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MODULAR JORDAN TYPE FOR $\mathbb{k}[x, y]/(x^m, y^n)$ FOR $m = 3, 4$

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Abstract. A sufficient condition for an Artinian complete intersection quotient $S = \mathbb{k}[x, y]/(x^m, y^n)$, where $\mathbb{k}$ is an algebraically closed field of a prime characteristic, to have the strong Lefschetz property (SLP) was proved by S. B. Glasby, C. E. Praezer, and B. Xia in [3]. In contrast, we find a necessary and sufficient condition on m, n satisfying $3 \le m \le n$ and $p > 2m-3$ for S to fail to have the SLP. Moreover we find the Jordan types for S failing to have SLP for $m \le n$ and $m = 3, 4$.

1. Introduction

Let $R = \mathbb{k}[x_1, \ldots, x_r] = \bigoplus_{i\ge 0} R_i$ be an r-variable polynomial ring over an algebraically closed field $\mathbb{k}$ of any characteristic, and let $A := R/I$, where I is a homogeneous ideal of R. The *Hilbert function of* A, $\mathrm{H}_A : \mathbb{N} \to \mathbb{N}$, is defined by

$$\mathbf{H}_A(t) := \dim_{\mathbb{k}} R_t - \dim_{\mathbb{k}} I_t$$

for $t \ge 0$. If I is a homogeneous ideal with $\sqrt{I} = (x_1, \ldots, x_r)$, and $c+1$ is the least positive integer such that $(x_1, \ldots, x_r)^{c+1} \subseteq I$, then

$$A = \mathbb{k} \oplus A_1 \oplus \cdots \oplus A_c \quad \text{where} \quad A_c \neq 0.$$

In this case, we call c *the socle degree of* A. For the Artinian graded ring A, the Hilbert function of A can be expressed as a vector

$$(h_0, h_1, \ldots, h_c) := (\mathbf{H}_A(0), \mathbf{H}_A(1), \ldots, \mathbf{H}_A(c)).$$

The Hilbert function $(h_0, h_1, \ldots, h_c)$ of A is *unimodal* if the vector $(h_0, h_1, \ldots, h_c)$ has only one local maximum, i.e.,

$$h_0 \le h_1 \le \cdots \le h_t = \cdots = h_s \ge h_{s+1} \ge \cdots \ge h_c.$$

We say that the vector $(h_0, h_1, \ldots, h_c)$ is *symmetric* if

$$h_i = h_{c-i} \quad \text{for} \quad i = 0, 1, \ldots, \lfloor \tfrac{c}{2} \rfloor.$$

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Let ℓ be a general enough linear form. We say that A has the *weak Lefschetz property* (WLP) if the homomorphism induced by multiplication by ℓ,

$$\times\ell : A_i \to A_{i+1},$$

has maximal rank for all i (i.e., it is injective or surjective for each i). We say that A has the *strong Lefschetz property* (SLP) if

$$\times\ell^d : A_i \to A_{i+d}$$

has maximal rank for all i and d (i.e., it is injective or surjective for each i and d). In this case, we call a linear form ℓ *the strong Lefschetz element of* A.

There is a way to characterize if an Artinian ring has the WLP or SLP based on Jordan type (see [5, 11]). Here the Jordan type $J_{\ell,M}$ of $\ell \in \mathfrak{m}$ acting on an A-module M is the partition, $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\lambda_1 \geq \cdots \geq \lambda_t$, giving the Jordan blocks of the multiplication map $\times\ell : M \to M$ ([9]). In particular, the *generic Jordan type* of A is the Jordan type of A for a general enough linear form ℓ. We introduce an important tool to verify if an Artinian ring has the WLP or SLP.

Lemma 1.1 ([5, Remark 3.63 and Proposition 3.64])**.** *Assume that the Artininan algebra A is standard-graded* (A *is generated by* A_1) *and that* H_A *is unimodal. Then*

(1) *The pair* (A, ℓ) *has the weak Lefschetz property if and only if the number of parts of the Jordan type* $J_{\ell,A} = \max_i\{\mathrm{H}_A(i)\}$. (*The Sperner number of* A);
(2) ℓ *is a strong Lefschetz element of A if and only if* $J_{\ell,A} = \mathrm{H}_A^\vee$, *where* $\mathrm{H}_S^\vee$ *is the conjugate of* H_S (*exchange rows and columns in the Ferrers diagram of* H_S).

Let $S := \Bbbk[x,y]/(x^m, y^n)$. When $m \leq n$, $\mathrm{H}_S = (1, 2, \dots, m-1, m, \dots, m_{n-1}, m-1, \dots, 2, 1)$. In characteristic 0, the Jordan type $J_{\ell,S} = (\lambda_1, \dots, \lambda_m)$ was shown to be the standard partition, i.e.,

$$J_{\ell,S} = (m+n-1, \dots, m+n-2i+1, \dots, n-m+1) \tag{1.1}$$

in 1934 by A. C. Aitken [1], in 1934 by W. E. Roth [16], and in 1936 by D. E. Littlewood [12], independently. When the characteristic of $\Bbbk$ is a prime p, the resulting formulas for $J_{\ell,S}$ were studied in 1954 by D. G. Higman [7], then in 1962 by J. A. Green [4], and in 1964 by B. Srinivasan [17]. In particular, B. Srinivasan proved that the Jordan type $J_{\ell,S} = (\lambda_1, \dots, \lambda_m)$ is the standard partition if the characteristic of $\Bbbk$ is $p > m+n-2$, and J. A. Green discussed the representation ring over $\mathbb{Z}_p$. The paper [17] seems to be the first paper emphasizing the characteristic p results in the present formulation related to the Clebsch-Gordan formula.

The WLP and SLP are strongly connected to many topics in algebraic geometry, commutative algebra, combinatorics, and representation theory. In 1980, R. Stanley showed in [18] using a topological method - the hard Lefschetz property - that if $\mathbb{k}$ is a field of characteristic 0 or greater than the socle degree of $A := \mathbb{k}[x_1, \dots, x_r]/(x_1^{a_1}, \dots, x_r^{a_r})$, then the Artinian complete intersection quotient A has the SLP. In 1987, J. Watanabe proved this again using the language 'representation theory' [19]. In [13], S. Lundqvist and L. Nicklasson find a necessary and sufficient condition of the SLP when the number of variables is ≥ 3. In 2013 J. Miglilore and U. Nagel surveyed recent works about Lefschetz properties [14]. Also in 2013, the book [5] by J. Watanabe et al. provided a comprehensive exploration of the Lefschetz properties from a different perspective, focusing on representation theory and combinatorial connections as well as commutative algebra methods. In 2018, A. Iarrobino, P. Marques, and C. McDaniel [9] explored a general invariant of an Artinian Gorenstein algebra A, or A-module M, which is the set of Jordan types of elements of the maximal ideal $\mathfrak{m}$ of A.

The generic Jordan type of a graded Artinian algebra A is that determined by a general enough element ℓ of A_1. For $S = \mathbb{k}[x, y]/(x^m, y^n)$ we may take $\ell = x+y$, so the Jordan type of S is the partition of mn giving the Jordan block decomposition of the multiplication by ℓ; this depends on the characteristic of $\mathbb{k}$.

When the characteristic of $\mathbb{k}$ is 0 or greater than or equal to $m + n$, the partitions are the Clebsch-Gordan formulas of invariant theory [8], which have many applications in physics and have been rediscovered or surveyed frequently ([1, 17], see also [6, Theorem 3.9] on Lefschetz properties of Artin algebras). The significance in representation theory is that each factor $\mathbb{k}[x]/(x^m)$ and $\mathbb{k}[x]/(x^n)$ is an irreducible representation of the Lie algebra $\mathfrak{sl}_2$, and that the Clebsch-Gordan formula (equation (1.1) above) of invariant theory [8] gives the decomposition of the tensor product into irreducible representations ([5, Section 3]).

The papers S. B. Glasby et al. [3] and K. I. Iima et al. [10] have obtained a very nice result in the direction of recursion formulas for the Jordan type $J_{\ell,S}$ in (m, n) for a fixed prime p. There are approaches to this problem from different directions and the S. B Glasby et al. paper [3], and briefly in Section 3.2 of A. Iarrobino et al. [9] include some survey of the previous characteristic p Clebsch-Gordan results. Moreover, S. B. Glasby et al. proved that if $m \leq n$, and $n \not\equiv 0, \pm 1, \dots, \pm(m-2) \pmod p$, then the Jordan type $J_{\ell,S} = (\lambda_1, \dots, \lambda_m)$ of mn, where $\lambda_1 \geq \cdots \geq \lambda_m$ is the standard partition of equation (1.1), whose i-th part is $\lambda_i = m+n-2i+1$ for $1 \leq i \leq m$. By Lemma 1.1, this is equivalent to S having the SLP for such m and n.

