

HARNACK ESTIMATES FOR NONLINEAR BACKWARD HEAT EQUATIONS WITH POTENTIALS ALONG THE RICCI-BOURGUIGNON FLOW

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ABSTRACT. In this paper, we derive various differential Harnack estimates for positive solutions to the nonlinear backward heat type equations on closed manifolds coupled with the Ricci-Bourguignon flow, which was done for the Ricci flow by J.-Y. Wu [30]. The proof follows exactly the one given by X.-D. Cao [4] for the linear backward heat type equations coupled with the Ricci flow.

1. Introduction and main results

The study of differential Harnack estimates for parabolic equations originated with the work of P. Li and S.-T. Yau [24] who developed a gradient estimate for positive solution of the heat equation on Riemannian manifolds with nonnegative Ricci curvature. They also derived a classical Harnack inequality by integrating the gradient estimate along a space-time path which could be used to compare the solution between different space-time points. This result was generalized to Harnack estimates for some nonlinear heat-type equations in [33] and for non-self-adjoint evolution equations in [34].

Apart from the work of P. Li and S.-T. Yau, many authors have also proved a variety of Harnack estimates for various equations under different geometric flows. It is well known that R. Hamilton proved Harnack estimates for the Ricci flow [20] and the mean curvature flow [22]. In dimension two, a Harnack estimate for the positive scalar curvature was obtained by R. Hamilton [19], and then extended by B. Chow [10] when the scalar curvature changed sign. The same techniques were used to obtain Harnack estimates for the Gauss curvature flow and the Yamabe flow in [11] and [12], respectively. B. Andrews [1] derived several Harnack estimates for general curvature flows on hypersurfaces. H.-D. Cao [3] proved a Harnack estimate for the Kähler-Ricci flow. R. Hamilton [21] generalized the Li-Yau Harnack estimate to a matrix form on Riemannian

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manifolds with nonnegative sectional curvature. In [26], L. Ni also derived a matrix Harnack estimate for the Kähler-Ricci flow by using the interpolation techniques.

On the other hand, the Harnack estimates for forward or backward heat-type equations coupled with the Ricci flow could be found in [6, 9, 13, 14, 16, 19, 31, 36], etc. Perhaps the most spectacular result is that G. Perelman [27] proved a Harnack estimate for the fundamental solution to the conjugate heat equation coupled with the Ricci flow without any curvature assumption (see also [25] or [28] for details). Perelman's Harnack estimate has an essential application in proving pseudolocality theorems. However, it does not apply for all positive solutions to the conjugate heat equation. Later, X.-D. Cao [4] and S.-L. Kuang and Q. S. Zhang [23] established a Harnack estimate that works for all positive solutions to the conjugate heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature.

Motivated by the above works, we study the Harnack estimates for positive solutions to the nonlinear backward heat-type equation

$$(1) \quad \frac{\partial f}{\partial t} = -\Delta f + f \ln f - \gamma Rf$$

on an n -dimensional closed manifold with the metric $g = g(t)$ evolving along the Ricci-Bourguignon flow

$$(2) \quad \frac{\partial g}{\partial t} = -2(Ric - \rho Rg),$$

where Ric and R denoting Ricci curvature tensor and scalar curvature, respectively. γ and ρ are real constants.

It is noticed that J.-Y. Wu [30] derived the Harnack estimates for the nonlinear backward heat-type equation (1) coupled with the Ricci flow, and pointed out that the equation (1) is closely related to the shrinking gradient Ricci soliton according to the arguments of X.-D. Cao and Z. Zhang [7]. For a general geometric flow, H.-X. Guo and M. Ishida proved the Harnack estimates for nonlinear forward and backward heat-type equations under various assumptions, see [18] and [17], respectively.

The evolution equation that defined by (2) is named as the Ricci-Bourguignon flow (shortly RB flow) which was proposed by J. P. Bourguignon (see [2], Question 3.24). As special cases, this family of geometric flows contains, the Ricci flow ($\rho = 0$), the Einstein flow ($\rho = \frac{1}{2}$), the traceless flow ($\rho = \frac{1}{n}$) and the Schouten flow ($\rho = \frac{1}{2(n-1)}$) on account of corresponding to Ric tensor, Einstein tensor, traceless Ric tensor and Schouten tensor, respectively. As stated in [8], when ρ is nonpositive, by a suitable rescaling in time, the RB flow can be seen as an interpolation between the Ricci flow and the Yamabe flow (see [35]), obtained as a limit when $\rho \rightarrow -\infty$. The authors of [8] also proved that, for any $\rho < \frac{1}{2(n-1)}$, the RB flow (2) has a unique solution for a positive time interval on closed manifold with any initial metric g_0 . On the contrary,

when $\rho > \frac{1}{2(n-1)}$, due to the principle symbol of the operator on the right hand side of (2) has negative eigenvalues, not allowing even a short time existence result for the flow for general initial metric. Later, A. Fischer [15] studied a conformal version of this flow where the scalar curvature is constrained along the flow.

For the nonlinear backward heat-type equation (1) with $\gamma = n\rho - 1$, i.e.,

$$(3) \quad \frac{\partial f}{\partial t} = -\Delta f + f \ln f + (1 - n\rho)Rf.$$

The corresponding linear version

$$\frac{\partial f}{\partial t} = -\Delta f + (1 - n\rho)Rf$$

is the conjugate heat equation with respect to the RB flow. Our main theorem is the following Harnack estimates for the equation (3).

