.

(2-) the one of simple convergence of support functions on E'.

From Theorem 3.4 in [5] a sequence (C_n) in $\mathcal{P}(E) \setminus \emptyset$ converges to some subset C in linear topology if and only if

$$\lim_n d(x, C_n) = d(x, C) \quad \text{and} \quad \lim_n \delta^*(x', C_n) = \delta^*(x', C) \quad \forall x' \in E'.$$

This topology is stronger than the Mosco topology.

Next denote by $L^1_E(\Omega, \mathcal{F}, \mu)$ the space of all (equivalence classes) of $\mathcal{F}$-measurable and Bochner integrable functions $X : \Omega \to E$. $L^\infty_{\mathbf{R}^+}(\Omega, \mathcal{F}, \mu)$ is the space of all equivalence classes of $\mathcal{F}$-measurable essentially bounded functions $X : \Omega \to \mathbf{R}^+$.

The map $X : \Omega \longrightarrow 2^E \setminus \emptyset$ is called be a multifunction (or set valued function, correspondence, etc). We say that it is scalarly measurable if for all $x' \in E'$, the map $\delta^*(x', X(\cdot))$ is measurable. X said to be Effros measurable (or measurable) if for every open subset U of E, the subset $X^-(U) = \{\omega, X(\omega) \cap U \neq \emptyset\}$ is a member of $\mathcal{F}$. The Effros measurability is stronger than the scalar measurability. Both notions of measurability coincide for more general classes of multifunctions (see Amri-Hess [12]). A measurable multifunction is called a random set. A function f from Ω into E is called a selection of X if for any $\omega \in \Omega$ one has $f(\omega) \in X(\omega)$ and denote by S_X the set of all measurable selections of

X. It is well known that every measurable $cl(E)$-valued multifunction X admits at least one measurable selection. Furthermore, a multifunction $X : \Omega \longrightarrow cl(E)$ is measurable if and only if there is a countable family of measurable selections (f_n) such that for each $\omega \in \Omega$ $X(\omega) = \overline{\{f_n(\omega), n \in \mathbf{N}\}}$ where the closure is taken with respect to the norm in E (see Castaing-Valadier [6, §3, p. 67]).

Let $\mathcal{L}^1_{cl(E)}(\mathcal{F}) := \mathcal{L}^1_{cl(E)}(\Omega, \mathcal{F}, \mu)$ be the space of all random sets X with values in $cl(E)$ such that $|X(\cdot)| \in L^1_R(\Omega, \mathcal{F}, \mu)$, and is called the space of integrably bounded random sets with values in $cl(E)$. Denote by S^1_X the set of all measurable and integrable selections of X and is non empty if and only if $d(0, X(\cdot)) \in L^1_R(\Omega, \mathcal{F}, \mu)$, in such a case we shall say that the multifunction X is integrable (see Hess [17] and Lemma 5.1 in Hess [18]).

A random set with values in $cl(E)$ is Aumann integrable if $S^1_X \neq \emptyset$, the Aumann integral of X is defined by

$$\int_A X d\mu = \{\int_A f d\mu, f \in S^1_X\}.$$

Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X be an integrable random variable defined in $(\Omega, \mathcal{F}, \mu)$ with values in a Banach space E. Then the conditional expectation of X with respect to $\mathcal{B}$ is the unique $\mathcal{B}$-measurable and integrable random variable $Y := E^{\mathcal{B}}(X)$ such that

$$\int_A X d\mu = \int_A Y d\mu \quad \text{for all } A \in \mathcal{B}.$$

It is well known that if X is a random set such that $X \in \mathcal{L}^1_{cl(E)}(\mathcal{F})$, Hiai-Umegaki [19] proved that $E^{\mathcal{B}}(X)$ exists and satisfying

$$S^1_{E^{\mathcal{B}}(X)}(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in S^1_X\}.$$

A random set $X : \Omega \to cl(E)$ is Pettis integrable whenever

1- $(\delta^*(x', X))^- \in L^1_R(\Omega, \mathcal{F}, \mu)$, where $(\delta^*(x', X))^-$ is the negative part.

2- For all $A \in \mathcal{F}$, there exists $C_A(X) \in cl(E)$ such that

$$\delta^*(x', C_A(X)) = \int_A \delta(x', X) d\mu \qquad \forall x' \in E',$$

$C_A(X)$ is called the Pettis or the weak integral of X over A and is denoted by $w - \int_A X d\mu$. A E-valued function f is Pettis integrable if and only if the set $\{\langle x', f\rangle, x' \in B'\}$ is uniformly integrable. This equivalence is not true in the multivalued case (see Amri-Hess [12]). Let us denote by $P^1_E(\mathcal{F}) := P^1_E(\Omega, \mathcal{F}, \mu)$ the space of all $\mathcal{F}$-measurable and Pettis integrable functions. In this space we define the following topologies:

1) The topology of the Pettis norm is defined as:

$$\|f\|_{Pe} = \sup_{x' \in B'} \int_\Omega |\langle x', f\rangle| d\mu,$$

which is equivalent to the norm $\sup_{A \in \mathcal{F}} \| \int_\Omega f d\mu\|$.

2) The topology induced by duality $(P^1_E(\mathcal{F}), L^\infty \otimes E')$, such the operator defined as $(v \otimes x', f) = \int_\Omega v(\omega)\langle x', f(\omega)\rangle d\mu$ is a bilinear form. This topology is denoted by $W - Pe$ and called the weak topology in $P^1_E(\mathcal{F})$. There is another topology in the space of scalarly integrable functions denoted by T_{Pe} and generated by the semi-norms defined as, for each $x' \in B'$, $p_{x'}(f) = \int_\Omega |\langle x', f\rangle| d\mu$, its trace on $P^1_E(\mathcal{F})$ is between $W - Pe$ and $\|\cdot\|_{Pe}$ (for more information see Godet-Thobie and Satco [16]).

Denote by $S^{Pe}_X(\mathcal{B})$ the set of all $\mathcal{B}$-measurable and Pettis integrable selections of X, if $\mathcal{B} = \mathcal{F}$ we can reduce this notation to S^{Pe}_X. A $cl(E)$-valued random set X is said to be Aumann-Pettis integrable if $S^{Pe}_X \neq \emptyset$. The Aumann-Pettis integral of X over A is defined by

$$\int_A X d\mu = \{\int_A f d\mu, f \in S^{Pe}_X\},$$

this integral is denoted by $I_A(X)$.

Remark 2.2. It is well known from [12] that if a $cc(E)$-valued random set X is Aumann Pettis integrable and $(\delta^*(x', X))^-$ is integrable, then X is Pettis integrable. Also if X is random set such that $\delta^*(x', X) \in L^1_{\mathbf{R}}$ and is Pettis integrable in $cc(E)$, then it is Pettis integrable in $ccb(E)$ (the result comes from the fact that a subset is bounded if and only if its support function is finite in each point of E').

Lemma 2.3 (see Godet-Thobie and Satco [16])**.** *If f is E-valued and scalarly integrable random variable. Then the following properties are equivalent*:

1) *f is Pettis integrable,*

2) *The set $\{\langle x', f\rangle, x' \in B'\}$ is uniformly integrable in L^1_R.*

3. Conditional expectation of Pettis integrable unbounded random sets

Several extensions of the Hiai and Umegaki [19] results on the existence of conditional expectation have been proved in the case of Pettis integrable random sets see the works of [1, 3, 13, 15, 28]. Recently, another result in the same direction have been proved in [2] for closed convex and bounded valued Pettis integrable random sets. In this section, using the decomposable sets technique, we continue this study with anther extension of this result dealing by closed convex and unbounded valued Pettis integrable random sets and a Banach space E without the $WRNP$ property. For the $WRNP$ property of a Banach space we refer to the well known papers of [21,22] and for decomposable sets we refer to [16]. We start this section by the following lemmas that will be useful in the proof of our main results. First we extend Lemma 1.3 of [19] to the Aumann-Pettis integrable random sets.