Recall that $S := \mathbb{k}[x, y]/(x^m, y^n) = \mathbb{k}[x]/(x^m) \otimes \mathbb{k}[y]/(y^n)$ for $m \leq n$, where $\mathbb{k}$ is an algebraically closed field of positive characteristic p. In this paper, we explore not only the Lefschetz property but also the Jordan type for S. We also study modular representations of finite cyclic p groups. Given two indecomposable modules $V(m-1)$ and $V(n-1)$ of a cyclic group order p^s, the Krull-Schmidt theorem implies that $V(m-1) \otimes V(n-1)$ is a sum of m indecomposable modules $V(\lambda_1 - 1) \oplus \cdots \oplus V(\lambda_m - 1)$. This is shown in [3, Lemma

9] and implies by Lemma 1.1 that S has (always) the WLP. Then there are forms $f_1, f_2, \ldots, f_m$ such that $\deg f_i = i-1$ for $0 \le i \le m-1$, and

$$f_i \mapsto f_i\ell \mapsto \cdots \mapsto f_i\ell^{\lambda_i - 1}$$

is a *string* of length λ_i. In other words, the ring S can be decomposed into irreducible $\mathfrak{sl}_2$-modules as

$$S := V(\lambda_1 - 1) \oplus \cdots \oplus V(\lambda_m - 1).$$

Suppose that either $3 \le m \le n$ and $p > 2m-3$ or $3 \le m < n$ and $p \ge 2m-3$. In this paper, we show that if $n \equiv 0, \pm1, \ldots, \pm(m-2) \pmod p$, then the Jordan type for S is not the standard partition, i.e., S fails to have the SLP for such m and n (see Theorem 2.5). This result has an important role to find the Jordan type for S with $m = 3, 4$. In Section 2, we prove a necessary and sufficient condition on m and n that S fails to have the SLP (see Corollary 2.6). In Section 3, we find other conditions that S fails to have the SLP for $m \le n$ and $m = 3, 4$. We also find the Jordan type for S for such m and n in Section 4. These results in Section 4 for $m = 3, 4$ are the same as the works in [3], but they [3] found the Jordan type for these rings using the representation theory of algebraic group. More precisely, they used new periodicity and duality result for $J_{\ell,S}$ that depend on the smallest p-power exceeding m. In addition, in [10], K. Iima and R. Iwamatsu found a recursive formula how to find the Jordan type for S. But, in this paper, we give a more direct proof in Section 4 without any recursive formula in [10] or any results in [3].

We are posting some calculations in the proofs of Theorems 4.4, 4.5, and 4.6 to Arxiv to make this paper shorter (see modular jordan type-full.pdf).

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2. A necessary and sufficient condition that $\Bbbk[x, y]/(x^m, y^n)$ fails to have the SLP

In this section, we find a necessary and sufficient condition for S to fail to have the SLP when $3 \le m \le n$ and $p > 2m-3$ or $3 \le m < n$ and $p \ge 2m-3$. In [15, Theorem 3.2], L. Nicklasson also find a necessary and sufficient condition of the SLP for S using the base p expansions of m, n.

We now recall the sufficient condition for S to have the SLP from [3].

Theorem 2.1 ([3, Theorem 2]). *Let* $S := \Bbbk[x, y]/(x^m, y^n)$ *with* $\operatorname{char} \Bbbk = p > 0$. *If* $0 < m \le n$ *and* $n \not\equiv 0, \pm1, \ldots, \pm(m-2) \pmod p$, *then S has the SLP.*

We shall show that if $p > 2m-3$ and $n \equiv 0, \pm1, \ldots, \pm(m-2) \pmod p$, then S fails to have the SLP. We first need the following two lemmas.

Lemma 2.2. *Suppose that $3 \le m \le n$ and p is a prime with $p > m-1$. If $n \equiv -k \pmod p$ with $0 \le k \le m-2$, then*

$$\binom{n+m-2}{m-1} \equiv 0 \pmod p.$$

Proof. By the assumption, $(m-1)! \not\equiv 0 \pmod p$ and $n+m-2 > m-1$. Since $n+k \equiv 0 \pmod p$, we have

$$\binom{n+m-2}{m-1} = \frac{(n+(m-2))(n+(m-3))\cdots(n+k)\cdots(n+1)n}{(m-1)!} \equiv 0 \pmod p,$$

as we wished. ☐

Lemma 2.3. *Let p be a prime. Suppose that either $3 \le m \le n$ and $2m-3 < p$ or $3 \le m < n$ and $2m-3 \le p$. If $n \equiv k \pmod p$ with $k = 1, 2, \ldots, m-2$, then the following hold.*

(a) *For any $1 \le \alpha \le k$ and $\alpha \le \beta \le \min\{k, n+\alpha-k-1\}$ with $m-k-\alpha+\beta < p$,*

$$\binom{n+m-2k-1}{m-k-\alpha+\beta} \equiv 0 \pmod p.$$

(b)

$$\binom{n+m-2k-1}{m-k-1} \not\equiv 0 \pmod p.$$

Proof. First note that, with given conditions,

$$n+m-2k-1 = (n-k)+(m-k)-1 \equiv m-k-1 \not\equiv 0 \pmod p.$$

(a) For $1 \le \alpha \le k$ and $\alpha \le \beta \le \min\{k, n+\alpha-k-1\}$, since $m-k-\alpha+\beta < p$, we get that

$$(m-k-\alpha+\beta)! \not\equiv 0 \pmod p.$$

Moreover, note that

$$n+m-2k-1 = (n-k)+(m-k-1) > n-k \equiv 0 \pmod p, \quad \text{and}$$
$$n-k+\alpha-\beta \le n-k.$$

This shows that

$$n+m-2k-1 > p > m-k-\alpha+\beta,$$

and thus

$$\binom{n+m-2k-1}{m-k-\alpha+\beta} = \frac{(n+m-2k-1)(n+m-2k-2)\cdots(n-k+\alpha-\beta)}{(m-k-\alpha+\beta)!} \equiv 0 \pmod p.$$

(b) Note that $m-k-1 < p$ and

$$n+m-2k-1 = (n-k)+(m-k-1) > m-k-1.$$

Since $1 \le k \le m-2$, for any $\gamma = 0, 1, \ldots, m-k-2$, we have

$$n+m-2k-1-\gamma = (n-k)+(m-k-1)-\gamma$$

$$\equiv (m-k-1) - \gamma \not\equiv 0 \pmod p.$$

This shows that

$$\binom{n+m-2k-1}{m-k-1} = \frac{(n+m-2k-1)(n+m-2k-2)\cdots(n-k+1)}{(m-k-1)!} \not\equiv 0 \pmod p.$$

This completes the proof. □

Remark 2.4. If $m = n = 3$, $k = 1$, and $p = 2m-3 = 3$, then the formula of Lemma 2.3(b) is not satisfied. Indeed,

$$\binom{n+m-2k-1}{m-k-1} = \binom{3}{1} \equiv 0 \pmod 3.$$

Theorem 2.5. *Let $S = \Bbbk[x,y]/(x^m, y^n)$, where $\Bbbk$ is a field of a prime characteristic p. Suppose that either $3 \le m \le n$ and $p > 2m-3$ or $3 \le m < n$ and $p \ge 2m-3$. If $n \equiv 0, \pm 1, \dots, \pm(m-2) \pmod p$, then S fails to have the SLP.*

Proof. First, note that since $n+m-2 > n \ge m$, both of x and y cannot be an SLP element for S. Thus it is enough to show that any linear form $\ell := x+y$ cannot be an SLP element of S.

(i) Suppose that $n \equiv -k \pmod p$ with $0 \le k \le m-2$. By Lemma 2.2, we have

$$(x+y)^{n+m-2} = \binom{n+m-2}{m-1} x^{m-1} y^{n-1} = 0.$$

Hence the first (largest) component of the Jordan type $J_{\ell,S}$ is $\le n+m-2$, i.e., the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\le n+m-2, \dots),$$

and thus S fails to have the SLP.

(ii) Now suppose that $n \equiv k \pmod p$ with $1 \le k \le m-2$. We shall show that the $(k+1)$-st component of $J_{\ell,S}$ cannot be $n+m-2k-1$. Let

$$P_k := b_0 x^k + b_1 x^{k-1} y + \cdots + b_{k-1} x y^{k-1} + b_k y^k$$

be a nonzero form of degree k in $\Bbbk[x,y]$. Let i be the smallest integer with $b_i \ne 0$, i.e., $P_k = b_i x^{k-i} y^i + \cdots + b_{k-1} x y^{k-1} + b_k y^k$. Since $x^m = 0$, $y^n = 0$ in S, we have

$$\begin{aligned} & P_k \cdot (x+y)^{n+m-2k-1} \\ =& \Big[b_i x^{k-i} y^i + b_{i+1} x^{k-i-1} y^{i+1} + \cdots + b_{k-1} x y^{k-1} + b_k y^k \Big] \cdot (x+y)^{n+m-2k-1} \\ =& \sum_{\alpha=1}^{k} \left(\sum_{\beta=u(\alpha)}^{v(\alpha)} b_\beta \binom{n+m-2k-1}{m-\alpha-k+\beta} \right) x^{m-\alpha} y^{n+\alpha-k-1}, \end{aligned}$$

where $u(\alpha) = \max\{i, -m + \alpha + k\}$, and $v(\alpha) = \min\{k, n + \alpha - k - 1\}$. Now consider the coefficient of $x^{m-(i+1)}y^{n+i-k}$ in $P_k \cdot (x+y)^{n+m-2k-1}$. Since

$$u(i+1) = \max\{i, -m + (i+1) + k\} = i, \quad \text{and}$$
$$v(i+1) = \min\{k, n + (i+1) - k - 1\} \geq i,$$

we get that by Lemma 2.3, the coefficient is

$$\sum_{\beta=i}^{v(i+1)} b_\beta \binom{n+m-2k-1}{m-k-(i+1)+\beta} = b_i \binom{n+m-2k-1}{m-k-1} \not\equiv 0 \pmod{p}.$$

(Here, note that $m - k - (i+1) + \beta < p$ for any $i \leq \beta \leq \min\{k, n+i-k\}$.) Hence

$$P_k \cdot (x+y)^{n+m-2k-1} \neq 0.$$

This shows that for a linear form $\ell \in R$, the Jordan type $J_{\ell,S}$ cannot be of the form

$$(\ldots, \overset{(k+1)\text{-st}}{n + m - (2k+1)}, \ldots).$$

Thus S fails to have the SLP.