Theorem 1.1. *Let $(M^n, g(t))_{t \in [0, T]}$ be a nontrivial solution to the RB flow (2) on an n -dimensional closed manifold and $f(x, t)$ be a positive solution to the equation (3). Let $u = -\ln f$, $\tau = T - t$, $\delta = R_{\max}^{-1}(0)$,*

$$A(n, \rho) = \frac{n(1 - n\rho)^2(1 - (n - 1)\rho)^2}{4((n - 1)(n + 2)\rho^2 - 2n\rho + 1)}$$

and

$$H = 2(1 - n\rho)(1 - (n - 1)\rho)\Delta u - ((n - 1)(n - 2)\rho^2 - 2(n - 1)\rho + 1)|\nabla u|^2 + (n\rho - 1)^2 R + c \frac{n}{\tau}.$$

(i) *If $\rho < 0$ and the curvature operator is nonnegative at the initial time, then*

$$(4) \quad H - A(n, \rho) \leq 0$$

for all $(x, t) \in M^n \times [0, T]$ with $T < \frac{\delta}{2(1-\rho)}$. Here

$$c = -\frac{2(n\rho - 1)^2}{n\rho} \left\{ \frac{2(n\rho - 1)(1 - (n - 1)\rho)^2}{(n - 1)(n + 2)\rho^2 - 2n\rho + 1} - 1 \right\}.$$

(ii) *If $\rho = 0$ and the scalar curvature is nonnegative at the initial time, then (4) holds for all $(x, t) \in M^n \times [0, T]$. Here $c = -2$, that is*

$$2\Delta u - |\nabla u|^2 + R - \frac{2}{\tau}n - \frac{n}{4} \leq 0.$$

(iii) *If $0 < \rho < \frac{1}{2(n-1)}$ and the curvature operator is nonnegative at the initial time, then (4) holds for all $(x, t) \in M^n \times [0, T]$ with $T < \frac{\delta}{2}$. Here*

$$c = -\frac{(1 - n\rho)^2}{2n} \left\{ \frac{(4n + 1)(1 - (n - 1)\rho)^2}{(n - 1)(n + 2)\rho^2 - 2n\rho + 1} + 1 + \rho \right\}.$$

Remark 1.2. (i) J.-Y. Wu [30] has also proved the same estimates as Theorem 1.1 for the case $\rho = 0$ (see [30, Theorem 1.6]). At this moment, The RB flow is the Ricci flow. However, our proof follows from a straightforward computation of evolution equation for more general Harnack quantity as was done for the Ricci flow by X.-D. Cao [4].

(ii) One interesting feature is that our Harnack estimates are not only like the Perelman's Harnack estimates, but also similar to the classical Li-Yau type Harnack estimates for the corresponding nonlinear heat equation. In fact, it is easy to see that our Harnack estimates have the following form

$$(5) \quad \alpha \Delta u - \beta |\nabla u|^2 + aR - C_1 \frac{n}{\tau} - C_2 \leq 0.$$

Since $u = -\ln f$, (5) can be written as

$$\frac{|\nabla f|^2}{f^2} - C_3 \left(\frac{f_\tau}{f} + \ln f + R \right) \leq C_4 \frac{n}{\tau} + C_5,$$

where $\alpha, \beta, a, C_1, C_2, C_3, C_4$ and C_5 are positive constants only depending on n and ρ , which is analogous to the classical Li-Yau type gradient estimate for the nonlinear heat-type equation

$$\frac{\partial f}{\partial t} = \Delta f - af \ln f - bf$$

in manifolds with fixed metrics (see [29] or [32] for details).

By means of the same arguments, we will obtain the following Harnack estimate for the nonlinear backward heat equation without any potential.

Theorem 1.3. *Suppose that $g(t)_{t \in [0, T]}$ evolve along the RB flow with $\rho \leq 0$ on an n -dimensional closed manifold, and $f(x, t) (< 1)$ be a positive solution to*

$$(6) \quad \frac{\partial f}{\partial t} = -\Delta f + f \ln f.$$

If the curvature operator is nonnegative at the initial time, then

$$\frac{|\nabla f|^2}{f^2} \leq \frac{1}{\tau} \ln \frac{1}{f}$$

for all $(x, t) \in M^n \times [0, T]$, where $\tau = T - t$.

Remark 1.4. (i) Theorem 1.3 can be regarded as an extension of the Harnack estimate for the equation (6) under the Ricci flow, which was obtained by J.-Y. Wu [30].

(ii) According to the proof, it is obvious that (i) and (iii) in Theorem 1.1 and Theorem 1.3 will hold whenever the Ricci curvature is nonnegative, but in general, the nonnegativity of the Ricci curvature is not preserved along the RB flow except the case of dimension three. Nevertheless, the nonnegativity of the curvature operator is preserved along the RB flow when $\rho < \frac{1}{2(n-1)}$ (see [8] for details).

The proof of the above theorems follows nearly from the techniques of X.-D. Cao [4] for the Ricci flow, where calculations of evolution equation for a general Harnack quantity and the maximum principle are employed. The main difference is that we derive the Harnack estimates for various nonlinear backward heat equations coupled with the RB flow.

This rest of paper is organized as follows. In Section 2, we will prove the evolution equation for a general Harnack quantity H (Lemma 2.1) and the estimates for scalar curvature R under the RB flow (Lemma 2.3), which play a key role in deriving main theorem. In Section 3, we will prove Theorem 1.1 and Theorem 1.3 by modifying the Harnack quantity H , and give an application of Theorem 1.3 which shows that any positive L^1 -solution of the nonlinear backward heat equation (6) can not blow up too fast.

2. Preliminaries

In this section, we shall derive the evolution equation of a general Harnack quantity H and the estimates for scalar curvature R under the RB flow, which are useful to prove the main theorem.

Throughout, M^n will be taken to be a closed manifold of dimension n . Let $g(t)$ evolving by the RB flow, and $f(x, t)$ be a positive solution to the nonlinear backward heat equation with potential term $-\gamma R$. That is, $(g(t), f(x, t))$ satisfies the system

$$(7) \quad \begin{cases} \frac{\partial g}{\partial t} = -2(Ric - \rho Rg), \\ \frac{\partial f}{\partial t} = -\Delta f + f \ln f - \gamma Rf, \end{cases}$$

where γ is a constant.

