East Asian Math. J.
Vol. 36 (2020), No. 1, pp. 025–034
http://dx.doi.org/10.7858/eamj.2020.003

NOTE ON STRONG LAW OF LARGE NUMBER UNDER SUB-LINEAR EXPECTATION

Kyo-Shin Hwang

Abstract. The classical limit theorems like strong law of large numbers, central limit theorems and law of iterated logarithms are fundamental theories in probability and statistics. These limit theorems are proved under additivity of probabilities and expectations. In this paper, we investigate strong law of large numbers under sub-linear expectation which generalize the classical ones. We give strong law of large numbers under sub-linear expectation with respect to the partial sums and some conditions similar to Petrov's. It is an extension of the classical Chung type strong law of large numbers of Jardas et al.'s result. As an application, we obtain Chung's strong law of large number and Marcinkiewicz's strong law of large number for independent and identically distributed random variables under the sub-linear expectation. Here the sub-linear expectation and its related capacity are not additive.

1. Introduction

The classical strong laws of large numbers are widely been known as fundamental limit theorems in the theory of probability and statistics. It plays fruitful role in the development of probability theory and its applications. However, many uncertain phenomena can not be well modeled by using additive probabilities and additive expectations. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measure of risk, super-hedge Pricing and modeling uncertainty in finance (c.f.[7],[8],[11],[12],[16]). Recently, motivated by the risk measures, superhedge pricing and modeling uncertainty in finance, the notions of independent and identically distributed random variables under the sublinear expectations is introduced by Peng([12-13],[15],[16]). Under Peng's framework, many limit theorems have been investigating. Chen([1]) proved a strong law of large numbers for independent and identically random variables under capacities induced by sublinear expectations. Chen, Hu and Zong([2]) obtained strong laws of large numbers for sub-linear

Received October 07, 2019; Accepted November 28, 2019.
2010 *Mathematics Subject Classification.* 60F15; 60F05.
Key words and phrases. sub-linear expectation, capacity, independence, strong law of large number.

(pISSN 1226-6973, eISSN 2287-2833)

expectation without independence. Zhang and Lin([22]) proved Marcinkiewicz's strong law of large numbers for nonlinear expectations. Many authors investigated on limit theorems under nonlinear expectation in the wide fields such as the strong law of large number([1],[3],[20]), the law of the iterated logarithm([3]) and the convergence of the infinite series of random variables([19],[23]). For the convergence of the sums of random variables, Zhang([23]) gave three series theorem on the sufficient and necessary conditions for the almost sure convergence of the infinite series $\sum_{n=1}^{\infty} X_n$ under the sub-linear expectation.

In this paper, we investigate strong law of large numbers under sub-linear expectation which generalize the classical ones. We give strong law of large numbers under sub-linear expectation with respect to the partial sums and some conditions similar to Petrov's([17-18]). It is an extension of the classical Chung type strong law of large numbers of Jardas et al.'s result([10]). As an application, we obtain Chung's strong law of large number and Marcinkiewicz's strong law of large number for independent and identically distributed random variables under the sub-linear expectation. Here the sub-linear expectation and its related capacity are not additive.

Our paper is organized as follows: we introduce some basic setting, definitions and proposition in Section 2. In Section 3, we state and prove our main result.

2. Second section

In this section, we introduce some basic definitions and notations about sub-linear expectation. Refer to Peng([15-16]) for more details.

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots,$ $X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(R_n)$, where $C_{l,Lip}(R^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x},\ \mathbf{y} \in R^n,$$

for some $C > 0$, $m \in \mathbf{N}$ depending on φ. Let $C_{b,Lip}(R^n)$ denote the linear space of bounded functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le C|x - y|, \quad \forall\ \mathbf{x},\ \mathbf{y} \in R^n,$$

for some $C > 0$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 1. ([12-16]) A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$ we have

(i) Monotonicity: If $X \ge Y$ then $\widehat{\mathbb{E}}[X] \ge \widehat{\mathbb{E}}[Y]$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$; whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$
Here $\bar{R} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] = -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that

$$\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X], \;\; \widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c \;\text{ and }\; \widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$$

for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Let $X = (X_1, X_2, \cdots, X_n)$ be a given n-dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. We define a functional on $C_{l,Lip}(R^n)$ by

$$\mathbb{F}_X[\varphi] := \widehat{\mathbb{E}}[\varphi(X)] : \varphi \in C_{l,Lip}(R^n) \to R.$$

$\mathbb{F}_X$ is called *the distribution of* X *under* $\widehat{\mathbb{E}}$.