This completes the proof. □

If we couple Theorem 2.5 with Theorem 2.1, we obtain the following corollary.

Corollary 2.6. *Let $S = \Bbbk[x,y]/(x^m, y^n)$ with $3 \leq m \leq n$ and $p > 2m-3$. Then a necessary and sufficient condition that S fails to have the SLP is $n \equiv 0, \pm 1, \ldots, \pm(m-2) \pmod{p}$.*

3. Other conditions that $\Bbbk[x,y]/(x^m, y^n)$ fails to have the SLP

In Section 2, we determined when $S = \Bbbk[x,y]/(x^m, y^n)$ fails to have SLP for $3 \leq m \leq n$ and $p > 2m-3$. In this section we consider the remaining cases when $m = 3$ or $m = 4$. Assume $m = 3, 4$ and $m \leq n$. Then we show that $S = \Bbbk[x,y]/(x^m, y^n)$ fails the SLP as summarized in the follow table:

Theorem	m	p	S fails the SLP
Theorem 3.2	3	2	$n \equiv 0, \pm 1 \pmod{4}$
Proposition 3.3	3	3	always
Theorem 3.4	3	$p \geq 3$	$n \equiv 0, \pm 1 \pmod{p}$
Theorem 3.5	4	2	always
Theorem 3.6	4	3	$n \not\equiv \pm 4 \pmod{9}$
Lemma 3.7	4	5	$n \geq 4$
Theorem 3.8	4	$p \geq 7$	$n \equiv 0, \pm 1, \pm 2 \pmod{p}$

Remark 3.1. Recall $S := \Bbbk[x,y]/(x^m, y^n)$ with $m \leq n$. As we mentioned in the introduction, for a linear form $\ell = x + y$, the Jordan type $J_{\ell,S}$ is of the form $(\lambda_1, \dots, \lambda_m)$ where $\lambda_1 + \cdots + \lambda_m = mn$. In this case there are forms $f_1, f_2, \dots, f_m$ such that $\deg f_i = i - 1$ for $0 \le i \le m-1$, and

$$f_i \mapsto f_i \ell \mapsto \cdots \mapsto f_i \ell^{\lambda_i - 1}$$

is a string of length λ_i. In other words, the ring S has the $\mathfrak{sl}_2$-module decomposition as follows.

$$S = \Bbbk[x,y]/(x^m, y^n) = \bigoplus_{i=1}^{m} V(\lambda_i - 1),$$

where $V(\lambda_i - 1)$ is a λ_i-dimensional irreducible $\mathfrak{sl}_2$-module for each i.

Recall that the Hilbert function of S is

$$\mathrm{H}_S(i) = \min\{i+1, m+n-1-i\} \quad \text{for} \quad i \ge 0.$$

In order for S to have the SLP we need that for each i satisfying $0 \le i \le m+n-2$ the following sets are linearly independent

$$\begin{cases} \{f_1\ell^i, f_2\ell^{i-1}, \dots, f_i\ell, f_{i+1}\} \\ \text{for } 0 \le i \le m-1, \\ \{f_1\ell^i, f_2\ell^{i-1}, \dots, f_{m-1}\ell^{i-(m-2)}, f_m\ell^{i-(m-1)}\} \\ \text{for } m \le i \le n-1, \\ \{f_1\ell^i, f_2\ell^{i-1}, \dots, f_{m+n-2-i}\ell^{2i+3-m-n}, f_{m+n-1-i}\ell^{2i+2-m-n}\} \\ \text{for } n \le i \le m+n-2. \end{cases} \tag{3.1}$$

However, if S fails to have the SLP, we have to find the different linearly independent sets for each degree-i based on Jordan type $J_{\ell,S} = (\lambda_1, \dots, \lambda_m)$. Fortunately, it is not hard to prove that those sets are linearly independent for $0 \le i \le m+n-2$. We shall omit the proof for the linear independence of the sets in general except for a few of cases (e.g., the proof of Theorem 3.6) in the rest of this paper.

3.1. char $\Bbbk \ge 2$ and $m = 3$

Theorem 3.2 is known by [2], and we give a different proof based on the Jordan type argument. We also investigate Jordan type when the ring $S = \Bbbk[x,y]/(x^3, y^n)$ fails to have the SLP for $n \ge 3$, i.e., it has only the WLP. Recall that if S has the SLP for a Lefschetz element ℓ, then the Jordan type $J_{\ell,S}$ for S is $(n+2, n, n-2)$ (see Lemma 1.1).

Theorem 3.2 (char $\Bbbk = 2$)**.** *Let $S := \Bbbk[x,y]/(x^3, y^n)$ with* char $\Bbbk = 2$ *and $n \ge 3$. Then S has the SLP if and only if $n = 2k$, where k is an odd positive integer with $k \ge 3$. In other words, S fails to have the SLP for $n \equiv 0, \pm 1$* (mod 4)*.*

Proof. By a computer calculation, one can show that for $3 \le n \le 5$, S does not have the SLP.

Now consider the case $(3,n)$ with $n \geq 6$. Then the socle degree of $R/(x^3, y^n)$ is $n+1$. Note that we have only three kind of linear forms, namely,

$$x, y, x+y.$$

But the strings from x and y are

$$\begin{aligned}&1 \mapsto x \mapsto x^2, \quad \text{and}\\ &1 \mapsto y \mapsto y^2 \mapsto \cdots \mapsto y^{n-1}.\end{aligned}$$

These two forms do not give a string of length $(n+2)$. Furthermore, the linear form $\ell = x+y$ satisfies

$$(x+y)^{n+1} = \binom{n+1}{2} x^2 y^{n-1}.$$

(i) If $4 \mid n$ or $4 \mid (n+1)$, then $x+y$ cannot give a string of length $(n+2)$. Thus $R/(x^3, y^n)$ does not have the SLP.

(ii) We now assume that $4 \nmid n$ and $4 \nmid (n+1)$.

- Let n be an odd. Since $4 \nmid (n+1)$, we get that $n = 4k+1$ for some $k \geq 2$. So $4 \mid (n-1)$.

  $$x(x+y)^n = x \cdot \binom{n}{1} x y^{n-1} = n x^2 y^{n-1} \neq 0.$$
  $$y(x+y)^{n-1} = y \cdot \binom{n-1}{2} x^2 y^{n-3} = \frac{(n-1)(n-2)}{2} x^2 y^{n-2} = 0.$$

  So the Jordan type $J_{\ell,S}$ is not of the form $(-, n, -)$ with a linear form $\ell = x+y$, i.e., $R/(x^3, y^n)$ does not have the SLP.
- Let $n = 2\alpha$ with α is an odd, so $n = 4k+2$ for some $k \geq 1$. Hence $4 \mid (n-2)$, and so the above two forms have to be 0. But,

  $$x(x+y)^{n-1} = x \cdot \binom{n-1}{1} x y^{n-2} = (n-1) x^2 y^{n-2} \neq 0,$$
  $$y^2(x+y)^{n-3} = y^{n-1} + (n-3) x y^{n-2} + \frac{(n-3)(n-4)}{2} x^2 y^{n-3} \neq 0.$$

  In degree $(n+1)$, a single form $x^2 y^{n-1}$ is obviously linearly independent. Now consider two forms in degree n. I.e.,

  $$\begin{aligned}(x+y)^n &= x^2 y^{n-2},\\ x(x+y)^{n-1} &= x^2 y^{n-2} + y^{n-1},\end{aligned}$$

  which are linearly independent. We now consider three forms in degree $(n-1)$. I.e.,

  $$\begin{aligned}(x+y)^{n-1} &= x^2 y^{n-3},\\ x(x+y)^{n-2} &= x^2 y^{n-3} + y^{n-1},\\ y^2(x+y)^{n-3} &= x^2 y^{n-3} - x y^{n-2} + y^{n-1},\end{aligned}$$

which are linearly independent as well. So the Jordan type $J_{\ell,S}$ is of the form $(n+2, n, n-2)$ with a linear form $\ell = x + y$. Therefore, $R/(x^3, y^n)$ has the SLP.