Let $u = -\ln f$ and $\tau = T - t$. By a direct computation, we have

$$(8) \quad \frac{\partial u}{\partial \tau} = \Delta u - |\nabla u|^2 - \gamma R - u.$$

Define the Harnack quantity

$$H = \alpha \Delta u - \beta |\nabla u|^2 + aR + b\frac{u}{\tau} + c\frac{n}{\tau},$$

where α, β, a, b, c are constants.

It is easy to see that the evolution equations for the Laplace-Beltrami operator Δ and the scalar curvature R with respect to the RB flow hold:

$$(9) \quad \left(\frac{\partial}{\partial t} \Delta\right)u = 2\langle Ric, \nabla \nabla u \rangle - 2\rho R \Delta u + (n - 2)\rho \langle \nabla R, \nabla u \rangle,$$

$$(10) \quad \frac{\partial R}{\partial t} = (1 - 2(n - 1)\rho) \Delta R + 2|Ric|^2 - 2\rho R^2.$$

Firstly, we derive the evolution equation of H which can be seen as a generalization of [4, Lemma 2.1].

Lemma 2.1. *Suppose that $(g(t), f(x, t))$ satisfies the system (7), and $\alpha > \beta > 0$. Then*

$$\begin{aligned}
(11) \quad \frac{\partial H}{\partial \tau} &\leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2\lambda(\alpha - \beta)}{\alpha\tau} + 2\right)H + (4\beta - 2\alpha)\text{Ric}(\nabla u, \nabla u) \\
&\quad + \left(\frac{b}{\tau} - \frac{2\beta\lambda(\alpha - \beta)}{\alpha\tau} - 2\beta\rho R\right)|\nabla u|^2 + \left(\frac{\alpha^2\rho(n\rho - 2)}{2(\alpha - \beta)} + 2a\rho\right)R^2 \\
&\quad + \left(\frac{2a\lambda(\alpha - \beta)}{\alpha} - \lambda\alpha(1 - n\rho) - b\gamma\right)\frac{R}{\tau} + \left(2a - \frac{\alpha^2}{2(\alpha - \beta)}(1 - n\rho)\right)R \\
&\quad + b\left(\frac{2\lambda(\alpha - \beta)}{\alpha} - 1\right)\frac{u}{\tau^2} + c\left(\frac{2\lambda(\alpha - \beta)}{\alpha} - 1\right)\frac{n}{\tau^2} + \frac{(\alpha - \beta)\lambda^2}{2} \cdot \frac{n}{\tau^2} \\
&\quad + (2a + 2\beta\gamma - (n - 2)\rho\alpha)\langle \nabla R, \nabla u \rangle - (\alpha\gamma + 2a - 2a(n - 1)\rho)\Delta R \\
&\quad + \left(\frac{\alpha^2}{2(\alpha - \beta)} - 2a\right)|\text{Ric}|^2 + b \cdot \frac{u}{\tau} + \left(2c + \frac{\lambda\alpha}{2}\right)\frac{n}{\tau} + \frac{n\alpha^2}{8(\alpha - \beta)},
\end{aligned}$$

where λ is a constant that will be chosen properly.

Proof. It is easy to calculate the first two terms in H by using (7), (8) and (9),

$$\begin{aligned}
\frac{\partial}{\partial \tau}(\Delta u) &= \left(\frac{\partial}{\partial \tau}\Delta\right)u + \Delta\frac{\partial u}{\partial \tau} \\
&= -2\langle \text{Ric}, \nabla \nabla u \rangle + 2\rho R\Delta u - (n - 2)\rho\langle \nabla R, \nabla u \rangle \\
&\quad + \Delta(\Delta u) - \Delta|\nabla u|^2 - \gamma\Delta R - \Delta u
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tau}|\nabla u|^2 &= -2(\text{Ric} - \rho Rg)(\nabla u, \nabla u) + 2\langle \nabla u, \nabla u_\tau \rangle \\
&= \Delta|\nabla u|^2 - 2|\nabla \nabla u|^2 - 4\text{Ric}(\nabla u, \nabla u) + 2\rho R|\nabla u|^2 \\
&\quad - 2\langle \nabla u, \nabla|\nabla u|^2 \rangle - 2\gamma\langle \nabla R, \nabla u \rangle - 2|\nabla u|^2,
\end{aligned}$$

here we used the Bochner formula

$$\Delta|\nabla u|^2 = 2|\nabla \nabla u|^2 + 2\langle \nabla u, \nabla \Delta u \rangle + 2\text{Ric}(\nabla u, \nabla u).$$

Using (8), (10) and the Bochner formula, we have

$$\begin{aligned}
\frac{\partial H}{\partial \tau} &= \alpha\frac{\partial}{\partial \tau}(\Delta u) - \beta\frac{\partial}{\partial \tau}|\nabla u|^2 + a\frac{\partial R}{\partial \tau} + \frac{b}{\tau}u_\tau - \frac{bu}{\tau^2} - \frac{cn}{\tau^2} \\
&= \alpha(\Delta(\Delta u) - \Delta|\nabla u|^2 + 2\rho R\Delta u - 2\langle \text{Ric}, \nabla \nabla u \rangle \\
&\quad - (n - 2)\rho\langle \nabla R, \nabla u \rangle - \gamma\Delta R) - \beta(\Delta|\nabla u|^2 - 2|\nabla \nabla u|^2 \\
&\quad - 4\text{Ric}(\nabla u, \nabla u) + 2\rho R|\nabla u|^2 - 2\langle \nabla u, \nabla|\nabla u|^2 \rangle \\
&\quad - 2\gamma\langle \nabla R, \nabla u \rangle) - a((1 - 2(n - 1)\rho)\Delta R + 2|\text{Ric}|^2 - 2\rho R^2) \\
&\quad + \frac{b}{\tau}(\Delta u - |\nabla u|^2 - \gamma R) - b\frac{u}{\tau^2} - c\frac{n}{\tau^2} - \alpha\Delta u + 2\beta|\nabla u|^2 - \frac{bu}{\tau}.
\end{aligned}$$