We adopt the following notion of independence and identical distribution for sub-linear expectation which is initiated by Peng([12-16]).

Definition 2. (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. X_1 and X_2 are called identically distributed, denoted by $\mathbf{X}_1 =^{\mathrm{d}} \mathbf{X}_2$, if

$$\widehat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)], \quad \forall \varphi \in C_{l,Lip}(R^n),$$

whenever the sub-linear expectation are finite.

Definition 3. (Independent) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}\varphi(x, \mathbf{Y})|_{x=\mathbf{X}}], \quad \forall \varphi \in C_{l,Lip}(R^m \times R^n)$$

whenever $\overline{\varphi}(x) = \widehat{\mathbb{E}}[|\varphi(x, \mathbf{Y})|] < \infty$ for all x and $\widehat{\mathbb{E}}[|\overline{\varphi}(\mathbf{X})|] < \infty$.

Definition 4. (IID random variables) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be independent, if X_{i+1} is independent to $(X_1, X_2, \cdots, X_i)$ for each $i \geq 1$. It is said to be identically distributed, if $X_i =^{\mathrm{d}} X_1$ for each $i \geq 1$.

In Peng([12-16]), the space of the test function φ is $C_{l,Lip}(R^n)$. When the considered random variables have finite moments of each order, i.e., $\widehat{\mathbb{E}}[\varphi(X)] < \infty$ for each $\varphi \in C_{l,Lip}(R^n)$, the test function φ in the definition is limit in the space of bounded Lipschitz function $C_{b,Lip}(R^n)$, since there exists $\varphi_k \in C_{b,Lip}(R^n)$ such that $\varphi_k \downarrow \varphi$ $(\varphi_k(x) = \sup_{y\in R^n} \{\varphi(y) - k|x-y|\})$.

Definition 5. (I) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called a capacity if $\mathbb{V}(\emptyset) = 0, \mathbb{V}(\Omega) = 1$ and $\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{F}$.

(II) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called to be countably sub-additive if

$$\mathbb{V}(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mathbb{V}(A_n) = 1, \quad \forall A_n \in \mathcal{F}.$$

Let $(\Omega, \mathcal{H}.\widehat{\mathbb{E}})$ be a sub-linear space. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) = 1 - \mathcal{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement se of A. Then

$$\widehat{\mathbb{E}}[f] \le \mathbb{V}(A) \le \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \le \mathcal{V}(A) \le \widehat{\mathcal{E}}[g],$$

if $f \le I_A \le g,\ f, g \in \mathcal{H}$. It is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \le \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X+Y] \le \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that

$$\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \ge \mathbb{V}(A^c) - \mathbb{V}(B) \ \text{ and } \ \widehat{\mathbb{E}}[-X-Y] \le \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[-Y].$$

Further, if X is not in $\mathcal{H}$, we define $\widehat{\mathbb{E}}$ by $\widehat{\mathbb{E}}[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \le Y, Y \in \mathcal{H}\}$. Then $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$.

In this paper we only consider the capacity generated by a sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, we define a capacity:

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \forall A \in \mathcal{F}$$

and also define the conjugate capacity:

$$\mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}.$$

It is clear that $\mathbb{V}$ is a sub-additive capacity and $\mathcal{V}(A) = \widehat{\mathcal{E}}[I_A]$.