This completes the proof. □

Proposition 3.3 (char $\Bbbk = 3$). *Let* $S := \Bbbk[x,y]/(x^3,y^n)$ *with* char $\Bbbk = 3$ *and* $n \geq 3$. *Then S fails to have the SLP.*

Proof. Note that

$$(x+y)^{n+1} = \binom{n+1}{2} x^2 y^{n-1}.$$

So if $n \equiv 0, -1 \pmod 3$, then the above equation is 0, i.e., a linear form $\ell = x + y$ does not give a string of length $(n+2)$.

If $n \equiv 1 \pmod 3$, then

$$x(x+y)^n = nx^2y^{n-1} = x^2y^{n-1} \neq 0,$$

and

$$y(x+y)^n = 0.$$

I.e., the Jordan type $J_{\ell,S}$ with $\ell = x + y$ cannot be of the form

$$J_{\ell,S} = (\lambda_1, n, \lambda_3),$$

and so S fails to have the SLP, as we wished. □

Theorem 3.4 (char $\Bbbk \geq 3$). *Let* $S := \Bbbk[x,y]/(x^3,y^n)$ *with* char $\Bbbk = p \geq 3$ *and* $n \geq 3$. *If* $n \equiv 0, \pm 1 \pmod p$, *then S fails to have the SLP. Otherwise, S has the SLP. In particular, if* char $\Bbbk = 3$, *then S fails to have the SLP for any* $n \geq 3$.

Proof. It is immediate that the two linear forms x and y do not give a string of length of $n+2$. So it is enough to consider a linear form $\ell = x + y$.

By Proposition 3.3, this theorem holds for char $\Bbbk = 3$. So we now suppose that char $\Bbbk \geq 5$.

(1) Let $n = p\alpha$, $p\alpha - 1$ and $\alpha \geq 1$. Then $p \mid \binom{n+1}{2}$ and $p \mid \binom{n+1}{3}$. So

$$(x+y)^{n+1} = 0,$$

i.e., for any linear form ℓ in R the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (< n+2, \dots).$$

This implies that S fails to have the SLP.

(2) Let $n = p\alpha + 1$. Then $p \mid \binom{n}{2}$, $p \mid \binom{n}{3}$, $p \mid (n-1)$, $p \mid \binom{n-1}{2}$, and $p \mid \binom{n-1}{3}$. Hence

$$x(x+y)^n = x^2y^{n-1} \neq 0,$$
$$y(x+y)^{n-1} = 0.$$

This shows that for any linear form $L = x + by$ with $b \in \Bbbk$,

$$L(x+y)^n \neq 0,$$

i.e., for a linear form $\ell = x + y \in R$, the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3)$$

with $\lambda_2 > n$. Thus S fails to have the SLP.

(3) Let $n \not\equiv 0, \pm 1 \pmod p$. By Theorem 2.1, S has the SLP. Hence for a linear form $\ell = x + y$, the Jordan type $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+2, n, n-2).}$$

This completes the proof. □

3.2. char $\Bbbk \geq 2$ and $m = 4$

Note that if $S = \Bbbk[x, y]/(x^4, y^n)$ has the SLP for a Lefschetz element ℓ, then the Jordan type $J_{\ell,S}$ for S is $(n+3, n+1, n-1, n-3)$. The following theorem is known by [2, Corollary 4.8], and we introduce a new proof based on Jordan type argument for a linear form $\ell = x + y$.

Theorem 3.5 (char $\Bbbk = 2$)**.** *Let $S := \Bbbk[x, y]/(x^4, y^n)$ and char $\Bbbk = 2$ and $n \geq 4$. Then S fails to have the SLP.*

Proof. Note that we have only three kind of linear forms, namely,

$$x, y, x + y.$$

But for a linear form x, y, the Jordan types are

$$J_x = (4, 4, \ldots, 4) := [4^n],$$
$$J_y = (n, n, n, n) := [n^4].$$

So two linear forms x and y are not strong Lefschetz elements. Now consider a linear form $\ell = x + y$, and note that

$$(x + y)^{n+3} = \binom{n+3}{3} x^3 y^{n-1}.$$

(a) If $n \equiv \pm 1, 2 \pmod 4$, then

$$(x + y)^{n+3} = \binom{n+3}{3} x^3 y^{n-1} = 0,$$

and so the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \ldots)$$

with $\lambda_1 \leq n + 2$, and thus S fails to have the SLP.

(b) We now assume that $n \equiv 0 \pmod 4$. By a simple calculation, the Jordan type is

$$J_{\ell,S} = (n, n, n, n) = [n^4].$$

This implies that S fails to have the SLP.

This completes the proof. □

Theorem 3.6 (char $\mathbb{k} = 3$)**.** *Let* $S := \mathbb{k}[x,y]/(x^4, y^n)$ *with* char $\mathbb{k} = 3$ *and* $n \geq 4$*. If* $n \not\equiv \pm 4 \pmod 9$*, then* S *fails to have the SLP. Otherwise* S *has the SLP.*

Proof. (1) Assume $n = 9\alpha, 9\alpha - 1, 9\alpha - 2$, with $\alpha \geq 1$. Note that $3 \mid \binom{n+2}{3}$. Then

$$(x+y)^{n+2} = \binom{n+2}{3} x^3 \cdot y^{n-1} = 0,$$

which implies that any linear form $x + y$ cannot give a string of length $(n+3)$. Thus the ring S fails to have the SLP.

(2) Let $n = 9\alpha + 1$ with $\alpha \geq 1$. Note that $3 \mid \binom{n}{2}$ and $3 \mid \binom{n}{3}$. So

$$\begin{aligned} y(x+y)^n &= \binom{n}{2} x^2 y^{n-1} + \binom{n}{3} x^3 y^{n-2} = 0, \quad \text{and} \\ x(x+y)^n &= \binom{n}{2} x^3 y^{n-2} = 0. \end{aligned}$$

Thus for any $a \in \mathbb{k} - \{0\}$,

$$(ax+y)(x+y)^n = 0,$$

as well. This implies that for a linear form $\ell = x + y$ the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 < n + 1$. Hence the ring S fails to have the SLP.

(3) Let $n = 9\alpha \pm 3$ with $\alpha \geq 1$. Note that $3 \mid \binom{n}{2}$ and $3 \nmid \binom{n+1}{3}$. So

$$y(x+y)^{n+1} = \binom{n+1}{3} x^3 y^{n-1} \neq 0, \quad \text{and}$$

$$x(x+y)^n = \binom{n}{2} x^3 y^{n-2} = 0.$$

Thus,

$$(x+y)(x+y)^{n+1} \neq 0.$$

This implies that for any linear form ℓ the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 > n + 1$. Hence the ring S fails to have the SLP.

(4) Let $n = 9\alpha + 2$ with $\alpha \geq 1$. Note that $3 \nmid (n-1) = (9\alpha + 1)$, $3 \mid \binom{n-2}{2}$, and $3 \mid \binom{n-2}{3}$. For every $a \in \mathbb{k} - \{0\}$,

$$x^2(x+y)^{n-1} = x^2 y^{n-1} + (n-1)x^3 y^{n-2} \neq 0,$$

$$xy(x+y)^{n-1} = (n-1)x^2 y^{n-1} \neq 0, \quad \text{and}$$

$$y^2(x+y)^{n-2} = (n-2)xy^{n-1} + \binom{n-2}{2} x^2 y^{n-2} + \binom{n-2}{3} x^3 y^{n-3} = 0.$$

Since one can easily show that the above two nonzero forms are linearly independent, we see that for any $(\gamma, \delta) \neq (0,0)$,

$$(\gamma x^2 + \delta xy)(x+y)^{n-1} \neq 0,$$

which implies that for any linear form ℓ the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_3 > n-1$. Thus the ring S fails to have the SLP.

(5) Let $n = 9\alpha + 4$ with $\alpha \geq 0$. Note that $3 \nmid \binom{n+2}{3}$ and $3 \mid \binom{n-1}{2}$. Let $\ell = x+y$. We shall find four forms L, Q, and C of degrees $1, 2$, and 3 which give strings of length $n+3, n+1, n-1$, and $n-3$, respectively.