Lemma 3.1. *Let X be a $cl(E)$-valued Aumann Pettis integrable random set. Let $\{f_n, n \geq 1\}$ be a sequence in S^{Pe}_X such that $X(\omega) = \overline{\{f_n(\omega), n \geq 1\}}$, $\forall\, \omega \in$*

Ω. *Then for each* $f \in S_X^{Pe}$ *and* $\epsilon > 0$*, there exist a finite partition* (A_i), $i = 1, \ldots, p$ *of* Ω *in* $\mathcal{F}$ *and* $\{f_{n_1}, f_{n_2}, \ldots, f_{n_p}\} \subset \{f_n, n \geq 1\}$ *such that*

$$\|f - \sum_{i=1}^{p} f_{n_i} \cdot \chi_{A_i}\|_{Pe} < \epsilon.$$

Proof. The proof of this Lemma is contained in the first part of the proof of Theorem 24 in [16], so the proof can be omitted. □

Remark 3.2. By Lemma 23 in [16] if X_1 and X_2 are two closed valued and Aumann Pettis integrable random sets such that

$$S_{X_1}^{Pe} = S_{X_2}^{Pe},$$

then

$$X_1 = X_2 \quad \mu \text{ a.s.}$$

A non-empty subset of $P_E^1(\mathcal{F})$ is decomposable (with respect to $\mathcal{F}$) if for every $f_1, f_2 \in M$ and $A \in \mathcal{F}$, we have

$$\chi_A \cdot f_1 + \chi_{\Omega \setminus A} \cdot f_2 \in M.$$

Then we have the following Lemma:

Lemma 3.3. *Let* E *be a separable Banach space, and* M *be a non-empty closed subset of* $P_E^1(\mathcal{F})$*. Then there exists a unique* $cl(E)$*-valued Aumann Pettis integrable random set* G *such that* $M = S_G^{Pe}$ *if and only if* M *is decomposable with respect to* $\mathcal{F}$*.*

Proof. ($\Rightarrow$) The proof is a modification of the proof of Theorem 3.1 in [19]. Indeed, it is well known that S_G^{Pe} is decomposable.

($\Leftarrow$) Since M is non-empty, then there is $f_0 \in M$ such that $f_0(\omega) \in E$, $\forall \omega \in \Omega$. Then

$$S_X^{Pe} \neq \emptyset, \text{ where } X(\omega) = E \text{ for } \omega \in \Omega.$$

Applying Lemma 23 in [16] there exists a sequence $(f_i)_{i \in \mathbf{N}}$ of Pettis integrable functions such that $(f_i(\omega))_{i \in \mathbf{N}}$ is dense in E for every $\omega \in \Omega$. For each $i \in \mathbf{N}$, let $\alpha_i = \inf_{g \in M}(\|\|f_i - g\|_E \wedge 1\|_1)$. Then for each $i, j \geq 1$, there is $g_{i,j}$ in M such that

$$|\|\|f_i - g_{i,j}\|_E \wedge 1\|_1 - \alpha_i| < \frac{1}{j}.$$

Let define G as

$$G(\omega) = \overline{\{g_{i,j}(\omega), i, j \geq 1\}}.$$

Now we prove that $M = S_G^{Pe}$. Let $f \in S_G^{Pe}$ and $\epsilon > 0$, then from Lemma 3.1 there exist a finite partition (A_n), $n = 1, \ldots, p$ of Ω in $\mathcal{F}$ and $\{g_1, g_2, \ldots, g_p\} \subset \{g_{i,j}(\omega), i, j \geq 1\}$ such that

$$\|f - \sum_{n=1}^{p} g_n \cdot \chi_{A_n}\|_{Pe} < \epsilon.$$

Since $\sum_{n=1}^{p} g_n \cdot \chi_{A_n} \in M$ and M is closed with respect to the Pettis norm, then $f \in M$. So $S_G^{Pe} \subset M$. Now we prove that $M \subset S_G^{Pe}$. By assume the converse, there is a $f \in M$ which does not belong to S_G^{Pe}. Then there exist $\delta > 0$ and $A \in \mathcal{F}$ with $\mu(A) > 0$ such that

$$\inf_{i,j}(\|f(\omega) - g_{i,j}(\omega)\|_E \wedge 1) > \delta \quad \forall \omega \in A.$$

Since for each $\omega \in \Omega$, $f(\omega) \in \overline{\{f_i(\omega), i \geq 1\}}$. Then we can write

$$\Omega = \cup_i \{\omega, (\|f(\omega) - f_i(\omega)\|_E \wedge 1) < \frac{\delta}{3}\},$$

so there is $i_0 \geq 1$ such that $B = A \cap \{\omega, \|f(\omega) - f_{i_0}(\omega)\|_E \wedge 1 < \frac{\delta}{3}\}$ has a positive measure. Define a sequence $(g_j)_{j \geq 1}$ as

$$g_j = \chi_B \cdot f + \chi_{\Omega \backslash B} g_{i_0,j}.$$

Since $(g_j) \subset M$ and

$$\begin{aligned}(\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) &\geq (\|f(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) - (\|f(\omega) - f_{i_0}(\omega)\|_E \wedge 1) \\ &> \frac{2\delta}{3} \quad \forall \omega \in B.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_\Omega (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu - \alpha_{i_0} \\ &\geq \int_\Omega (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu - \int_\Omega (\|f_{i_0}(\omega) - g_j(\omega)\|_E \wedge 1) d\mu,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu - \alpha_{i_0} \\ &\geq \int_B (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu + \int_{\Omega \backslash B} (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu \\ &\quad - \int_B (\|f_{i_0}(\omega) - g_j(\omega)\|_E \wedge 1) d\mu - \int_{\Omega \backslash B} (\|f_{i_0}(\omega) - g_j(\omega)\|_E \wedge 1) d\mu \\ &= \int_B (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu - \int_B (\|f_{i_0}(\omega) - f(\omega)\|_E \wedge 1) d\mu > \frac{\delta}{3}\mu(B).\end{aligned}$$

Letting j go to the infinity, then we have a contradiction with the fact that the sequence $(\int_\Omega (\|f_{i_0}(\omega) - g_{i_0,j}(\omega)\|_E \wedge 1) d\mu)_{j \in \mathbf{N}^*}$ converges to α_{i_0}. Consequently we have

$$M = S_G^{Pe}.$$

The uniqueness of G follows from Remark 3.2. □

Lemma 3.4. *Let X be a $cl(E)$-valued Aumann Pettis integrable random set. Then we have*

$$\overline{S_{\overline{co}(X)}^{Pe}}^{\|\cdot\|_{Pe}} = \overline{co}(S_X^{Pe}).$$

Proof. It clear that $S_X^{Pe} \subset S_{\overline{co}(X)}^{Pe}$, then $\overline{co}(S_X^{Pe}) \subset \overline{S_{\overline{co}(X)}^{Pe}}^{\|\cdot\|_{Pe}}$. Now to prove the converse assume that the inclusion is strict, so there is $f \in \overline{S_{\overline{co}(X)}^{Pe}}^{\|\cdot\|_{Pe}}$ and $f \notin \overline{co}(S_X^{Pe})$. Then by using the separation theorem there is $v \otimes x' \in (P_E^1)^* = L^\infty \otimes E'$ such that

$$\begin{aligned} \delta^*(v \otimes x', S_X^{Pe}) &= \delta^*(v \otimes x', \overline{co}(S_X^{Pe})) \\ &< (v \otimes x', f) = \int_\Omega v(\omega) \cdot \langle x', f(\omega) \rangle d\mu. \end{aligned} \tag{3.4.1}$$