A straightforward computation gives

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \Delta H - 2\alpha \langle \nabla u, \nabla \Delta u \rangle - (\alpha\gamma + a + a(1 - 2(n-1)\rho)) \Delta R \\ &\quad + 2\beta \langle \nabla u, \nabla |\nabla u|^2 \rangle + (2\beta\gamma - (n-2)\rho\alpha) \langle \nabla R, \nabla u \rangle - 2\alpha R_{ij} u_{ij} \\ &\quad - 2(\alpha - \beta) |u_{ij}|^2 - 2a |Ric|^2 + (4\beta - 2\alpha) Ric \langle \nabla u, \nabla u \rangle \\ &\quad - \frac{b}{\tau} |\nabla u|^2 - \frac{b\gamma}{\tau} R - \frac{bu}{\tau^2} - \frac{cn}{\tau^2} + 2\alpha\rho R \Delta u - 2\beta\rho R |\nabla u|^2 + 2a\rho R^2 \\ &\quad - \alpha \Delta u + 2\beta |\nabla u|^2 - \frac{bu}{\tau}. \end{aligned}$$

By the definition of H , we have

$$\nabla H \cdot \nabla u = \alpha \langle \nabla u, \nabla \Delta u \rangle - \beta \langle \nabla u, \nabla |\nabla u|^2 \rangle + a \langle \nabla R, \nabla u \rangle + \frac{b}{\tau} |\nabla u|^2.$$

This implies

$$-2\alpha \langle \nabla u, \nabla \Delta u \rangle + 2\beta \langle \nabla u, \nabla |\nabla u|^2 \rangle = -2\nabla H \cdot \nabla u + 2a \langle \nabla R, \nabla u \rangle + \frac{2b}{\tau} |\nabla u|^2.$$

Moreover, since

$$\begin{aligned} &-2(\alpha - \beta) |u_{ij}| + \frac{\alpha}{2(\alpha - \beta)} (R_{ij} - \rho R g_{ij})^2 \\ &= -2(\alpha - \beta) |u_{ij}|^2 - 2\alpha R_{ij} u_{ij} + 2\alpha\rho R \Delta u \\ &\quad - \frac{\alpha^2}{2(\alpha - \beta)} |Ric|^2 - \frac{\alpha^2 \rho (n\rho - 2)}{2(\alpha - \beta)} R^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \Delta H - 2\nabla H \cdot \nabla u - 2(\alpha - \beta) |u_{ij}| + \frac{\alpha}{2(\alpha - \beta)} (R_{ij} - \rho R g_{ij})^2 \\ &\quad + (2a + 2\beta\gamma - (n-2)\rho\alpha) \langle \nabla R, \nabla u \rangle - (\alpha\gamma + 2a - 2a(n-1)\rho) \Delta R \\ (12) \quad &\quad + (4\beta - 2\alpha) Ric \langle \nabla u, \nabla u \rangle + \left(\frac{b}{\tau} - 2\beta\rho R\right) |\nabla u|^2 - \alpha \Delta u + 2\beta |\nabla u|^2 \\ &\quad + \left(\frac{\alpha^2}{2(\alpha - \beta)} - 2a\right) |Ric|^2 + \left(\frac{\alpha^2 \rho (n\rho - 2)}{2(\alpha - \beta)} + 2a\rho\right) R^2 \\ &\quad - \frac{b\gamma}{\tau} R - \frac{bu}{\tau^2} - \frac{cn}{\tau^2} - \frac{bu}{\tau}. \end{aligned}$$

Notice that

$$\begin{aligned} &-2(\alpha - \beta) |u_{ij}| + \frac{\alpha}{2(\alpha - \beta)} (R_{ij} - \rho R g_{ij}) - \frac{\lambda}{2\tau} g_{ij}^2 \\ &= -2(\alpha - \beta) |u_{ij}| + \frac{\alpha}{2(\alpha - \beta)} (R_{ij} - \rho R g_{ij})^2 + \frac{2\lambda(\alpha - \beta)}{\tau} \left(\Delta u + \frac{\alpha(1 - n\rho)}{2(\alpha - \beta)} R\right) \\ &\quad - \frac{(\alpha - \beta)n\lambda^2}{2\tau^2}, \end{aligned}$$

$$-\frac{2\lambda(\alpha-\beta)}{\tau}\Delta u = -\frac{2\lambda(\alpha-\beta)}{\alpha\tau}(H + \beta|\nabla u|^2 - aR - b\frac{u}{\tau} - c\frac{n}{\tau}),$$

and

$$\begin{aligned} -\alpha\Delta u + 2\beta|\nabla u|^2 &= -2H + \alpha\left(\Delta u + \frac{\alpha}{2(\alpha-\beta)}(1-n\rho)R - \frac{n\lambda}{2\tau}\right) + \frac{2bu}{\tau} \\ &\quad + \left(2a - \frac{\alpha^2}{2(\alpha-\beta)}(1-n\rho)\right)R + \left(2c + \frac{\lambda\alpha}{2}\right)\frac{n}{\tau}. \end{aligned}$$

Using the elementary inequality and $\alpha > \beta > 0$, then

$$\begin{aligned} &-2(\alpha-\beta)|u_{ij}| + \frac{\alpha}{2(\alpha-\beta)}(R_{ij} - \rho Rg_{ij}) - \frac{\lambda}{2\tau}|g_{ij}|^2 \\ &\quad + \alpha\left(\Delta u + \frac{\alpha}{2(\alpha-\beta)}(1-n\rho)R - \frac{n\lambda}{2\tau}\right) \\ &\leq -\frac{2(\alpha-\beta)}{n}\left(\Delta u + \frac{\alpha}{2(\alpha-\beta)}(1-n\rho)R - \frac{n\lambda}{2\tau}\right)^2 \\ &\quad + \alpha\left(\Delta u + \frac{\alpha}{2(\alpha-\beta)}(1-n\rho)R - \frac{n\lambda}{2\tau}\right) \\ &\leq \frac{n\alpha^2}{8(\alpha-\beta)}. \end{aligned}$$