First, let $\ell = x+y$. Then

$$(x+y)^{n+2} = \binom{n+2}{3} x^3 y^{n-1} = 2x^3y^{n-1} \neq 0.$$

Since

$$x(x+y)^n = nx^2y^{n-1} + \binom{n}{2} x^3 y^{n-2} = x^2y^{n-1}, \quad \text{and}$$

$$y(x+y)^n = \binom{n}{2} x^2 y^{n-1} + \binom{n}{3} x^3 y^{n-2} = x^3 y^{n-2},$$

we can take $L = x - y \nmid x+y$. Then

$$(x-y)(x+y)^n = x^2y^{n-1} - x^3y^{n-2} \neq 0, \quad \text{and}$$

$$(x-y)(x+y)^{n+1} = \binom{n+1}{2} x^3 y^{n-1} = 0.$$

Now let $Q = \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2$ for some $\alpha_i \in \Bbbk$, and assume that

$$Q \cdot (x+y)^{n-2} \neq 0, \quad \text{and}$$
$$Q \cdot (x+y)^{n-1} = 0.$$

By a simple calculation, one can find that $Q = xy \nmid x+y$. Indeed,

$$\begin{aligned} xy(x+y)^{n-2} &= xy^{n-1} + (n-2)xy^{n-2} + \binom{n-2}{2} x^3 y^{n-3} \\ &= xy^{n-1} - x^2y^{n-2} + x^3y^{n-3} \neq 0, \quad \text{and} \end{aligned}$$

$$xy(x+y)^{n-1} = (xy^{n-1} - x^2y^{n-2} + x^3y^{n-3})(x+y) = 0.$$

We now find a cubic form $C = \beta_1 x^3 + \beta_2 x^2 y + \beta_3 xy^2 + \beta_4 y^3$ with $\beta_i \in \Bbbk$ such that

$$C \cdot (x+y)^{n-4} \neq 0, \quad \text{and}$$
$$C \cdot (x+y)^{n-3} = 0.$$

By a simple calculation, we find $C = x^3 - xy^2 + xy^2 - y^3$. In fact, since $3 \mid \binom{n-4}{2}$ and $3 \mid \binom{n-4}{3}$, we have

$$x^3(x+y)^{n-4} = x^3y^{n-4},$$

$$x^2y(x+y)^{n-4} = x^2y^{n-3},$$

$$xy^2(x+y)^{n-4} = xy^{n-2}, \quad \text{and}$$

$$y^3(x+y)^{n-4} = y^{n-1}.$$

In other words,

$$(x^3 - xy^2 + xy^2 - y^3)(x+y)^{n-4} = x^3y^{n-4} - x^2y^{n-3} + xy^{n-2} - y^{n-1} \neq 0, \text{ and}$$
$$(x^3 - xy^2 + xy^2 - y^3)(x+y)^{n-3} = (x^3y^{n-4} - x^2y^{n-3} + xy^{n-2} - y^{n-1})(x+y) = 0.$$

We now prove that the four forms

$$(x+y)^{n-1}, L \cdot (x+y)^{n-2}, Q \cdot (x+y)^{n-3}, C \cdot (x+y)^{n-4}$$

are linearly independent. Assume that for some $\alpha_i \in \Bbbk$

$$\alpha_1(x+y)^{n-1} + \alpha_2 L \cdot (x+y)^{n-2} + \alpha_3 Q \cdot (x+y)^{n-3} + \alpha_4 C \cdot (x+y)^{n-4} = 0.$$

After we multiply by $(x+y)^3$ to the above equation, we obtain that

$$\alpha_1(x+y)^{n+2} = 0, \quad \text{i.e.,} \quad \alpha_1 = 0.$$

By a similar argument, we can easily show that

$$\alpha_2 = \alpha_3 = \alpha_4 = 0$$

as well. This shows that the above four forms are linearly independent. By an analogous argument as above, one can easily show that the following three sets

$$\{(x+y)^n, L \cdot (x+y)^{n-1}, Q \cdot (x+y)^{n-2}\},$$
$$\{(x+y)^{n+1}, L \cdot (x+y)^n\}, \quad \text{and}$$
$$\{(x+y)^{n+2}\}$$

are linearly independent, respectively. Thus the Jordan type $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+3, n+1, n-1, n-3)}$$

and hence the ring S has the SLP.

(6) Let $n = 9\alpha + 5$ with $\alpha \geq 0$. Note that $3 \nmid \binom{n+2}{3}$ and $3 \mid \binom{n-1}{2}$, and $3 \mid \binom{n+1}{2}$. Let $\ell = x + y$. By an analogous argument as in Case (5), one can find that

$$L = x, \quad Q = x^2 - xy - y^2, \quad C = x^3 - xy^2 - y^3.$$

Indeed,

$$(x+y)^{n+2} = \binom{n+2}{3} x^3y^{n-1} = 2x^3y^{n-1} \neq 0,$$

$$x(x+y)^n = nx^2y^{n-1} + \binom{n}{2}x^3y^{n-2} = 2y^{n-1} + x^3y^{n-2} \neq 0, \quad \text{and}$$

$$x(x+y)^{n+1} = \binom{n+1}{2}x^3y^{n-1} = 0.$$

Moreover, note that

$$\begin{aligned} x^2(x+y)^{n-2} &= x^2y^{n-2}, \\ xy(x+y)^{n-2} &= xy^{n-1}, \quad \text{and} \\ y^2(x+y)^{n-2} &= x^3y^{n-3}, \end{aligned}$$

which implies that

$$\begin{aligned} (x^2-xy-y^2)(x+y)^{n-2} &= -xy^{n-1} + x^2y^{n-2} - x^3y^{n-3} \neq 0, \quad \text{and} \\ (x^2-xy-y^2)(x+y)^{n-1} &= (-xy^{n-1} + x^2y^{n-2} - x^3y^{n-3})(x+y) = 0. \end{aligned}$$

Since $3 \mid \binom{n-4}{2}$ and $3 \mid \binom{n-4}{3}$, we get that

$$\begin{aligned} x^3(x+y)^{n-4} &= x^3y^{n-4}, \\ xy^2(x+y)^{n-4} &= xy^{n-2} + x^2y^{n-3}, \quad \text{and} \\ y^3(x+y)^{n-4} &= y^{n-1} + xy^{n-2}, \end{aligned}$$

i.e.,

$$\begin{aligned} (x^3-xy^2-y^3)(x+y)^{n-4} &= -y^{n-1} + xy^{n-2} - x^2y^{n-3} + x^3y^{n-4} \neq 0, \quad \text{and} \\ (x^3-xy^2-y^3)(x+y)^{n-3} &= (-y^{n-1} + xy^{n-2} - x^2y^{n-3} + x^3y^{n-4})(x+y) = 0. \end{aligned}$$

By a similar argument as in Case (5), one can show that the following four sets

$$\begin{aligned} &\{(x+y)^{n-1}, L\cdot(x+y)^{n-2}, Q\cdot(x+y)^{n-3}, C\cdot(x+y)^{n-4}\}, \\ &\{(x+y)^{n}, L\cdot(x+y)^{n-1}, Q\cdot(x+y)^{n-2}\}, \quad \text{and} \\ &\{(x+y)^{n+1}, L\cdot(x+y)^{n}\}, \\ &\{(x+y)^{n+2}\} \end{aligned}$$

are linearly independent, respectively. Thus the Jordan type $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+3, n+1, n-1, n-3)}$$

as we wished, and hence the ring S has the SLP.

This completes the proof. □

We now move on to $\operatorname{char} \Bbbk \geq 5$. Let $S := \Bbbk[x,y]/(x^4, y^n)$ with $\operatorname{char} \Bbbk = 5$ and $n \geq 4$. Then

$$\mathbf{H}_S^\vee = (n+3, n+1, n-1, n-3).$$

Note that

$$x(x+y)^{n+2} = y(x+y)^{n+2} = 0.$$

Hence two linear forms x and y cannot give a string of length $(n+3)$. So we shall assume that a linear form is $\ell = x+y$ for the rest of this section.

Lemma 3.7 ($\operatorname{char} \Bbbk = 5$). *Let $S := \Bbbk[x,y]/(x^4, y^n)$ with $\operatorname{char} \Bbbk = 5$ and $n \geq 4$. Then S fails to have the SLP for every $n \geq 4$.*

Proof. If $n = 4$, then

$$(x+y)^6 = 0,$$

i.e., the Jordan type $J_{\ell,S}$ cannot be of the form

$$J_{\ell,S} = (7,5,3,1).$$

Furthermore, since $p = 5 \geq 2 \cdot 4 - 3 = 2 \cdot m - 3$, by Theorem 2.5 for every $n \equiv 0, \pm 1, \pm 2 \pmod 5$, i.e., for every $n \geq 5$, S fails to have the SLP. This completes the proof. □

We now classify the Jordan type for an Artinian ring $S := \Bbbk[x,y]/(x^4, y^4)$ for any characteristic $p > 0$. Recall that S has the SLP for $p = 3$ and $(m,n) = (4,4)$ (see Theorem 3.6), but S fails to have the SLP for $p = 5$ and $(m,n) = (4,4)$ (see Lemma 3.7). So we assume that $\operatorname{char} \Bbbk = p \geq 7$ for the following theorem.

Recall that by Theorem 2.5 and Lemma 3.7 the ring $S := \Bbbk[x,y]/(x^4, y^n)$ with $\operatorname{char} \Bbbk \geq 5$ fails to have the SLP for any $n \geq 4$ with $n \equiv 0, \pm 1, \pm 2 \pmod p$. By Theorems 2.1 and 2.5, the following theorem is immediate, and thus we omit the proof.