On the other hand since $f \in \overline{S_{\overline{co}(X)}^{Pe}}^{\|\cdot\|_{Pe}}$ there is a sequence (f_n) in $S_{\overline{co}(X)}^{Pe}$ such that

$$\lim_n \|f_n - f\|_{Pe} = 0.$$

This implies that

$$\lim_n \langle x', f_n \rangle = \langle x', f \rangle \quad \text{in } L^1_{\mathbf{R}}.$$

Hence

$$\lim_n (v \otimes x', f_n) = (v \otimes x', f).$$

So by using this and the fact that $(v \otimes x', f_n) \leq \delta^*(v \otimes x', S_{\overline{co}(X)}^{Pe})$, it follows that

$$(v \otimes x', f) \leq \delta^*(v \otimes x', S_{\overline{co}(X)}^{Pe}).$$

Then

$$\begin{aligned} \int_\Omega v(\omega) \cdot \langle x', f(\omega) \rangle d\mu &\leq \sup_{h \in S_{\overline{co}(X)}^{Pe}} (v \otimes x', h) \\ &= \sup_{h \in S_{\overline{co}(X)}^{Pe}} \int_\Omega v(\omega) \cdot \langle x', h(\omega) \rangle d\mu \\ &\leq \int_\Omega \sup_{x \in \overline{co}(X)(\omega)} (v(\omega) \cdot \langle x', x \rangle) d\mu \\ &= \int_\Omega \delta^*(v \otimes x'(\omega), \overline{co}(X)(\omega)) d\mu \\ &= \int_\Omega \delta^*(v \otimes x'(\omega), X(\omega)) d\mu \\ &= \delta^*(v \otimes x', S_X^{Pe}) = \delta^*(v \otimes x', \overline{co}(S_X^{Pe})). \end{aligned}$$

This contradicts what is proved in (3.4.1). Consequently we have

$$\overline{S_{\overline{co}(X)}^{Pe}}^{\|\cdot\|_{Pe}} = \overline{co}(S_X^{Pe}). \qquad \square$$

Theorem 3.5. *Let X be a $cc(E)$-valued Aumann Pettis integrable random set, and $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$. Assume that for every selection $f \in S_X^{Pe}$,*

$E^{\mathcal{B}}(f)$ *exists. Then there exists a unique* (μ a.s.) $cc(E)$*-valued,* $\mathcal{B}$*-measurable and Aumann Pettis integrable random set* Y *such that*

$$\overline{S_Y^{Pe}(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}.$$

Proof. Let us define a set M as

$$M = \{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}.$$

It is clear that M is well defined. As S_X^{Pe} is nonempty and decomposable, then also M is a nonempty and decomposable subset in the set of all $\mathcal{B}$-measurable and Pettis integrable functions. From [16], $\overline{M}^{\|\cdot\|_{Pe}}$ is also decomposable with respect to $\mathcal{B}$. Then by applying Lemma 3.3 there exists a unique $cl(E)$-valued $\mathcal{B}$-measurable and Aumann Pettis integrable random set G such that

$$S_G^{Pe} = \overline{M}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}.$$

Now let us define Y as $Y(\omega) = \overline{co}(G)(\omega)$ for each $\omega \in \Omega$. Since $S_G^{Pe}(\mathcal{B}) \neq \emptyset$, from Lemma 23 in [16] there is a sequence $(g_n)_{n\geq 1}$ in $S_G^{Pe}(\mathcal{B})$ such that

$$G(\omega) = cl\{g_n(\omega), n \geq 1\}, \quad \omega \in \Omega.$$

Define

$$U = \{g, g = \sum_{i=1}^{m} \alpha_i . g_i : \alpha_i \text{ rational} \geq 0,\ \sum_{i=1}^{m} \alpha_i = 1, m \geq 1\}.$$

Then U is a countable set of $\mathcal{B}$-measurable functions and for each $\omega \in \Omega$, $Y(\omega) = cl\{g(\omega), g \in U\}$. Hence by Theorem 1.0 in Hiai and Umegaki [19], Y is a $cc(E)$-valued and $\mathcal{B}$-measurable multifunction. Also it is not difficult to see that Y is Aumann Pettis integrable. Again since G is Aumann-Pettis integrable, so by applying Lemma 3.4 we get

$$\overline{S_Y^{Pe}(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{co}(S_G^{Pe}(\mathcal{B})).$$

Combining this with the properties of closed convex hull and the convexity of the set $\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}$, it follows that

$$\begin{aligned}\overline{S_Y^{Pe}(\mathcal{B})}^{\|\cdot\|_{Pe}} &= \overline{co}(S_G^{Pe}(\mathcal{B})) = \overline{co}(\overline{S_G^{Pe}(\mathcal{B})}^{\|\cdot\|_{Pe}}) \\ &= \overline{co}(\overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}) = \overline{co}(\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}) \\ &= \overline{co(\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}.\end{aligned}$$

Finally we have

$$\overline{S_Y^{Pe}(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}.$$

□

Before starting the main theorem noting that:

Remark 3.6. If X is a $cc(E)$-valued Pettis integrable random set and $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$. Then the following properties are equivalent:

1) $E^{\mathcal{B}}(|X|) < +\infty \quad \mu$ a.s.

2) There exists a partition $(B_m)_{m\geq 1}$ of Ω in $\mathcal{B}$ such that for each m, $\int_{B_m} |X| d\mu < +\infty$.

It is obvious to see that one of both properties implies that X is (μ a.s.) with bounded values.

Definition 3.7. A random set $X : \Omega \to cl(E)$ is said D-countably supported if one can find a countable dense subset D in B' such that

$$X(\omega) = \bigcap_{y\in D} \{x, \langle y, x\rangle \leq \delta^*(y, X(\omega))\} \quad \forall\, \omega \in \Omega.$$

Remark 3.8. From [12] a D-countably supported random set may be with unbounded values, and from the same author if X is of bounded values and E' is separable, then it is D-countably supported. Then this condition is weaker than condition $E^{\mathcal{B}}(|X|) < +\infty \ \mu$ a.s. which is based in the works of authors [1, 2, 13, 28].

Theorem 3.9. *Assume that* $(\Omega, \mathcal{F}, \mu)$ *is a probability measure space, and* E' *is separable. Let* $\mathcal{B}$ *be a sub-*σ*-algebra of* $\mathcal{F}$ *and* X *a* $cc(E)$*-valued scalarly integrable random set such that*

(i) $S^{Pe}_X \neq \emptyset$ *and* $E^{\mathcal{B}}(f)$ *exists for each* $f \in S^{Pe}_X$,

(ii) X *is* D*-countably supported* (D *is a countable dense subset in* B').