Substituting these into (12) and rearranging each terms, the desired result (11) follows. \square

In the same way. Let $v = -\ln f - h(\tau)$, where $h(\tau)$ is any smooth function on τ . Define

$$F = \alpha\Delta v - \beta|\nabla v|^2 + aR + b\frac{v}{\tau} + c\frac{n}{\tau}.$$

Then the following holds:

Lemma 2.2. *Suppose $(g(t), f(x, t))$ satisfies (7) and $\alpha > \beta > 0$. Then*

$$\begin{aligned} \frac{\partial F}{\partial \tau} &\leq \Delta F - 2\nabla F \cdot \nabla v - \left(\frac{2\lambda(\alpha-\beta)}{\alpha\tau} + 2\right)F + \left(\frac{b}{\tau} - \frac{2\beta\lambda(\alpha-\beta)}{\alpha\tau} - 2\beta\rho R\right)|\nabla v|^2 \\ &\quad + b\left(\frac{2\lambda(\alpha-\beta)}{\alpha} - 1\right)\frac{v}{\tau^2} + c\left(\frac{2\lambda(\alpha-\beta)}{\alpha} - 1\right)\frac{n}{\tau^2} \\ &\quad + \left(\frac{2a\lambda(\alpha-\beta)}{\alpha} - \lambda\alpha(1-n\rho) - b\gamma\right)\frac{R}{\tau} + (4\beta - 2\alpha)\text{Ric}(\nabla v, \nabla v) \\ &\quad + \left(\frac{\alpha^2}{2(\alpha-\beta)} - 2a\right)|\text{Ric}|^2 + \left(\frac{\alpha^2\rho(n\rho-2)}{2(\alpha-\beta)} + 2a\rho\right)R^2 \\ &\quad + \frac{(\alpha-\beta)\lambda^2}{2} \cdot \frac{n}{\tau^2} + \frac{bv}{\tau} + \left(2a - \frac{\alpha^2}{2(\alpha-\beta)}(1-n\rho)\right)R \\ &\quad + \left(2c + \frac{\lambda\alpha}{2}\right)\frac{n}{\tau} + \frac{n\alpha^2}{8(\alpha-\beta)} + (2a + 2\beta\gamma - (n-2)\rho\alpha)\langle\nabla R, \nabla v\rangle \\ &\quad - (\alpha\gamma + 2a - 2a(n-1)\rho)\Delta R - \frac{bh(\tau)}{\tau} - \frac{bh'(\tau)}{\tau}, \end{aligned}$$

where λ is a constant.

Proof. In fact, notice that

$$v = u - h(\tau), \quad F = H - \frac{bh(\tau)}{\tau}.$$

Therefore

$$\Delta v = \Delta u, \quad \nabla v = \nabla u, \quad \Delta F = \Delta H, \quad \nabla F = \nabla H,$$

and

$$\frac{\partial F}{\partial \tau} = \frac{\partial H}{\partial \tau} + \frac{bh(\tau)}{\tau^2} - \frac{bh'(\tau)}{\tau}.$$

Following from the same direct computations as in the proof of Lemma 2.1, we obtain the desired result. \square

For the purpose of proving main theorem, we also have the following results about the estimates of scalar curvature R which come from the evolution equation of R under the RB flow and the maximum principle.

Lemma 2.3. *Suppose that $g(t)_{t \in [0, T]}$ evolve by the RB flow on a closed manifold M^n of dimension n , and the curvature operator is nonnegative at the initial time. Let $\delta = R_{\max}^{-1}(0) > 0$. Then*

(a) *If $\rho < 0$ and $T < \frac{\delta}{2(1-\rho)}$, then*

$$0 \leq R < -\frac{1}{\rho\tau}$$

for all $(x, t) \in M^n \times [0, T)$.

(b) *If $0 < \rho < \frac{1}{2(n-1)}$ and $T < \frac{\delta}{2}$, then*

$$0 \leq R < \frac{1}{2\tau}$$

for all $(x, t) \in M^n \times [0, T)$.

Proof. The evolution of R under the RB flow is

$$\frac{\partial R}{\partial t} = (1 - 2(n - 1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2.$$

We have known that the nonnegativity of the curvature operator is preserved by the RB flow. This implies that $Ric \geq 0, R \geq 0$, and we have $|Ric|^2 \leq R^2$. the evolution equation of scalar curvature satisfies

$$\frac{\partial R}{\partial t} \leq (1 - 2(n - 1)\rho)\Delta R + 2(1 - \rho)R^2.$$

Applying the maximum principle to this inequality yields

$$(13) \quad R \leq \frac{1}{\delta - 2(1 - \rho)t}$$

on $[0, T)$, where $T < \frac{\delta}{2(1-\rho)}$.