Theorem 3.8 ($\operatorname{char} \Bbbk = p \geq 7$). *Let $S := \Bbbk[x,y]/(x^4, y^n)$ with $\operatorname{char} \Bbbk = p \geq 7$ and $n \geq 4$. Then S has the SLP for $n \equiv \pm 3, \ldots, \pm \frac{p-1}{2} \pmod p$. Otherwise, S fails to have the SLP.*

4. The Jordan type for rings $\Bbbk[x,y]/(x^m, y^n)$ failing to have the SLP when m is 3 or 4

In this section, we determine the Jordan type for an Artinian complete intersection quotient $S := \Bbbk[x,y]/(x^m, y^n)$ for $m = 3, 4$ with $\operatorname{char} \Bbbk = p > 0$. In order to shorten the paper, we are posting full calculations for proofs of some Theorems of this section on the arXiv version of the paper (see modular jordan type-full.pdf).

4.1. char $\Bbbk \geq 2$ and $m = 3$

Theorem 4.1 ($\operatorname{char} \Bbbk = 2$). *Let $S := \Bbbk[x,y]/(x^3, y^n)$ with $\operatorname{char} \Bbbk = 2$ and $n \equiv 0, \pm 1 \pmod 4$. Then for a linear form $\ell = x+y$, the Jordan type $J_{\ell,S}$ is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod 4$	(n,n,n)
$n \equiv -1 \pmod 4$	$(n+1, n+1, n-2)$
$n \equiv 1 \pmod 4$	$(n+2, n-1, n-1)$

Proof. Recall that S fails to have the SLP for $n \equiv 0, \pm 1 \pmod 4$ and S has the SLP for $n \equiv 2 \pmod 4$ (see Theorem 3.2). Since there is no quadratic form Q such that the product

$$Q \cdot (x+y)^{n-4} \neq 0, \quad \text{and}$$
$$Q \cdot (x+y)^{n-3} = 0,$$

$J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, \lambda_2, \lambda_3)$$

with $n+2 \geq \lambda_2 \geq \lambda_3 \geq n-2$.

(a) Assume $n \equiv 0 \pmod 4$. Let $4 \mid n$ with $n \geq 4$. Note that

$$(x+y)^{n-1} = (n-1)xy^{n-2} + y^{n-1} \neq 0,$$
$$(x+y)^n = 0.$$

In other words,

$$\boxed{J_{\ell,S} = (n,n,n).}$$

(b) Let $n \equiv 1 \pmod 4$. Let $\ell = x+y$ with $n \geq 4$. But S fails to have the SLP, i.e., $J_{\ell,S}$ is not of the form

$$J_{\ell,S} = (n+2, n, n-2).$$

Furthermore, it is easy to prove that each of the following three sets

$$\{(x+y)^{n-1}, y(x+y)^{n-2}, y^2(x+y)^{n-3}\},$$
$$\{(x+y)^n, y^2(x+y)^{n-2}\},$$
$$\{(x+y)^{n+1}\},$$

is linearly independent, respectively. In other words,

$$\boxed{J_{\ell,S} = (n+2, n-1, n-1).}$$

(c) Let $n \equiv -1 \pmod 4$. Since there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^n = 0,$$

and

$$(x+y)^n = x^2y^{n-2} + xy^{n-1} \neq 0,$$
$$(x+y)^{n+1} = 0,$$

$J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, \geq n+1, \geq n-2).$$

So $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+1, n+1, n-2).}$$

This completes the proof. □

Theorem 4.2 (char $\Bbbk = 3$)**.** *Let* $S := \Bbbk[x,y]/(x^3, y^n)$ *with* char $\Bbbk = 3$ *and* $n \geq 3$*. Then for a linear form* $\ell = x + y$*, the Jordan type* $J_{\ell,S}$ *is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod 3$	(n, n, n)
$n \equiv -1 \pmod 3$	$(n+1, n+1, n-2)$
$n \equiv 1 \pmod 3$	$(n+2, n-1, n-1)$

Proof. Recall that S fails to have the SLP (see Proposition 3.3). Note that there is no quadratic form Q such that

$$Q \cdot (x+y)^{n-3} = 0.$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3)$$

with $\lambda_3 \geq n-2$.

(a) Assume $n \equiv 0 \pmod 3$. Note that

$$\begin{aligned}(x+y)^{n-1} &= x^2y^{n-3} + xy^{n-2} + y^{n-1} \neq 0,\\ (x+y)^n &= 0.\end{aligned}$$

In other words,

$$\boxed{J_{\ell,S} = (n, n, n).}$$

(b) Let $n \equiv 1 \pmod 3$. Note that

$$\begin{aligned}(x+y)^{n+1} &= x^2y^{n-1} \neq 0,\\ (x+y)^{n+2} &= 0,\end{aligned}$$

$J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, \lambda_2, \lambda_3)$$

with $\lambda_3 \geq n-2$. Since S does not have the SLP, $J_{\ell,S}$ cannot be of the form

$$J_{\ell,S} = (n+2, n, n-2).$$

Hence $J_{\ell,S}$ is of the form

$$\boxed{J_{\ell,S} = (n+2, n-1, n-1).}$$

(c) Let $n \equiv -1 \pmod 3$. Note that

$$\begin{aligned}(x+y)^n &= x^2y^{n-2} - xy^{n-1} \neq 0,\\ (x+y)^{n+1} &= 0.\end{aligned}$$

Hence $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, \lambda_2, \lambda_3)$$

with $\lambda_3 \geq n-2$. Since there is no linear form $L \neq x+y$ such that

$$\begin{aligned}L \cdot (x+y)^{n-1} &\neq 0,\\ L \cdot (x+y)^n &= 0,\end{aligned}$$

$J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, > n, n-1).$$

So we get that

$$\boxed{J_{\ell,S} = (n+1, n+1, n-2).}$$

This completes the proof. $\square$

Theorem 4.3 (char $\mathbb{k} = p \geq 5$)**.** *Let* $S := \mathbb{k}[x,y]/(x^3, y^n)$ *with* char $\mathbb{k} = p \geq 5$. *For a linear form* $\ell = x + y$ *and for* $n \equiv 0, \pm 1 \pmod p$*, the Jordan type* $J_{\ell,S}$ *is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod p$	(n, n, n)
$n \equiv -1 \pmod p$	$(n+1, n+1, n-2)$
$n \equiv 1 \pmod p$	$(n+2, n-1, n-1)$

Proof. Recall that by Theorem 3.4, S fails to have the SLP for $n \equiv 0, \pm 1 \pmod p$.

(a) Assume $n \equiv 0 \pmod p$. Note that

$$(x+y)^{n-1} = x^2y^{n-3} - xy^{n-2} + y^{n-1} \neq 0,$$
$$(x+y)^n = 0.$$

In other words,

$$\boxed{J_{\ell,S} = (n, n, n).}$$

(b) Let $n \equiv 1 \pmod p$. Note that

$$(x+y)^{n+1} = x^2y^{n-1} \neq 0,$$
$$(x+y)^{n+2} = 0.$$

Hence $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, \lambda_2, \lambda_3)$$

with $\lambda_3 \geq n-2$. Since S fails to have the SLP, $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, n-1, n-1).$$

(c) Let $n \equiv -1 \pmod p$. Note that

$$(x+y)^n = x^2y^{n-2} - xy^{n-1} \neq 0,$$
$$(x+y)^{n+1} = 0.$$

Hence $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, \lambda_2, \lambda_3)$$

with $\lambda_3 \geq n-2$. Note that there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^n = 0,$$

and for $Q = 3x^2 + 3xy + y^2$,

$$y(x+y)^n = x^2y^{n-1} \neq 0,$$

$$\begin{aligned} y(x+y)^{n+1} &= 0, \\ Q \cdot (x+y)^{n-3} &= x^2y^{n-3} - xy^{n-2} + y^{n-1} \neq 0, \quad \text{and} \\ Q \cdot (x+y)^{n-2} &= 0. \end{aligned}$$

So

$$\boxed{J_{\ell,S} = (n+1, n+1, n-2).}$$

This completes the proof. □

4.2. char $\Bbbk \geq 2$ and $m = 4$

Theorem 4.4 (char $\Bbbk = 2$)**.** *Let $S = \Bbbk[x,y]/(x^4, y^n)$ with char $\Bbbk = 2$ and $n \geq 4$. For a linear form $\ell = x+y$, the Jordan type $J_{\ell,S}$ is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod 4$	(n,n,n,n)
$n \equiv -1 \pmod 4$	$(n+1, n+1, n+1, n-3)$
$n \equiv 2 \pmod 4$	$(n+2, n+2, n-2, n-2)$
$n \equiv 1 \pmod 4$	$(n+3, n-1, n-1, n-1)$

Proof. Recall that S fails to have the SLP for $n \geq 4$ (see Theorem 3.5). Note that there is no cubic form C such that

$$C \cdot (x+y)^{n-4} = 0.$$

Hence the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$.