Then there exists a unique (μ a.s.) $cc(E)$*-valued,* $\mathcal{B}$*-measurable and Aumann Pettis integrable random set* Y *denoted by* $Y = E^{\mathcal{B}}(X)$ *such that*

$$\overline{S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

Consequently Y *is Pettis integrable and we have*

$$\overline{\int_A X d\mu} = \overline{\int_A Y d\mu} = \overline{\int_A E^{\mathcal{B}}(X) d\mu} \qquad \forall A \in \mathcal{B}.$$

Proof. By applying Theorem 3.5 there exists a unique (μ a.s.) $cc(E)$-valued, $\mathcal{B}$-measurable and Aumann Pettis integrable random set Y denoted by $E^{\mathcal{B}}(X)$ such that

$$\overline{S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

Now we prove the Pettis integrability of X and $Y = E^{\mathcal{B}}(X)$. The Pettis integrability of X is an immediate consequence of Theorem 3.9 and Corollary 3.10 in Amri-Hess [12]. Now since from condition (i) $S^{Pe}_X \neq \emptyset$, then also $S^{Pe}_{E^{\mathcal{B}}(X)} \neq \emptyset$, then there is a Pettis integrable selection g in $S^{Pe}_{E^{\mathcal{B}}(X)}$. The Pettis integrability of g implies that the set $\{(\langle x', g\rangle)^-, x' \in B'\}$ is uniformly integrable in $L^1_{\mathbf{R}}$, hence it is bounded. In turn, it implies that

$$\int_\Omega (\delta^*(x', Y))^- d\mu \leq \int_\Omega (\langle x', g\rangle)^- d\mu < +\infty.$$

Also by applying Theorem 3.9 and Corollary 3.10 in Amri-Hess [12] we get the Pettis integrability of $Y = E^{\mathcal{B}}(X)$. Now we prove the second part of theorem, first observe that for all $A \in \mathcal{B}$,

$$\begin{aligned}\delta^*(x', \int_A X d\mu) &= \delta^*(x', \{\int_A f d\mu : f \in S_X^{Pe}\}) \\ &= \delta^*(x', \{\int_A E^{\mathcal{B}}(f) d\mu : f \in S_X^{Pe}\}) \\ &\le \delta^*(x', \{\int_A g d\mu : g \in S_{E^{\mathcal{B}}(X)}^{Pe}\}) \\ &= \delta^*(x', \int_A E^{\mathcal{B}}(X) d\mu).\end{aligned}$$

Hence

$$\overline{\int_A X d\mu} \subset \overline{\int_A E^{\mathcal{B}}(X) d\mu}. \tag{3.9.1}$$

To prove the reverse inclusion. Indeed, for each $A \in \mathcal{B}$, let $Y_A = \int_A g d\mu \in \int_A E^{\mathcal{B}}(X) d\mu$ for some $g \in S_{E^{\mathcal{B}}(X)}^{Pe}(\mathcal{B})$. Since

$$g \in S_{E^{\mathcal{B}}(X)}^{Pe}(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}}.$$

Then there is a sequence $(f_n)_{n \in \mathbf{N}} \subset S_X^{Pe}$ such that $E^{\mathcal{B}}(f_n)$ converges in the Pettis norm to g. Now the fact that the Pettis norm is equivalent to norm defined by $\sup_{A \in \mathcal{F}} \| \int_A (\cdot) d\mu \|$ (see Musial [21], p. 198). It follows that

$$\lim_{n \to +\infty} \sup_{A \in \mathcal{B}} \| \int_A E^{\mathcal{B}}(f_n) d\mu - \int_A g d\mu \| = 0.$$

Hence

$$\lim_{n \to +\infty} \int_A f_n d\mu = \int_A g d\mu \quad \forall A \in \mathcal{B}. \tag{3.9.2}$$

Since $(f_n)_{n \in \mathbf{N}} \subset S_X^{Pe}$ and X is D-countably supported, then Proposition 4.1(a) in [12] implies that

$$(\int_A f_n d\mu)_{n \in \mathbf{N}} \subset \int_A X d\mu \quad \forall A \in \mathcal{B}.$$

This with (3.9.2) implies

$$y_A = \int_A g d\mu \in \overline{\int_A X d\mu} \quad \forall A \in \mathcal{B}.$$

Combining this with (3.9.1) we get

$$\overline{\int_A X d\mu} = \overline{\int_A Y d\mu} = \overline{\int_A E^{\mathcal{B}}(X) d\mu} \qquad \forall A \in \mathcal{B}.$$

□

If the space $P^1_E(\mathcal{F})$ is separable the result comes directly from Theorem 25 in [16] and Lemma 3.4 respectively, without using Lemma 3.3 and Theorem 3.5.

Corollary 3.10. *Assume that* $(\Omega, \mathcal{F}, \mu)$ *is a probability measure space,* E' *and* $P^1_E(\mathcal{F})$ *are separable. Let* $\mathcal{B}$ *be a sub-*σ*-algebra of* $\mathcal{F}$ *and* X *a* $cc(E)$*-valued scalarly integrable random set such that*

(i) $S^{Pe}_X \neq \emptyset$ *and* $E^{\mathcal{B}}(f)$ *exists for each* $f \in S^{Pe}_X$,

(ii) X *is* D*-countably supported* (D *is a countable dense subset in* B').

Then there exists a unique (μ a.s.) $cc(E)$*-valued,* $\mathcal{B}$*-measurable and Aumann Pettis integrable random set* Y *denoted by* $Y = E^{\mathcal{B}}(X)$ *such that*

$$\overline{S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})} = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

Consequently Y *is Pettis integrable and we have*

$$\overline{\int_A X d\mu} = \overline{\int_A Y d\mu} = \overline{\int_A E^{\mathcal{B}}(X) d\mu} \qquad \forall A \in \mathcal{B}.$$

Proof. Let us define a set M as

$$M = \{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}.$$

From the linearity of $E^{\mathcal{B}}(\cdot)$ and condition (i), M is well defined and non-empty decomposable subset in the set of $\mathcal{B}$-measurable and Pettis integrable functions. From [16] also $\overline{M}^{\|\cdot\|_{Pe}}$ is decomposable with respect to $\mathcal{B}$. Then by applying Theorem 25 in [16] there exists a unique (μ a.s.) $cl(E)$-valued, $\mathcal{B}$-measurable and Aumann Pettis integrable random set G such that

$$\overline{S^{Pe}_G(\mathcal{B})}^{\|\cdot\|_{Pe}} = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

The rest of the proof is the same as of the proof of Lemma 3.4 and Theorem 3.9. □

Remark 3.11. If the random set X is $ccb(E)$-valued and the space E' is strongly separable. From Amri-Hess [12, pp. 346–347] X is countably supported, then condition (ii) in Theorem 3.9 can be omitted.

The following theorem is an extension of Theorem 5.5 in Hiai and Umegaki [19]. Also the theorem gives a necessary and sufficient condition that the multivalued Pettis conditional expectation can be represented in the form as a sequence of real valued conditional expectations. Before starting our theorem remark that if X is $cc(E)$-valued and Aumann Pettis integrable, then from Corollary 3.10 in [12] we have

$$C_A(X) = w - \int_A X d\mu = \overline{\int_A X d\mu} \quad \forall A \in \mathcal{F}.$$

Theorem 3.12. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X a $cc(E)$-valued random set satisfies all conditions in Theorem 3.9. If for every $A \in \mathcal{B}$, $I_A(E^{\mathcal{B}}(X))$ is closed. Then the following properties are equivalent*:

1) $E^{\mathcal{B}}(X)(\omega) = \bigcap_{n\geq 1}\{x, \langle x'_n, x\rangle \leq E^{\mathcal{B}}(\delta^*(x'_n, X)(\omega))\}$ *for* μ a.s. $\omega \in \Omega$.