(a) When $\rho < 0$ and $T < \frac{\delta}{2(1-\rho)}$, we have

$$\frac{\delta + \rho T}{2 - \rho} - T = \frac{\delta - 2(1 - \rho)T}{2 - \rho} > 0,$$

that is,

$$0 \leq t < T < \frac{\delta + \rho T}{2 - \rho}.$$

This is equivalent to

$$\frac{1}{\delta - 2(1 - \rho)t} < -\frac{1}{\rho\tau}.$$

Hence

$$0 \leq R < -\frac{1}{\rho\tau}.$$

(b) When $0 < \rho < \frac{1}{2(n-1)}$ and $T < \frac{\delta}{2}$, (13) also holds on $[0, T)$ with $T < \frac{\delta}{2}$. Moreover, since

$$\frac{\delta}{2(1 - \rho)} > \frac{\delta}{2} > T > t \geq 0 > \frac{2T - \delta}{2\rho}.$$

This is equivalent to

$$0 < 2\tau < \delta - 2(1 - \rho)t.$$

Hence

$$0 \leq R < \frac{1}{2\tau}. \quad \square$$

3. Proof of Theorem 1.1 and Theorem 1.3

In this section, Theorem 1.1 and Theorem 1.3 will be proved which mainly based on the above lemmas and the maximum principle. We will take suitable constants $\alpha, \beta, a, b, c, \lambda$ and derive the desired conclusions.

Proof of Theorem 1.1. Fixing $\gamma = n\rho - 1$ and taking into account eliminating the terms $\langle \nabla R, \nabla u \rangle$ and ΔR in (11). Let

$$\begin{cases} 2a + 2\beta(n\rho - 1) - (n - 2)\rho\alpha = 0, \\ \alpha(n\rho - 1) + 2a - 2a(n - 1)\rho = 0. \end{cases}$$

Choosing

$$\alpha = 2(1 - n\rho)(1 - (n - 1)\rho), \quad \beta = (n - 1)(n - 2)\rho^2 - 2n\rho + 2\rho + 1,$$

then $a = (1 - n\rho)^2$ and

$$\alpha - \beta = (n - 1)(n + 2)\rho^2 - 2n\rho + 1 > 0 \text{ when } \rho < \frac{1}{2(n - 1)}.$$

Setting $b = 0, \lambda = \frac{\alpha}{\alpha - \beta}$, then (11) can be simplified as

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H - 2\beta\left(\rho R + \frac{1}{\tau}\right)|\nabla u|^2 + \frac{n\alpha^2}{8(\alpha - \beta)}$$

$$\begin{aligned}
 & + \left(c + \frac{\alpha^2}{2(\alpha - \beta)}\right) \frac{n}{\tau^2} + \left(2c + \frac{\alpha^2}{2(\alpha - \beta)}\right) \frac{n}{\tau} + (4\beta - 2\alpha) Ric(\nabla u, \nabla u) \\
 (14) \quad & + \left(2a - \frac{\alpha^2}{\alpha - \beta}(1 - n\rho)\right) \frac{R}{\tau} + \left(\frac{\alpha^2}{2(\alpha - \beta)} - 2a\right) |Ric|^2 \\
 & + \left(\frac{\alpha^2 \rho(n\rho - 2)}{2(\alpha - \beta)} + 2a\rho\right) R^2 + \left(2a - \frac{\alpha^2}{2(\alpha - \beta)}(1 - n\rho)\right) R.
 \end{aligned}$$

(i) $\rho < 0$ and the curvature operator is nonnegative at the initial time. In this case, a straightforward computation gives

$$\alpha > \beta > 0, \lambda > 0, 4\beta - 2\alpha < 0, \frac{\alpha^2}{2(\alpha - \beta)} - 2a < 0.$$

Since the nonnegativity of the curvature operator is preserved along the RB flow when $\rho < 0$, which implies $Ric \geq 0, R \geq 0$. Moreover, we have $|Ric|^2 \leq R^2$. By dropping some negative terms of (14), (14) can be reduced to

$$\begin{aligned}
 \frac{\partial H}{\partial \tau} & \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H - 2\beta(\rho R + \frac{1}{\tau})|\nabla u|^2 + \frac{\alpha^2 \rho(n\rho - 2)}{2(\alpha - \beta)} R^2 \\
 & + \frac{2aR}{\tau} + \left(c + \frac{\alpha^2}{2(\alpha - \beta)}\right) \frac{n}{\tau^2} + 2aR + \left(2c + \frac{\alpha^2}{2(\alpha - \beta)}\right) \frac{n}{\tau} + \frac{n\alpha^2}{8(\alpha - \beta)}.
 \end{aligned}$$

Since $0 \leq R < -\frac{1}{\rho\tau}$ which comes from (a) in Lemma 2.3, we have

$$\begin{aligned}
 \frac{\partial H}{\partial \tau} & \leq \Delta H - 2\nabla H \cdot \nabla u + \left(c + \frac{\alpha^2}{2(\alpha - \beta)} + \frac{\alpha^2(n\rho - 2)}{2n\rho(\alpha - \beta)} - \frac{2a}{n\rho}\right) \frac{n}{\tau^2} \\
 & - \left(\frac{2}{\tau} + 2\right)H + 2\left(c + \frac{\alpha^2}{4(\alpha - \beta)} - \frac{a}{n\rho}\right) \frac{n}{\tau} + \frac{n\alpha^2}{8(\alpha - \beta)}.
 \end{aligned}$$

Choosing $c = -\left(\frac{\alpha^2}{2(\alpha - \beta)} + \frac{\alpha^2(n\rho - 2)}{2n\rho(\alpha - \beta)} - \frac{2a}{n\rho}\right) < 0$, then

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H + \frac{n\alpha^2}{8(\alpha - \beta)}.$$

Adding $-\frac{n\alpha^2}{16(\alpha - \beta)}$ to H , then

$$\begin{aligned}
 \frac{\partial}{\partial \tau} \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) & \leq \Delta \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) - 2\nabla \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) \cdot \nabla u \\
 & - \left(\frac{2}{\tau} + 2\right) \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) - \frac{n\alpha^2}{8\tau(\alpha - \beta)} \\
 & \leq \Delta \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) - 2\nabla \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right) \cdot \nabla u \\
 & - \left(\frac{2}{\tau} + 2\right) \left(H - \frac{n\alpha^2}{16(\alpha - \beta)}\right).
 \end{aligned}$$

It is easy to see that $H - \frac{n\alpha^2}{16(\alpha-\beta)} < 0$ for τ small enough. By using the maximum principle yields

$$H - \frac{n\alpha^2}{16(\alpha-\beta)} \leq 0$$

for all τ , hence for all $(x, t) \in M^n \times [0, T]$ with $T < \frac{\delta}{2(1-\rho)}$.