(a) Let $n \equiv 0 \pmod 4$. Then

$$(x+y)^n = 0,$$

and thus the Jordan type $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n,n,n,n).}$$

(b) Let $n \equiv 1$. For any linear form L,

$$L \cdot (x+y)^{n+2} = 0.$$

Moreover, if for a linear form L

$$L \cdot (x+y)^{n+1} = 0,$$

then $L = y$, and thus

$$L \cdot (x+y)^n = L(x+y)^{n-1} = 0,$$

as well. This shows that $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+3, n-1, n-1, n-1).}$$

(c) Let $n \equiv -1$. Then

$$(x+y)^n = x^3y^{n-3} + x^2y^{n-2} + xy^{n-1} \neq 0,$$
$$(x+y)^{n+1} = 0.$$

Since there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^n = 0.$$

So the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, n+1, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Moreover, if $Q \cdot (x+y)^n = 0$ for a quadratic form Q, then $x+y \mid Q$. we get that $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+1, n+1, n+1, n-3).}$$

(d) Let $n \equiv 2$. Then

$$(x+y)^{n+1} = x^3y^{n-2} + x^2y^{n-1} \neq 0,$$
$$(x+y)^{n+2} = 0.$$

Since there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^{n+1} = 0.$$

So the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, n+2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Moreover, since there is no a cubic form C such that $x+y \nmid C$ and

$$C \cdot (x+y)^{n-3} = 0,$$

we get that $J_{\ell,S}$ is

$$\boxed{J_{\ell,S} = (n+2, n+2, n-2, n-2).}$$

This completes the proof. □

Theorem 4.5 (char $\Bbbk = 3$)**.** *Let* $S = \Bbbk[x,y]/(x^4, y^n)$ *with* char $\Bbbk = 3$ *and* $n \geq 4$. *For a linear form* $\ell = x+y$ *and for* $n \not\equiv \pm 4 \pmod 9$, *the Jordan type* $J_{\ell,S}$ *is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod 9$	(n, n, n, n)
$n \equiv -1 \pmod 9$	$(n+1, n+1, n+1, n-3)$
$n \equiv -2 \pmod 9$	$(n+2, n+2, n-1, n-3)$
$n \equiv -3 \pmod 9$	$(n+3, n, n, n-3)$
$n \equiv 1 \pmod 9$	$(n+3, n-1, n-1, n-1)$
$n \equiv 2 \pmod 9$	$(n+3, n+1, n-2, n-2)$
$n \equiv 3 \pmod 9$	$(n+3, n, n, n-3)$

Proof. Recall that by Theorem 3.6, for $n \not\equiv \pm 4 \pmod 9$, S fails to have the SLP. Otherwise, S has the SLP. First note that there is no cubic form C such that

$$C \cdot (x+y)^{n-4} = 0.$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$.

(a) Let $n \equiv 0 \pmod 9$. Note that

$$\begin{aligned}(x+y)^{n-1} &= -x^3y^{n-4} + x^2y^{n-3} - xy^{n-2} + y^{n-1} \neq 0,\\ (x+y)^n &= 0.\end{aligned}$$

In other words,

$$\boxed{J_{\ell,S} = (n,n,n,n).}$$

(b) Let $n \equiv 1 \pmod 9$. Note that

$$\begin{aligned}(x+y)^{n+2} &= x^3y^{n-1} \neq 0,\\ (x+y)^{n+3} &= 0.\end{aligned}$$

Thus $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Moreover there is no quadratic form Q such that

$$Q \cdot (x+y)^{n-2} = 0.$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_3 \geq n-1$ and $\lambda_4 \geq n-3$. Since the sum of the components of $J_{\ell,S}$ is $4n$, the second component of $J_{\ell,S}$ has to be $\leq n+1$. But for $k = n+1, n$, and for some linear form L,

$$(x+y)^k = x^{k-n+1}y^{n-1},$$

we see that $L \cdot (x+y)^k = 0$ implies that $L \cdot (x+y)^{k-1} = 0$. Hence we conclude that

$$\boxed{J_{\ell,S} = (n+3, n-1, n-1, n-1).}$$

(c) Let $n \equiv -1 \pmod 9$. Note that

$$\begin{aligned}(x+y)^n &= x^2y^{n-2} - xy^{n-1} \neq 0,\\ (x+y)^{n+1} &= 0.\end{aligned}$$

Hence $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+1, -, -, \geq n-3).$$

Note that

$$x(x+y)^n = nx^2y^{n-1} + \frac{n(n-1)}{2}x^3y^{n-2} \neq 0,$$

$$y(x+y)^n = \frac{n(n-1)}{2}x^2y^{n-1} + \frac{n(n-1)(n-2)}{6}x^3y^{n-2} \;\neq\; 0,$$
$$(x+2y)(x+y)^n = n^2x^2y^{n-1} + \frac{n(n-1)(2n-1)}{6}x^3y^{n-2} \;\neq\; 0,$$

which implies that there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^n = 0.$$

This shows that the Jordan type $J_{\ell,S}$ has to be of the form

$$J_{\ell,S} = (n+1, n+1, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Furthermore, it is not hard to show that if for a quadric form Q

$$Q \cdot (x+y)^{n-1} \neq 0, \quad \text{and}$$
$$Q \cdot (x+y)^n = 0,$$

then $Q = y(x+y)$. This implies that the third component of the Jordan type $J_{\ell,S}$ has to be $\geq n+1$, i.e.,

$$\boxed{J_{\ell,S} = (n+1, n+1, n+1, n-3).}$$

(d) Let $n \equiv 2 \pmod 9$. Note that

$$(x+y)^{n+2} = x^3y^{n-1} \neq 0,$$
$$(x+y)^{n+3} = 0.$$

Hence the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Suppose $C = ax^3 + bx^2y + cxy^2 + dy^3$ for some $a, b, c, d \in \Bbbk$ such that

$$C \cdot (x+y)^{n-4} \neq 0, \quad \text{and}$$
$$C \cdot (x+y)^{n-3} = 0.$$

Then

$$ax^3 \cdot (x+y)^{n-3} = ax^3y^{n-3},$$
$$bx^2y \cdot (x+y)^{n-3} = bx^2y^{n-2} + (n-3)bx^3y^{n-3},$$
$$cxy^2 \cdot (x+y)^{n-3} = cxy^{n-1} + (n-3)cx^2y^{n-2} + \frac{(n-3)(n-4)}{2}cx^3y^{n-3},$$
$$dy^3 \cdot (x+y)^{n-3} = (n-3)dxy^{n-1} + \frac{(n-3)(n-4)}{2}dx^2y^{n-2} + \frac{(n-3)(n-4)(n-5)}{6}dx^3y^{n-3}.$$

First, since $n \equiv 2 \pmod 9$, we have $n \equiv 2 \pmod 3$, i.e., $n-3 \equiv 2 \pmod 3$, $n-4 \equiv 1 \pmod 3$, and $n-5 \equiv 0 \pmod 3$.

(1) $c+(n-3)d=0$ implies that $c=d$.
(2) $b+(n-3)c+\frac{(n-3)(n-4)}{2}d=0$ with $c=d$, we have that $b=0$.
(3) $a+(n-3)b+\frac{(n-3)(n-4)}{2}c+\frac{(n-3)(n-4)(n-5)}{6}d=0$ with $b=0$, and $c=d$ yield $a=0$.

In other words, $(x+y)\mid C=y^2(x+y)$. Thus the last component of the Jordan type $J_{\ell,S}$ has to be $\geq n-2$, i.e.,

$$J_{\ell,S}=(n+3,\lambda_2,\lambda_3,\lambda_4)$$

with $\lambda_4\geq n-2$. Moreover, there is no linear form $L\neq x+y$ such that

$$L\cdot(x+y)^n=0.$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S}=(n+3,\lambda_2,\lambda_3,\lambda_4)$$

with $\lambda_2\geq n+1$ and $\lambda_4\geq n-2$, i.e.,

$$\boxed{J_{\ell,S}=(n+3,n+1,n-2,n-2).}$$

(e) Let $n\equiv -2 \pmod 9$ and $\ell=x+y$. Note that

$$\begin{aligned}(x+y)^{n+1}&=-x^3y^{n-2}+x^2y^{n-1}\neq 0,\\ (x+y)^{n+2}&=0,\end{aligned}$$

this shows that the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S}=(n+2,\lambda_2,\lambda_3,\lambda_4)$$

with $\lambda_4\geq n-3$. Furthermore, there is no linear form $L\neq x+y$ such that

$$L\cdot(x+y)^{n+1}=0,$$

and no quadratic form Q such that

$$Q\cdot(x+y)^{n-2}=0,$$

we see that $J_{\ell,S}$ is of the form

$$J_{\ell,S}=(n+2,\lambda_2,\lambda_3,\lambda_4)$$

with $\lambda_2\geq n+2$, $\lambda_3\geq n-1$, and $\lambda_4\geq n-3$, i.e.,

$$\boxed{J_{\ell,S}=(n+2,n+2,n-1,n-3).}$$

(f) Let $n\equiv 3 \pmod 9$. Note that

$$\begin{aligned}(x+y)^{n+2}&=x^3y^{n-1}\neq 0,\\ (x+y)^{n+3}&=0, \quad \text{and}\end{aligned}$$

there is no quadratic form Q such that

$$Q\cdot(x+y)^{n-1}=0.$$

So the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S}=(n+3,\lambda_2,\lambda_3,\lambda_4)$$

with $\lambda_3 \geq n$ and $\lambda_4 \geq n-3$, i.e., $J_{\ell,S}$ has to be

$$\boxed{J_{\ell,S} = (n+3, n, n, n-3).}$$

(g) Let $n \equiv -3 \pmod 9$. Note that

$$(x+y)^{n+2} = -x^3y^{n-1} \neq 0,$$
$$(x+y)^{n+3} = 0,$$

there is no quadratic form Q such that

$$Q \cdot (x+y)^{n-1} = 0.$$

So the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_3 \geq n$ and $\lambda_4 \geq n-3$, i.e.,

$$\boxed{J_{\ell,S} = (n+3, n, n, n-3).}$$

This completes the proof. □

Theorem 4.6 (char $\Bbbk \geq 5$ and $m = 4$)**.** *Let* $S := \Bbbk[x,y]/(x^4, y^n)$ *with* char $\Bbbk = p \geq 5$ *and* $n \geq 4$*. For a linear form* $\ell = x + y$ *and for* $n \equiv 0, \pm 1, \pm 2 \pmod p$*,* S *fails to have the SLP, and the Jordan type* $J_{\ell,S}$ *is as follows.*