2) $S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})$ *is closed with respect to the Pettis norm, i.e.,*

$$S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

Proof. We prove that 1) implies 2). By using the existence of $E^{\mathcal{B}}(\delta^*(x', X))$ and $E^{\mathcal{B}}(X)$, and the Pettis integrability of X and $E^{\mathcal{B}}(X)$ we get for all $x' \in E'$,

$$\begin{aligned}\int_A \delta^*(x', E^{\mathcal{B}}(X))d\mu &= \delta^*(x', \int_A E^{\mathcal{B}}(X)d\mu) = \delta^*(x', \int_A X d\mu)\\ &= \int_A \delta^*(x', X)d\mu = \int_A E^{\mathcal{B}}(\delta^*(x', X))d\mu \quad \forall A \in \mathcal{B}.\end{aligned}$$

Then from the uniqueness of conditional expectation for every $x' \in E'$, we have

$$E^{\mathcal{B}}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{B}}(X)) \quad \mu \text{ a.s.}$$

Let $D = \{x'_n, n \in \mathbf{N}^*\}$ be a countable strong dense sequence in B', then for every $n \in \mathbf{N}^*$, there is a null set N_n such that

$$E^{\mathcal{B}}(\delta^*(x'_n, X)(\omega)) = \delta^*(x'_n, E^{\mathcal{B}}(X)(\omega)) \quad \forall\, \omega \in \Omega \setminus N_n.$$

Set $N = \cup_n N_n$, then for all $n \in \mathbf{N}^*$, we have

$$E^{\mathcal{B}}(\delta^*(x'_n, X)(\omega)) = \delta^*(x'_n, E^{\mathcal{B}}(X)(\omega)) \quad \forall\, \omega \in \Omega \setminus N.$$

This with condition 1) implies that

$$E^{\mathcal{B}}(X)(\omega) = \bigcap_{n\geq 1}\{x, \langle x'_n, x\rangle \leq \delta^*(x'_n, E^{\mathcal{B}}(X)(\omega))\} \quad \mu \text{ a.s.}$$

Hence $E^{\mathcal{B}}(X)$ is μ a.s. D-countably supported. Now applying Proposition 4.2 in [12] it follows that $S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})$ is closed with respect to the Pettis norm. So

$$S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

Now we prove that 2) implies 1). Set

$$Y(\omega) = \bigcap_{n\geq 1}\{x, \langle x'_n, x\rangle \leq E^{\mathcal{B}}(\delta^*(x'_n, X)(\omega))\} \quad \mu \text{ a.s.}$$

It is clear that Y is $\mathcal{B}$-measurable. Then to prove this implication it suffice to prove that

$$S^{Pe}_Y(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(f) : f \in S^{Pe}_X\}}^{\|\cdot\|_{Pe}}.$$

If $f \in S^{Pe}_X$, then

$$\langle x'_n, E^{\mathcal{B}}(f)\rangle = E^{\mathcal{B}}(\langle x'_n, f\rangle) \leq E^{\mathcal{B}}(\delta^*(x'_n, X)(\omega)) = \delta^*(x'_n, E^{\mathcal{B}}(X)(\omega)) \quad \mu \text{ a.s.}$$

so $E^{\mathcal{B}}(f) \in S_Y^{Pe}(\mathcal{B})$. Conversely, let $g \in S_Y^{Pe}(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|.\|_{Pe}}$, then there is a sequence $(f_n)_{n\in\mathbf{N}} \subset S_X^{Pe}$ such that

$$\lim_n \|E^{\mathcal{B}}(f_n) - g\|_{Pe} = 0.$$

Hence

$$\lim_n \int_A f_n d\mu = \int_A g d\mu \quad \forall A \in \mathcal{B}.$$

Since $(\int_A f_n d\mu)_{n\in\mathbf{N}} \subset \int_A X d\mu$, $\forall A \in \mathcal{B}$ and X is D-countably supported, then

$$\int_A g d\mu \in \overline{\int_A X d\mu} = \overline{\int_A E^{\mathcal{B}}(X) d\mu} = \int_A E^{\mathcal{B}}(X) d\mu \quad \forall A \in \mathcal{B}.$$

So this implies that $g \in S^{Pe}_{E^{\mathcal{B}}(X)}(\mathcal{B})$. Thus the reverse implication is proved. □

If the random sets X are $cwk(E)$-valued and integrably bounded, Klei-Assani [4] proved that $E^{\mathcal{B}}(X)$ exists, furthermore is $cwk(E)$-valued and satisfying

$$S^1_{E^{\mathcal{B}}(X)}(\mathcal{B}) = \{E^{\mathcal{B}}(f) : f \in S^1_X\}.$$

Here we present an extension of this result to the $cwk(E)$-valued Pettis integrable random sets.

Lemma 3.13. *Let E' be separable, $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$ and X a $cwk(E)$-valued scalarly and Aumann Pettis integrable random set such that $E^{\mathcal{B}}(X)$ exists and $|E^{\mathcal{B}}(X)| < +\infty$. Then*

$$E^{\mathcal{B}}(X) \in cwk(E) \quad \mu \text{ a.s.}$$

Proof. From Theorem 5.4 in [12], we have for every $A \in \mathcal{B}$,

$$M(A) = \int_A X d\mu = \{\int_A f d\mu,\ f \in S_X^{Pe}\} \in cwk(E).$$

By using the Pettis integrability of X we have for every $x' \in E'$,

$$\delta^*(x', M(A)) = \int_A \delta^*(x', X) d\mu.$$

Then $A \to \delta^*(x', M(A))$ is a signed measure who is absolutely continuous with respect to μ. Hence $M(\cdot)$ is a set valued measure with values in $cwk(E)$ which is also absolutely continuous with respect to μ. Then from Theorem 4.4 in [27] there is a $cwk(E)$-valued and weakly integrable random set G such that

$$M(A) = \int_A G d\mu.$$

Thus

$$\int_A G d\mu = \int_A E^{\mathcal{B}}(X) d\mu \quad \forall A \in \mathcal{B}.$$

This shows that for all $x' \in E'$, we have

$$\int_A \delta^*(x', G) d\mu = \int_A \delta^*(x', E^{\mathcal{B}}(X)) d\mu \quad \forall A \in \mathcal{B}.$$

Then

$$\delta^*(x', G(\omega)) = \delta^*(x', E^{\mathcal{B}}(X)(\omega)) \quad \forall \omega \in \Omega \setminus N_{x'},$$

where $\mu(N_{x'}) = 0$. Let $(x'_n)_{n\in\mathbf{N}}$ be a dense sequence in E' and $N = \cup_n N_{x'_n}$ ($\mu(N) = 0$). Let $x' \in E'$ and $(x_k)_{k\in\mathbf{N}}$ a subsequence of $(x'_n)_{n\in\mathbf{N}}$ such that $x'_k \to x'$ strongly. Then for every $\omega \in \Omega \setminus N$, we have

$$\begin{aligned}
&|\delta^*(x', E^{\mathcal{B}}(X)(\omega)) - \delta^*(x', G(\omega))| \\
&\leq |\delta^*(x', E^{\mathcal{B}}(X)(\omega)) - \delta^*(x'_k, E^{\mathcal{B}}(X)(\omega))| + |\delta^*(x'_k, G(\omega)) - \delta^*(x', G(\omega))| \\
&\leq \|x'_k - x'\|(|G(\omega)| + |E^{\mathcal{B}}(X)(\omega)|) \to 0, k \to +\infty.
\end{aligned}$$

So for every $\omega \in \Omega \setminus N$, $\mu(N) = 0$ we have

$$\delta^*(x', E^{\mathcal{B}}(X)(\omega)) = \delta^*(x', G(\omega)).$$

Consequently

$$E^{\mathcal{B}}(X)) = G \in cwk(E) \quad \mu \text{ a.s.}$$

□

Proposition 3.14. *Assume that $(\Omega, \mathcal{F}, \mu)$ is a probability measure space, and E' is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X is a $cwk(E)$-valued scalarly integrable random set such that*