(ii) $\rho = 0$ and the scalar curvature is nonnegative at the initial time.

In this case, $\alpha = 2, \beta = 1, a = 1$ and $\lambda = 2$. Hence (14) can be simplified as

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H - \frac{2R}{\tau} + (c+2)\frac{n}{\tau^2} + (2c+2)\frac{n}{\tau} + \frac{n}{2}.$$

It is well known that the nonnegativity of the scalar curvature is preserved along the Ricci flow. Choosing $c = -2$ and adding $-\frac{n}{4}$ to H yield

$$\frac{\partial}{\partial \tau} \left(H - \frac{n}{4}\right) \leq \Delta \left(H - \frac{n}{4}\right) - 2\nabla \left(H - \frac{n}{4}\right) \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)\left(H - \frac{n}{4}\right).$$

Notice that

$$H - \frac{n}{4} \leq 0$$

for τ small enough. Applying the maximum principle yields

$$H - \frac{n}{4} \leq 0$$

for all $(x, t) \in M^n \times [0, T]$.

(iii) $0 < \rho < \frac{1}{2(n-1)}$ and the initial curvature operator is nonnegative.

In this case, according to the same arguments as (i), we have

$$\alpha > \beta > 0, \lambda > 0, 4\beta - 2\alpha > 0, \frac{\alpha^2}{2(\alpha-\beta)} - 2a > 0,$$

and

$$Ric \geq 0, R \geq 0, |Ric|^2 \leq R^2.$$

Moreover,

$$(4\beta - 2\alpha)Ric(\nabla u, \nabla u) \leq (4\beta - 2\alpha)|Ric||\nabla u|^2 \leq 4\beta R|\nabla u|^2.$$

Hence (14) can be simplified as

$$\begin{aligned} \frac{\partial H}{\partial \tau} &\leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H + 4\beta\left(R - \frac{1}{2\tau}\right)|\nabla u|^2 \\ &\quad + \left(c + \frac{\alpha^2}{2(\alpha-\beta)}\right)\frac{n}{\tau^2} + \left(2c + \frac{\alpha^2}{2(\alpha-\beta)}\right)\frac{n}{\tau} + 2aR \\ &\quad + 2a \cdot \frac{R}{\tau} + \left(\frac{\alpha^2}{2(\alpha-\beta)} - 2a + 2a\rho\right)R^2 + \frac{n\alpha^2}{8(\alpha-\beta)}. \end{aligned}$$

Since $0 \leq R < \frac{1}{2\tau}$ which is based on (b) in Lemma 2.3, we obtain

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H + 2\left(c + \frac{\alpha^2}{4(\alpha-\beta)} + \frac{a}{2n}\right)\frac{n}{\tau}$$

$$+ \left(c + \frac{\alpha^2}{2(\alpha - \beta)} + \frac{\alpha^2}{8n(\alpha - \beta)} + \frac{a}{2n} + \frac{a\rho}{2n} \right) \frac{n}{\tau^2} + \frac{n\alpha^2}{8(\alpha - \beta)}.$$

Choosing $c = -\left(\frac{\alpha^2}{2(\alpha - \beta)} + \frac{\alpha^2}{8n(\alpha - \beta)} + \frac{a}{2n} + \frac{a\rho}{2n}\right) < 0$, then

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2\right)H + \frac{n\alpha^2}{8(\alpha - \beta)}.$$

The following proof is exactly the same as (i). Therefore,

$$H - \frac{n\alpha^2}{16(\alpha - \beta)} \leq 0$$

for all $(x, t) \in M^n \times [0, T]$ with $T < \frac{\delta}{2}$. □

Remark 3.1. (i) From Lemma 2.2, we can see that Theorem 1.1 also hold if u is replaced by v only by changing u into v in the conclusions.

(ii) We can derive classical Harnack inequalities by integrating the Harnack estimate in Theorem 1.1 along a space-time path. Since the method is standard, we omit it.

In the following, we consider the positive solution to the nonlinear backward heat equation (6) without any potential when the metric evolved by the RB flow on a closed manifold M^n . Assume $0 < f < 1$. Note that this property is preserved as time t evolves. In fact, let $u = -\ln f$ and $\tilde{u}(x, \tau) = u(x, T - \tau) = u(x, t)$, then

$$\frac{\partial \tilde{u}}{\partial \tau} = \Delta \tilde{u} - |\nabla \tilde{u}|^2 - \tilde{u}.$$

If the initial value $0 < f(x, 0) < 1$, i.e., $\tilde{u}(x, 0) > 0$. Using the maximum principle and the Grönwall inequality, one can show that

$$u(x, t) = \tilde{u}(x, \tau) \geq \tilde{u}_{\min}(\tau) \geq e^{-\tau} \tilde{u}_{\min}(0) > 0,$$

that is,

$$0 < f(x, t) < 1$$

for all $(x, t) \in M^n \times [0, T]$.