	$J_{\ell,S}$
$n \equiv 0 \pmod p$	(n, n, n, n)
$n \equiv -1 \pmod p$	$(n+1, n+1, n+1, n-3)$
$n \equiv -2 \pmod p$	$(n+2, n+2, n-1, n-3)$
$n \equiv 1 \pmod p$	$(n+3, n-1, n-1, n-1)$
$n \equiv 2 \pmod p$	$(n+3, n+1, n-2, n-2)$

Proof. Recall that by Theorem 2.5, if $n \equiv 0, \pm 1, \pm 2 \pmod p$, S fails to have the SLP. Otherwise, S has the SLP (see Theorem 2.1). First, note that there is no cubic form C such that

$$C \cdot (x+y)^{n-4} = 0.$$

So the Jordan type is of the form

$$J_{\ell,S} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$.

(a) Let $n \equiv 0 \pmod p$. Then

$$(x+y)^{n-1} = -x^3y^{n-4} + x^2y^{n-3} - xy^{n-2} + y^{n-1} \neq 0, \quad \text{and}$$
$$(x+y)^n = 0.$$

So $J_{\ell,S}$ is of the form

$$\boxed{J_{\ell,S} = (n, n, n, n).}$$

(b) Let $n \equiv 1 \pmod{p}$. Note that

$$(x+y)^{n+2} = x^3y^{n-1} \neq 0, \quad \text{and}$$
$$(x+y)^{n+3} = 0.$$

Note that for any linear form L

$$L \cdot (x+y)^{n+2} = 0,$$

so we have

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \leq n+2$ and $\lambda_4 \geq n-3$. If for a quadratic form Q

$$Q \cdot (x+y)^{n-2} = 0,$$

then

$$Q = xy + y^2 = (x+y)y = \ell \cdot y.$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2 n+2, \lambda_3 \geq n-1, \lambda_4 \geq n-3).$$

But the second component $n+2$ of $J_{\ell,S}$ is not possible. Moreover, since S does not have the SLP, $J_{\ell,S}$ is not of the form

$$J_{\ell,S} = (n+3, n+1, n-1, n-3).$$

Furthermore, there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^{n-1} \neq 0,$$
$$L \cdot (x+y)^{n} = 0,$$

and thus $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \leq n-1$, $\lambda_3 \geq n-1$, and $\lambda_4 \geq n-3$, i.e., $J_{\ell,S}$ is of the form

$$\boxed{J_{\ell,S} = (n+3, n-1, n-1, n-1).}$$

(c) Let $n \equiv 2 \pmod{p}$. Note that

$$(x+y)^{n+2} = 4x^3y^{n-1} \neq 0, \quad \text{and}$$
$$(x+y)^{n+3} = 0.$$

Note that for any linear form L

$$L \cdot (x+y)^{n+2} = 0.$$

So

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \leq n+2$ and $\lambda_4 \geq n-3$. If for a cubic form C

$$C \cdot (x+y)^{n-3} = 0,$$

then

$$C = y^2 \cdot (x+y) = y^2 \cdot \ell.$$

This implies that

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \leq n+2$ and $\lambda_4 \geq n-2$ and so

$$J_{\ell,S} = (n+3, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \leq n+1$ and $\lambda_4 \geq n-2$. Since there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^n = 0,$$

$J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+3, n+1, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-2$, i.e.,

$$\boxed{J_{\ell,S} = (n+3, n+1, n-2, n-2).}$$

(d) Let $n \equiv -1 \pmod{p}$. Note that

$$\begin{aligned} (x+y)^n &= -x^3y^{n-3} + x^2y^{n-2} - xy^{n-1} \neq 0, \quad \text{and} \\ (x+y)^{n+1} &= 0. \end{aligned}$$

So

$$J_{\ell,S} = (n+1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Furthermore there is no quadratic form Q such that $(x+y) \nmid Q$ and

$$Q \cdot (x+y)^n = 0,$$

so,

$$J_{\ell,S} = (n+1, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_3 \geq n+1$ and $\lambda_4 \geq n-3$, i.e.,

$$\boxed{J_{\ell,S} = (n+1, n+1, n+1, n-3).}$$

(e) Let $n \equiv -2 \pmod{p}$. Note that

$$\begin{aligned} (x+y)^{n+1} &= -x^3y^{n-2} + x^2y^{n-1} \neq 0, \quad \text{and} \\ (x+y)^{n+2} &= 0. \end{aligned}$$

So $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_4 \geq n-3$. Now consider a quadratic form $Q = ax^2 + bxy + xy^2$ with $a, b, c \in \Bbbk$ such that

$$Q(x+y)^{n-2} = 0.$$

Note that

$$\begin{aligned} (x+y)^{n-2} = y^{n-1} + (n-2)xy^{n-3} + \frac{(n-2)(n-3)}{2}x^2y^{n-4} \\ + \frac{(n-2)(n-3)(n-4)}{6}x^3y^{n-5}. \end{aligned}$$

This implies that

$$\begin{aligned} ax^2(x+y)^{n-2} &= ax^2y^{n-2} + (n-2)ax^3y^{n-3}, \\ bxy(x+y)^{n-2} &= bxy^{n-1} + (n-2)bx^2y^{n-2} + \frac{(n-2)(n-3)}{2}bx^3y^{n-3}, \\ cy^2(x+y)^{n-2} &= (n-2)cxy^{n-1} + \frac{(n-2)(n-3)}{2}cx^2y^{n-2} \\ &\quad + \frac{(n-2)(n-3)(n-4)}{6}cx^3y^{n-3}. \end{aligned}$$

Moreover, $Q(x+y)^{n-2} = 0$ yields

$$\begin{aligned} b + (n-2)c = 0 \quad &\text{if and only if} \quad b = 4c, \\ a + (n-2)b + \tfrac{(n-2)(n-3)}{2}c = 0 \quad &\text{if and only if} \quad a = 6c. \end{aligned}$$

Hence we may take that $a = 6$, $b = 4$, and $c = 1$. But,

$$\begin{aligned} &(n-2)a + \frac{(n-2)(n-3)}{2}b + \frac{(n-2)(n-3)(n-4)}{6}c \\ &= (n-2)\cdot 6 + \frac{(n-2)(n-3)}{2}\cdot 4 + \frac{(n-2)(n-3)(n-4)}{6} \neq 0, \end{aligned}$$

which follows that there is no quadratic form Q such that

$$Q \cdot (x+y)^{n-2} = 0.$$

In other words, $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, , \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_3 \geq n-1$ and $\lambda_4 \geq n-3$. Note that there is no linear form $L \neq x+y$ such that

$$L \cdot (x+y)^{n+1} = 0.$$

So the Jordan type $J_{\ell,S}$ is of the form

$$J_{\ell,S} = (n+2, \lambda_2, \lambda_3, \lambda_4)$$

with $\lambda_2 \geq n+2$, $\lambda_3 \geq n-1$, and $\lambda_4 \geq n-3$, i.e.,

$$\boxed{J_{\ell,S} = (n+2, n+2, n-1, n-3).}$$

This completes the proof of Theorem 4.6. □

Remark 4.7. We found a general formula for characteristic $p \geq 2m-3$, but not for low characteristic $p < 2m-3$, which were discussed individually in Sections 3 and 4. It has been explored when $S = \Bbbk[x,y]/(x^m, y^n)$ has the SLP using a different language 'representation theory' for $m \leq n$ and $m = 3, 4$ in [3]. As we mentioned in the introduction, there is a recursive formula how to find the Jordan type for S [10]. However, not much is known about the Jordan type of $S = \Bbbk[x_1, \ldots, x_r]/(x_1^{m_1}, \ldots, x_r^{m_r})$ for $r \geq 3$ over a field $\Bbbk$ of a prime characteristic p smaller than the socle degree $j = (\sum_i m_i) - r$, except for the strong Lefschetz case treated in [2] and completed in [13].

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