(i) $S_X^{Pe} \neq \emptyset$ *and* $E^{\mathcal{B}}(|X|) < +\infty$ μ *a.s.*

Then there exists a unique (μ a.s.) $cwk(E)$-valued, $\mathcal{B}$-measurable and Aumann Pettis integrable random set Y denoted by $E^{\mathcal{B}}(X)$ such that

$$S_Y^{Pe}(\mathcal{B}) = \{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}.$$

Consequently Y is Pettis integrable and we have

$$\int_A X d\mu = \int_A E^{\mathcal{B}}(X) d\mu \qquad \forall A \in \mathcal{B}.$$

Proof. Since $E^{\mathcal{B}}(|X|) < +\infty$ μ *a.s.*, then also $E^{\mathcal{B}}(|f|) < +\infty$ μ *a.s.* $\forall f \in S_X^{Pe}$. This and Theorem 3.6 in [13] implies that $E^{\mathcal{B}}(f)$ exists. Since X is $cwk(E)$-valued and E' is separable, then X is D-countably supported. Hence all conditions of Theorem 3.9 are satisfied, so there exists a unique (μ a.s.) $cc(E)$-valued, $\mathcal{B}$-measurable and Aumann Pettis integrable random set Y denoted by $E^{\mathcal{B}}(X)$ such that

$$\overline{S_{E^{\mathcal{B}}(X)}^{Pe}(\mathcal{B})} = \overline{\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}}^{\|\cdot\|_{Pe}},$$

and

$$\overline{\int_A X d\mu} = \overline{\int_A E^{\mathcal{B}}(X) d\mu} \qquad \forall A \in \mathcal{B}.$$

Now we prove that the set $\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}$ is $\|\cdot\|_{Pe}$-closed. Indeed, let $(E^{\mathcal{B}}(f_n))_{n\in\mathbf{N}}$ a sequence in $\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}$ such that $E^{\mathcal{B}}(f_n)$ converges in the Pettis norm to some g, since the Pettis norm is equivalent to norm defined by $\sup_{A\in\mathcal{F}} \| \int_A (\cdot) d\mu\|$ (see Musial [21, p. 198]). It follows that

$$\lim_{n\to+\infty} \sup_{A\in\mathcal{B}} \| \int_A E^{\mathcal{B}}(f_n) d\mu - \int_A g d\mu\| = 0.$$

Thus

$$\lim_{n\to+\infty} \sup_{A\in\mathcal{B}} \| \int_A f_n d\mu - \int_A g d\mu\| = 0.$$

Hence

$$\lim_{n\to+\infty} \int_A f_n d\mu = \int_A g d\mu \quad \forall A \in \mathcal{B}. \tag{3.14.1}$$

Since X is $cwk(E)$-valued, by Theorem 3.7 in [12] it is Pettis integrable. So by Theorem 5.4 in [12], the set $\{\langle x', f\rangle, x' \in B', f \in S_X^{P_e}\}$ is uniformly integrable. Hence from Theorem 3.6 in [7], $S_X^{P_e}$ is relatively sequentially compact with respect to the weak topology of $P_E^1(\Omega, \mathcal{F}, \mu)$. Since from [16] $S_X^{P_e}$ is closed, then it is sequentially weakly compact. Combining this property with the fact that $(f_n)_{n\in\mathbf{N}} \subset S_X^{P_e}$, there is a subsequence $(f_{\varphi(n)})_{n\in\mathbf{N}}$ of $(f_n)_{n\in\mathbf{N}}$ and $f \in S_X^{P_e}$ such that for every $x' \in E'$, we have

$$\lim_{n\to+\infty} \langle x', \int_A f_{\varphi(n)} d\mu\rangle = \langle x', \int_A f d\mu\rangle \qquad \forall A \in \mathcal{F}.$$

In particulary

$$\lim_{n\to+\infty} \langle x', \int_A f_{\varphi(n)} d\mu\rangle = \langle x', \int_A f d\mu\rangle \qquad \forall A \in \mathcal{B}.$$

Combining this with (3.14.1) we get for every $x' \in E'$,

$$\langle x', \int_A f d\mu\rangle = \langle x', \int_A g d\mu\rangle \quad \forall A \in \mathcal{B}.$$

Consequently

$$\int_A f d\mu = \int_A g d\mu \quad \forall A \in \mathcal{B}.$$

From this and the uniqueness of conditional expectation we get $g = E^{\mathcal{B}}(f)$. Consequently the set

$$\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\},$$

is closed with respect to the Pettis norm, hence

$$\overline{S_{E^{\mathcal{B}}(X)}^{Pe}}(\mathcal{B}) = \{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\} = E^{\mathcal{B}}(S_X^{Pe}).$$

Now since from condition (ii) and Lemma 3.13, $E^{\mathcal{B}}(X)$ is $cwk(E)$-valued, then combining this with the separability of E' it follows from ([12, p. 346]) that $E^{\mathcal{B}}(X)$ is D-countably supported. So by applying Proposition 4.2 in [12], $S_{E^{\mathcal{B}}(X)}^{Pe}(\mathcal{B})$ is closed with respect to the Pettis norm, then

$$S_{E^{\mathcal{B}}(X)}^{Pe}(\mathcal{B}) = \{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\} = E^{\mathcal{B}}(S_X^{Pe}).$$

Now if we go back to the Proposition 5.2 and Theorem 5.4 in [12], respectively we get

$$\int_A X d\mu = \int_A E^{\mathcal{B}}(X) d\mu \qquad \forall A \in \mathcal{B}.$$

□

4. Properties of $E^{\mathcal{B}}(\cdot)$ and Lévy's convergence theorem

The Lévy's theorem is one of the most useful convergence results in stochastic analysis theory. The original theorem introduced by Lévy and is: if $(\mathcal{F}_n)_{n\in\mathbf{N}*}$ is an increasing sequence of sub-σ-algebra of $\mathcal{F}$ and f a real or vector valued integrable random variable, then one has

$$\lim_{n\to+\infty} E^{\mathcal{F}_n}(f) = E^{\mathcal{F}}(f) = f,$$

where $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$. This theorem has been extended to the multivalued random sets whose values are convex and weakly compact (resp. closed convex and bounded). Let us mention the works of [9, 10, 14, 24, 26] in the Bochner integrable case and the works of [1, 2] in the Pettis integrable case. In this section we present an extension of this theorem: One for closed convex and unbounded Pettis integrable random sets dealing with the Slice topology, and the other for closed convex and bounded Pettis integrable random sets dealing with the linear topology.

Before starting those results we begin by the following Jensen's inequalities for the unbounded Pettis integrable random sets.

Lemma 4.1. *Let X be a $cc(E)$-valued random set satisfying all conditions of Theorem 3.9. Then we have the following properties*:

(1) $\forall x \in E,\ d(x, E^{\mathcal{B}}(X)) \le E^{\mathcal{B}}(d(x, X))\quad \mu$ a.s.,

(2) $|E^{\mathcal{B}}(X)| \le E^{\mathcal{B}}(|X|)\quad \mu$ a.s.