The following representation theorem for sub-linear expectation is very useful(see Peng([15-16]) for the Proof):

Proposition 2.1. *Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sublinear expectation space.*

(I) There exists a family of finitely additive probability measures $\{P_\theta : \theta \in \Theta\}$ defined on $(\Omega, \mathcal{F})$ such that for each $X \in \mathcal{H}$

$$\widehat{\mathbb{E}}[X] = \max_{\theta \in \Theta} E_{P_\theta}[X].$$

(II) For any fixed random variables $X \in \mathcal{H}$, there exists a family of probability measures $\{\mu_\theta\}_{\theta\in\Theta}$ defined on $(R, \mathcal{B}(R))$ such that for each $\varphi \in C_{l,Lip}(R)$,

$$\widehat{\mathbb{E}}[\varphi(X)] = \sup_{\theta\in\Theta} \int_R \varphi(x)\mu_\theta(dx).$$

For any random variable X and constant c, define $X^c = (-c) \vee (X \wedge c)$.

3. Main result and Proof

In this section, we give the main results. We first recall some related important lemmas in sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Then we give strong law of large numbers under sub-linear expectation by using a theorem similar to Kolmogorov's three series theorem in classical probability theory.

Lemma 3.1. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Suppose that $\mathbb{V}$ is countably sub-additive. Then $\sum_{n=1}^\infty X_n$ converges almost surely in capacity if the following three conditions hold for some $c > 0$;*

(S_1) $\sum_{n=1}^\infty \mathbb{V}(|X_n| > c) < \infty$,

(S_2) $\sum_{n=1}^\infty \widehat{\mathbb{E}}[X_n^c]$ and $\sum_{n=1}^\infty \widehat{\mathbb{E}}[-X_n^c]$ are both convergent,

(S_3) $\sum_{n=1}^\infty \widehat{\mathbb{E}}[(X_n^c - \widehat{\mathbb{E}}[X_n^c])^2)] < \infty$ and $\sum_{n=1}^\infty \widehat{\mathbb{E}}[(X_n^c + \widehat{\mathbb{E}}[-X_n^c])^2)] < \infty$

Conversely, if $\mathbb{V}$ is continuous and $\sum_{n=1}^\infty X_n$ is convergent almost surely in capacity $\mathbb{V}$, then $(S_1), (S_2), (S_3)$ will hold for all $c > 0$

Proof. The proof of Lemma 3.1 can be found in Zhang([23]). □

The following lemma can be found in Chen([4]).

Lemma 3.2. (Chebyshev's inequality) *Let X be a real measurable random variable in sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Let $f(x) > 0$ be a nondecreasing function on R. Then for any x,*

$$\mathbb{V}(X \geq x) \leq \frac{\widehat{\mathbb{E}}[f(x)]}{f(x)}, \quad \mathcal{V}(X \geq x) \leq \frac{\widehat{\mathcal{E}}[f(x)]}{f(x)}$$

Let $f(x) > 0$ be an even function and nondecreasing on $(0, \infty)$. Then for any $x > 0$,

$$\mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}[f(x)]}{f(x)}, \quad \mathcal{V}(|X| \geq x) \leq \frac{\widehat{\mathcal{E}}[f(x)]}{f(x)}.$$

Lemma 3.3. (Kronecker's lemma) *For real sequcence* $\{x_n, n \geq 1\}$ *and* $\{b_n, n \geq 1\}$*, with* $0 < b_n \uparrow \infty$*, if* $\sum_{n=1}^{\infty} x_k$ *converges, then*

$$\frac{1}{b_n}\sum_{k=1}^{n} b_k x_k \to 0 \quad as \quad n \to \infty$$

The following theorem is strong law of large numbers under sub-linear expectation, from which we can deduce the strong convergence of a random series under sub-linear expectation. It is an extension of the classical Chung type strong law of large numbers of Jardas et al.'s result([10]).