Proof of Theorem 1.3. In the proof of Lemma 2.1, let us take $\alpha = 0, \beta = -1, a = 0, b = -1, c = 0$ and $\gamma = 0$, then

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Comparing with the equation (12), we have

$$\frac{\partial H}{\partial \tau} = \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{\tau} + 1\right)H - 2|\nabla \nabla u|^2 - |\nabla u|^2 - 4Ric(\nabla u, \nabla u) + 2\rho R|\nabla u|^2.$$

The nonnegativity of the initial curvature operator implies $Ric \geq 0$ and $R \geq 0$. Combining with the assumption of $\rho \leq 0$, we obtain

$$\frac{\partial H}{\partial \tau} \leq \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{\tau} + 1\right)H.$$

Since

$$H = |\nabla u|^2 - \frac{u}{\tau} < 0$$

holds for τ small enough. Applying the maximum principle yields

$$|\nabla u|^2 - \frac{u}{\tau} = \frac{|\nabla f|^2}{f^2} - \frac{1}{\tau} \ln \frac{1}{f} \leq 0$$

for all $(x, t) \in M^n \times [0, T]$. \square

Remark 3.2. Following the same arguments, we show that Theorem 1.3 also holds if $1 \leq f < A$ (A is a constant) only need to set $u = \ln \frac{A}{f}$, i.e.,

$$(15) \quad \frac{|\nabla f|^2}{f^2} \leq \frac{1}{\tau} \ln \frac{A}{f}.$$

By means of the inequality (15), we can derive the following corollary which shows that any positive L^1 -solution of the equation (6) cannot blow up too fast.

Corollary 3.3. *Suppose that $g(t)_{t \in [0, T]}$ evolve by the RB flow with $\rho \leq 0$ on an n -dimensional closed manifold M^n , and the initial curvature operator is nonnegative. $f(x, t)$ (≥ 1) be a L^1 -solution to the equation (6). Let $\tau = T - t$, then there exists a constant C depending on the geometry of $g(t)_{t \in [0, T]}$ such that*

$$f(x, t) \leq \frac{C}{\tau^{\frac{n}{2}}}$$

for all $(x, \tau) \in M^n \times (0, \min\{1, T\}]$.

Proof. Let $\tilde{f}(x, \tau) = f(x, T - \tau) = f(x, t)$, then

$$\frac{\partial \tilde{f}}{\partial \tau} = \Delta \tilde{f} - \tilde{f} \ln \tilde{f}.$$

Since the solution and the flow are well defined in $M^n \times [0, T]$, there exists $(x_0, \tau_0) \in M^n \times [0, T]$, such that

$$\max_{M^n \times [0, T]} \tau^{\frac{n}{2}} \tilde{f}(x, \tau) = \tau_0^{\frac{n}{2}} \tilde{f}(x_0, \tau_0).$$

In particular,

$$\max_{M^n \times [\frac{\tau_0}{2}, \tau_0]} \tilde{f}(x, \tau) \leq \frac{\tau_0^{\frac{n}{2}}}{\tau^{\frac{n}{2}}} \tilde{f}(x_0, \tau_0) \leq 2^{\frac{n}{2}} \tilde{f}(x_0, \tau_0).$$

Let $A = 2^{\frac{n}{2}} \tilde{f}(x_0, \tau_0) > 1$. Applying (15) to $\tilde{f}(x, \tau)$ in $M^n \times [\frac{\tau_0}{2}, \tau_0]$, then

$$(16) \quad \tau_0 \frac{|\nabla \tilde{f}|^2}{\tilde{f}^2}(x, \tau_0) \leq \ln \frac{A}{\tilde{f}(x, \tau_0)}.$$

Let $h(x, \tau) = \ln \frac{A}{\tilde{f}(x, \tau)}$, (16) can be rewritten as

$$|\nabla \sqrt{h(x, \tau_0)}| \leq \frac{1}{2\sqrt{\tau_0}}.$$

Let $\gamma(s) : [0, 1] \rightarrow B_{\tau_0}(x_0, \sqrt{\tau_0})$ be a minimal geodesic between $\gamma(0) = (x_0, \tau_0)$ and $\gamma(1) = (x, \tau_0)$ for any $x \in B_{\tau_0}(x_0, \sqrt{\tau_0})$, where $B_{\tau_0}(x_0, \sqrt{\tau_0})$ denoting the ball of radius $\sqrt{\tau_0}$ measured by $g(\tau_0)$ around the point x_0 . Integrating along $\gamma(s)$, then

$$\begin{aligned} \sqrt{h(x, \tau_0)} - \sqrt{h(x_0, \tau_0)} &= \int_0^1 \frac{d}{ds} \sqrt{h(\gamma(s), \tau_0)} ds \\ &\leq \int_0^1 |\nabla \sqrt{h(\gamma(s), \tau_0)}| |\gamma'(s)| ds \\ &\leq \frac{1}{2\sqrt{\tau_0}} \cdot \sqrt{\tau_0} = \frac{1}{2}. \end{aligned}$$

Hence

$$\sup_{B_{\tau_0}(x_0, \sqrt{\tau_0})} \sqrt{h(x, \tau_0)} \leq \sqrt{h(x_0, \tau_0)} + \frac{1}{2} = \sqrt{\frac{n}{2} \ln 2} + \frac{1}{2}.$$

By the definitions of A and $h(x, \tau)$, we obtain

$$\tilde{f}(x, \tau_0) \geq C_1 \tilde{f}(x_0, \tau_0),$$

where C_1 is a constant only depending on n .

Notice that there exists a constant C_2 depending on the geometry of $(M^n, g(\tau_0))$ such that (cf. [5])

$$\text{Vol}_{g(\tau_0)}(B_{\tau_0}(x_0, \sqrt{\tau_0})) \geq C_2 \tau_0^{\frac{n}{2}}$$

as long as $0 < \tau_0 \leq 1$. Therefore,

$$C_1 C_2 \tau_0^{\frac{n}{2}} \tilde{f}(x_0, \tau_0) \leq \int_{B_{\tau_0}(x_0, \sqrt{\tau_0})} \tilde{f}(x, \tau_0) d\mu_{\tau_0}(x) \leq \int_{M^n} \tilde{f}(x, \tau_0) d\mu_{\tau_0}(x).$$

By the choice of (x_0, τ_0) and $f \in L^1$, then

$$f(x, t) = \tilde{f}(x, \tau) \leq \frac{C}{\tau^{\frac{n}{2}}}. \quad \square$$

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