Proof. To prove (1). Indeed, since X is measurable, then the mapping: $\omega \longmapsto d(x, X(\omega))$ is measurable for any $x \in E$ (see [6, §3, p. 67]). Now for a given $x \in E$ and $\epsilon > 0$, let us define a positive random variable r as

$$r(\omega) = d(x, X(\omega)) + \epsilon, \quad \omega \in \Omega,$$

and a multifunction G as

$$G(\omega) = X(\omega) \cap \overline{B}(x, r(\omega)), \quad \omega \in \Omega.$$

It is clear that $G(\omega)$ is with non-empty closed values in a complete space E and from Proposition 3.3.3 in Hess [17], G is a measurable multifunction. Hence by applying Theorem III.6 in [6] there is a measurable selection g of G such that

$$\|x - g(\omega)\| \le d(x, X(\omega)) + \epsilon. \tag{4.4.1}$$

Since g is also a selection of X, then the scalar integrability of X implies that g is scalarly integrable. Now by taking the conditional expectation in (4.4.1) we have

$$E^{\mathcal{B}}(\|x - g\|) \le E^{\mathcal{B}}(d(x, X)) + \epsilon \quad \mu \text{ a.s.}$$

Furthermore, from Lemma 3.3 in [2], we get

$$\begin{aligned}\|x - E^{\mathcal{B}}(g)\| = \|E^{\mathcal{B}}(x-g)\| &\leq E^{\mathcal{B}}(\|x-g\|) \quad \mu \,\text{a.s} \\ &\leq E^{\mathcal{B}}(d(x,X)) + \epsilon.\end{aligned}$$

Thus the property $E^{\mathcal{B}}(g) \in E^{\mathcal{B}}(X)$ implies

$$d(x, E^{\mathcal{B}}(X)) \leq E^{\mathcal{B}}(d(x,X)) + \epsilon \quad \mu \,\text{a.s.},$$

where ϵ is arbitrary. Then for all $x \in E$,

$$d(x, E^{\mathcal{B}}(X)) \leq E^{\mathcal{B}}(d(x,X)) \quad \mu \,\text{a.s.}$$

Finally to prove (2), let D' be a countable dense subset in B' with respect to the norm topology in E'. Then we have

$$\begin{aligned}E^{\mathcal{B}}|X| &= E^{\mathcal{B}}(\sup_{x' \in D'} (\delta^*(x', X)) \\ &\geq E^{\mathcal{B}}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{B}}(X)) \quad \mu \,\text{a.s.} \quad \forall\, x' \in D'.\end{aligned}$$

Thus

$$E^{\mathcal{B}}|X| \geq \sup_{x' \in D'} \delta^*(x', E^{\mathcal{B}}(X)) = |E^{\mathcal{B}}(X)| \quad \mu \,\text{a.s.}$$

□

A sequence of random sets (X_n) is called a martingale (resp. supermartingale, submartingale) if for all $m < n$, we have

$$E^{\mathcal{F}_m}(X_n) = X_m \ \ (\text{resp.}\, E^{\mathcal{F}_m}(X_n) \subset X_m,\ E^{\mathcal{F}_m}(X_n) \supset X_m),$$

where $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of elements of $\mathcal{F}$.

Theorem 4.2. *Let X be a $cc(E)$-valued random set satisfying all conditions of Theorem 3.9, and $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of elements of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$. Assume that:*

1) *$E^{\mathcal{F}_n}(X)$ is majorized by a w-ball compact set R,*

2) *There is a countable partition $(B_m)_{m\in \mathbf{N}}$ of Ω in $\mathcal{F}_1$ such that for each $m \in \mathbf{N}$,*

$$\int_{B_m} d(0,X) d\mu < +\infty.$$

Then

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X) = X \quad \mu \,\text{a.s.}$$

Proof. For each $m \in \mathbf{N}^*$, let us define a positive integrable random variable as $r^m(\omega) = d(0, (X/B_m)(\omega)) + m$ and a closed convex valued integrable supermartingale as

$$X_n^m(\omega) = E^{\mathcal{F}_n}(X) \cap \overline{B(0, r^m(\omega))}, \quad \omega \in \Omega,$$

where X/B_m is the restriction of X to each B_m. Since for each n and m in $\mathbf{N}^*$ we have $X/B_m = E^{\mathcal{F}_n}(X/B_m)$, it is clear that $X_n^m(\omega) \neq \emptyset$. Then from conditions 1) and 2) and Theorem 3.1 in [20] for each integer $m \in \mathbf{N}^*$, there is a random set $X_{+\infty}^m$ with values in R_w such that

$$T_{Slice} - \lim_n X_n^m = X_{+\infty}^m \quad \mu \,\text{a.s.}$$

Thus

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X) \cap B(0, r^m) = X^m_{+\infty} \quad \mu \text{ a.s.}$$

Now by applying Lemma 5.7 in [20] we get

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X)(\omega) = X_{+\infty}(\omega) \qquad \forall \omega \in \Omega \setminus N,$$

where

$$\begin{aligned} X_{+\infty}(\omega) &= \cup_m X^m_{+\infty}(\omega) \text{ if } \omega \in \Omega \setminus N, \\ X_{+\infty}(\omega) &= \{0\} \text{ if } \omega \in N. \end{aligned}$$

Also for each $m \in \mathbf{N}^*$, we have

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X/B_m)(\omega) = X_{+\infty}/B_m \quad \mu \text{ a.s.}$$

Hence for each $m \in \mathbf{N}^*$ and every $x \in E$,

$$d(x, X_{+\infty}/B_m) = \lim_n d(x, E^{\mathcal{F}_n}(X/B_m)).$$

According to condition 1) in Lemma 4.1 we have for each $m \in \mathbf{N}^*$ and for every $x \in E$,

$$\begin{aligned} d(x, X_{+\infty}/B_m) &= \lim_n d(x, E^{\mathcal{F}_n}(X/B_m)) \\ &\leq \lim_n E^{\mathcal{F}_n}(d(x, X/B_m)) \\ &= (d(x, X/B_m)), \end{aligned}$$

and this shows that for each $m \in \mathbf{N}^*$, $X/B_m \subset X_{+\infty}/B_m$, so $X \subset X_{+\infty}$.

Now we prove the reverse inclusion. Since

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X) = X_{+\infty} \qquad \mu \text{ a.s.}$$

Then also we have

$$T_{Mosco} - \lim_n E^{\mathcal{F}_n}(X) = X_{+\infty} \qquad \mu \text{ a.s.},$$

hence

$$X_{+\infty} = w - lsE^{\mathcal{F}_n}(X) \quad \mu \text{ a.s.}$$

Let $D = \{x'_k, k \in \mathbf{N}\}$ be a dense sequence in B' with respect to strong topology in E'. Since

$$\delta^*(x'_k, E^{\mathcal{F}_n}(X)) = E^{\mathcal{F}_n}(\delta^*(x'_k, X)) \quad \mu \text{ a.s.}$$

for every $k \in \mathbf{N}$ and $n \in \mathbf{N}^*$. By the Lévy's theorem for $L^1_{\mathbf{R}}$ we have

$$\lim_n E^{\mathcal{F}_n}(\delta^*(x'_k, X)) = \delta^*(x'_k, E^{\mathcal{F}_n}(X)) = \delta^*(x'_k, X) \quad \mu \text{ a.s.}$$

Let $\omega \in \Omega$ such that the previous equalities are satisfied. Let $f \in w - lsE^{\mathcal{F}_n}(X)(\omega)$, so there is $f_j \in E^{\mathcal{F}_{n_j}}(X)(\omega)$, such that $f = w - \lim_j f_j$. Hence

$$\begin{aligned} \langle x'_k, f \rangle &= \lim_j \langle x'_k, f_j \rangle \\ &\leq \limsup_j \delta^*(x'_k, E^{\mathcal{F}_{n_j}}(X)(\omega)) \end{aligned}$$

$$= \limsup_j E^{\mathcal{F}_{n_j}}(\delta^*(x'_k, X(\omega)))$$
$$\leq \lim_j E^{\mathcal{F}_{n_j}}(\delta^*(x'_k, X(\omega))) = (x'_k, X(\omega)).$$

Since X is D-countably supported (X satisfying all conditions of Theorem 3.9), then from the previous formula we conclude that $f \in X(\omega)$, hence $X_{+\infty} \subset X$.