Theorem 3.4. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables with* $\mathbb{E}[X_n] = 0 = \mathcal{E}[X_n], \quad n = 1, 2, \cdots$ *in the sub-linear expectation space*$(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$*, and let* $\mathbb{V}$ *be countably sub-additive. Let* $\{a_n, n \geq 1\}$ *be a non-zero sequence tending to infinity. Let* $\{g_n(x), n \geq 1\}$ *be a sequence of locally Lipschitz, positive and non-decreasing for* $x > 0$*. Let* $\alpha_n \geq 1, \beta_n \leq 2, K_n \geq 1, M_n \geq 1 \ \ (n \in N)$ *be constants satisfying for* $t_1 \leq t_2$*,*

$$\frac{g_n(t_1)}{{t_1}^{\alpha_n}} \leq K_n \frac{g_n(t_2)}{{t_2}^{\alpha_n}} \quad and \quad \frac{{t_1}^{\beta_n}}{g_n(t_1)} \leq M_n \frac{{t_2}^{\beta_n}}{g_n(t_2)} \tag{1}$$

If

$$\sum_{n=1}^{\infty} K_n \frac{\widehat{\mathbb{E}}[g_n(X_n)]}{g_n(|a_n|)} < \infty \quad and \quad \sum_{n=1}^{\infty} M_n \frac{\widehat{\mathbb{E}}[g_n(X_n)]}{g_n(|a_n|)} < \infty. \tag{2}$$

Then

$$\frac{S_n}{a_n} \to 0 \quad a.s. \quad \mathbb{V}$$

Proof. It would be sufficient for our purpose to get the conditions $(S_1), (S_2), (S_3)$ of Lemma 3.1. We first show condition (S_1). By Lemma 3.2, we have

$$\mathbb{V}(|X_n| \geq |a_n|) \leq \frac{\widehat{\mathbb{E}}[g_n(|X_n|)]}{g_n(|a_n|)}.$$

and hence by (2)

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_n| \geq |a_n|) \leq \sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[g_n(|X_n|)]}{g_n(|a_n|)} < \infty.$$

It follows that condition (S_1) in Lemma 3.1 is proved.

Our next claim is that condition (S_2) in Lemma 3.1 holds. Define

$$X_n^c = (-|a_n|) \vee (X_n \wedge |a_n|), \quad n = 1, 2, \cdots .$$

We just need to show

$$\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n^c] < \infty.$$

Because of considering $\{-X_n, n \geq 1\}$ instead of $\{X_n, n \geq 1\}$ in Lemma 3.1, we can obtain

$$\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[-X_n^c] < \infty.$$

By Proposition 2.1, for any fixed random variable $X \in \mathcal{H}$, there exists a family of probability measure $\{\mu_\theta\}_{\theta\in\Theta}$ defined on $(R, \mathcal{B}(R))$ such that for each $\varphi \in C_{l,Lip}(R)$

$$\widehat{\mathbb{E}}[\varphi(X)] = \sup_{\theta\in\Theta} \int_R \varphi(x)\mu_\theta(dx).$$

Using the fact that $\widehat{\mathbb{E}}[X_n] = 0 = \widehat{\mathcal{E}}[X_n]$ for $n = 1, 2, \cdots$, we have $X_n - X_n^c \leq (|X_n| - |a_n|)^+ \leq |X_n| \cdot I_{\{|X_n|>|a_n|\}}$ and also $|X_n| \cdot I_{\{|X_n|>|a_n|\}} \in C_{l,Lip}(R)$, we have

$$\begin{aligned}
\widehat{\mathbb{E}}[X_n^c] &= \widehat{\mathbb{E}}[X_n^c] - \widehat{\mathbb{E}}[X_n] \leq \widehat{\mathbb{E}}[X_n^c - X_n] \leq \widehat{\mathbb{E}}[|X_n - X_n^c|] \leq \widehat{\mathbb{E}}[(|X_n| - |a_n||] \\
&\leq \widehat{\mathbb{E}}[|X_n| \cdot I\{|X_n| > |a_n|\}]
\end{aligned}$$

and also

$$\widehat{\mathcal{E}}[X_n^c] \leq \widehat{\mathcal{E}}[|X_n| \cdot I\{|X_n| > a_n\}].$$