Consequently

$$T_{Slice} - \lim_n E^{\mathcal{F}_n}(X) = X. \qquad \square$$

If the random sets are of bounded values, the conditions 1) and 2) are omitted and we get a convergence with the linear topology.

Proposition 4.3. *Let X be a $cc(E)$-valued random set satisfying all conditions of Theorem 3.9. Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$. Assume that $\sup_n E^{\mathcal{F}_n}(|X|) < +\infty$ μ a.s.. Then*

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \quad \mu\, a.s.$$

Proof. Since X satisfying all conditions of Theorem 3.9, then for every $n \geq 1$, by using the existence of $E^{\mathcal{F}_n}(\delta^*(x', X))$ and $E^{\mathcal{F}_n}(X)$, and the Pettis integrability of X and $E^{\mathcal{F}_n}(X)$ we get for all $x' \in E'$,

$$\int_A \delta^*(x', E^{\mathcal{F}_n}(X))d\mu = \delta^*(x', \int_A E^{\mathcal{F}_n}(X)d\mu) = \delta^*(x', \int_A X d\mu)$$
$$= \int_A \delta^*(x', X)d\mu = \int_A E^{\mathcal{F}_n}(\delta^*(x', X))d\mu \qquad \forall A \in \mathcal{F}_n.$$

Then from the uniqueness of conditional expectation for every $n \geq 1$ and $x' \in E'$, we have

(4.3.1) $$E^{\mathcal{F}_n}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{F}_n}(X)) \quad \mu \text{ a.s.}$$

Then by applying the classical Lévy's martingale convergence theorem and (4.3.1) it follows that

$$\lim_{n\to+\infty} \delta^*(x', E^{\mathcal{F}_n}(X))(\omega) = \lim_{n\to+\infty} E^{\mathcal{F}_n}(\delta^*(x', X))(\omega) = \delta^*(x', X)(\omega)$$

for all $\omega \in \Omega \setminus N_{x'}, \mu(N_{x'}) = 0$. Let $D' = \{x'_n, n \in \mathbf{N}^*\}$ be a countable dense sequence in E' for the strong topology. Then we have

$$\lim_{m\to+\infty} E^{\mathcal{F}_m}(\delta^*(x'_n, X))(\omega) = \delta^*(x'_n, X)(\omega) \quad \forall \omega \in \Omega \setminus N_{x'_n}.$$

Let us define the set $N = \cup_{n\geq 1}(N_{x'_n})$ ($\mu(N) = 0$), then for all $\omega \in \Omega \setminus N$ and every $k \geq 1$, we have

(4.3.2) $$\lim_{m\to+\infty} E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega) = \delta^*(x'_k, X)(\omega).$$

Then for $\epsilon > 0$ given , $k \geq 1$, and $\omega \in \Omega \setminus N$, there is $m_{\omega,k}$ such that $\forall m \geq m_{\omega,k}$, we have

$$| E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega) - \delta^*(x'_k, X)(\omega) | < \epsilon.$$

Now let x' be a fixed point in E', and let $\epsilon > 0$, then there exists a sequence $(x'_k)_{k\geq 1}$ in D' such that for large $k \geq k_0$, we have $\|x'_k - x'\| < \epsilon$. Then we have

$$\begin{aligned}
&|E^{\mathcal{F}_m}(\delta^*(x', X))(\omega) - \delta^*(x', X)(\omega)| \\
&\leq |E^{\mathcal{F}_m}(\delta^*(x', X))(\omega) - E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega)| \\
&\quad + |E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega) - \delta^*(x'_k, X)(\omega)| + |\delta^*(x'_k, X)(\omega) - \delta^*(x', X)(\omega)| \\
&\leq E^{\mathcal{F}_m}|(\delta^*(x', X)(\omega) - \delta^*(x'_k, X)(\omega))| + |E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega) - \delta^*(x'_k, X)(\omega)| \\
&\quad + |\delta^*(x'_k - x', X)(\omega) + \delta^*(x' - x'_k, X)(\omega)| \\
&\leq 2 \cdot E^{\mathcal{F}_m}(|X|)(\omega)\|x'_k - x'\| + |E^{\mathcal{F}_m}(\delta^*(x'_k, X))(\omega) - \delta^*(x'_k, X)(\omega)| \\
&\quad + 2 \cdot |X|(\omega)\|x'_k - x'\| \\
&\leq (2 \cdot E^{\mathcal{F}_m}(|X|)(\omega) + 2 \cdot |X|(\omega) + 1) \cdot \epsilon.
\end{aligned}$$

Consequently if $k \geq k_0$ and $m \geq m_{\omega,k}$ we have

$$\lim_m \delta^*(x', E^{\mathcal{F}_m}(X)(\omega)) = \delta^*(x', X)(\omega) \ \ \forall\omega \in \Omega \setminus N, \ \forall x' \in E'. \tag{4.3.3}$$

Now we prove that for every $x \in E$,

$$\lim_n d(x, E^{\mathcal{F}_n}(X)) = d(x, X) \quad \mu \text{ a.s.}$$

Indeed, let $x \in E$, $x' \in B'$, and $\omega \in \Omega$, then we can write

$$d(x, E^{\mathcal{F}_n}(X)(\omega)) \geq (\langle x', x\rangle - \delta^*(x', E^{\mathcal{F}_n}(X)(\omega))). \tag{4.3.4}$$

Now applying (4.3.3) and (4.3.4) we get

$$\liminf_n d(x, E^{\mathcal{F}_n}(X)) \geq (\langle x', x\rangle - \delta^*(x', X)) \ \ \mu \text{ a.s.}$$

and by taking the supremum on B' we have

$$\liminf_n d(x, E^{\mathcal{F}_n}(X)) \geq d(x, X) \quad \mu \text{ a.s.} \tag{4.3.5}$$

Next we define $A_m = \{m-1 \leq E^{\mathcal{F}_1}(|X|) < m\}$. It is clear that for each $m \geq 1$, for every $x \in E$, $d(x, X) \cdot \chi_{A_m} \in L^1_R$. Then, from Lemma 3.4(2), the classical Lévy's theorem for $L^1_{\mathbf{R}}$ we have

$$\begin{aligned}
\limsup_n d(x, E^{\mathcal{F}_n}(X)) \cdot \chi_{A_m} &\leq \limsup_n E^{\mathcal{F}_n}(d(x, X)) \cdot \chi_{A_m} \\
&= \limsup_n E^{\mathcal{F}_n}(d(x, X) \cdot \chi_{A_m}) \\
&= d(x, X) \cdot \chi_{A_m} \quad \mu \text{ a.s.}
\end{aligned}$$

The fact that (A_m) is a partition of Ω in $\mathcal{F}_1$, it is clear that for every $x \in E$,

$$\limsup_n d(x, E^{\mathcal{F}_n}(X)) \leq d(x, X) \quad \mu \text{ a.s.}$$

Combining this with (4.3.5) we get for every $x \in E$,

$$\lim_n d(x, E^{\mathcal{F}_n}(X)) = d(x, X) \quad \mu \text{ a.s.}$$

The μ a.s. convergence follows from the fact that

$$\limsup_n d(x, E^{\mathcal{F}_n}(X)) \geq \liminf_n d(x, E^{\mathcal{F}_n}(X)).$$

Consequently we have,

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \qquad \mu \text{ a.s.} \qquad \square$$

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Mohamed El Harami
University Moulay Ismail
Higher School of Technology
Km 5, Road of Agouray, P1, Meknes 50040, Morocco
Email address: haramimohamed@live.fr