For $|x| > |a_n|$, we have by (1)

$$\frac{|x|}{|a_n|} \leq \frac{|x|^{\alpha_n}}{|a_n|^{\alpha_n}} \leq K_n \frac{g_n(|x|)}{g_n(|a_n|)}.$$

Thus we have by Proposition 2.1

$$\begin{aligned}
|\widehat{\mathbb{E}}[X_n^c]| &\leq \sup_{\theta\in\Theta} \int_{|x|>|a_n|} |x| \; \mu_{n,\theta}(dx) \\
&\leq \sup_{\theta\in\Theta} \int_{|x|>|a_n|} K_n \frac{|a_n|}{g_n(|a_n|)} g_n(|x|)\mu_{n,\theta}(dx) \\
&\leq K_n \frac{|a_n|}{g_n(|a_n|)} \sup_{\theta\in\Theta} \int_{|x|>|a_n|} g_n(|x|)\mu_{n,\theta}(dx) \\
&\leq K_n \frac{|a_n|}{g_n(|a_n|)} \widehat{\mathbb{E}}[g_n(|X_n|)]
\end{aligned}$$

and by (2)

$$\sum_{n=1}^{\infty} \frac{|\widehat{\mathbb{E}}[X_n^c]|}{|a_n|} < \infty.$$

We only need to show condition (S_3) in Lemma 3.1. For $|x| < |a_n|$, we have

$$\frac{|x|^2}{|a_n|^2} \leq \frac{|x|^{\beta_n}}{|a_n|^{\beta_n}} \leq M_n \frac{g_n(|x|)}{g_n(|a_n|)},$$

hence

$$\begin{aligned}\widehat{\mathbb{E}}[X_n^{c\,2}] &= \sup_{\theta\in\Theta}\int_{|x|\le|a_n|} x^2\mu_{n,\theta}(dx)\\ &\le M_n\frac{|a_n|^2}{g_n(|a_n|)}\sup_{\theta\in\Theta}\int_{|x|\le|a_n|} g_n(|x|)\mu_{n,\theta}(dx)\\ &\le M_n\frac{|a_n|^2}{g_n(|a_n|)}\widehat{\mathbb{E}}[g_n(|X_n|)]\end{aligned}$$

and by (2)

$$\sum_{n=1}^{\infty}\widehat{\mathbb{E}}\left[\left(\frac{X_n^c}{|a_n|}\right)^2\right]<\infty.$$

From Lemma 3.1, it follows that the series $\sum_{n=1}^{\infty} X_n/a_n$ is convergent a.s. in capacity. By Lemma 3.3, we obtain

$$\frac{S_n}{a_n}\to 0 \quad \text{a.s.} \quad \mathbb{V},$$

which completes the proof. □

We have the following corollaries. The following Corollary 3.5 is Chung type strong law of large numbers under sub-linear expectation (c.f. [6],[17-18])

Corollary 3.5. *Let $\{X_n, n\ge 1\}$ be a sequence of independent random variables in the sub-linear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$, and let $\mathbb{V}$ be countably sub-additive. Let $\{a_n, n\ge 1\}$ be a positive increasing sequence tending to infinity. Let $g(x)$ be a locally Lipschhitz, even function, positive and non-decreasing for $x>0$. Let one of the following two conditions hold:*

(a) $x/g(x)$ is non-decreasing for $x>0$,

(b) $x/g(x)$ and $g(x)/x^2$ are non-increasing for $x>0$.

If the series

$$\sum_{n=1}^{\infty}\frac{\widehat{\mathbb{E}}[g(X_n)]}{g(a_n)}<\infty$$

is satisfied. Moreover, suppose that

$$\widehat{\mathbb{E}}[X_n]=0=\widehat{\mathcal{E}}[X_n], \quad n=1,2,\cdots$$

when condition (b) is satisfied. Then

$$\frac{S_n}{a_n}\to 0 \quad a.s. \quad \mathbb{V}$$

The following Corollary 3.6 is very similar to Marcinkiew's strong law of large numbers for nonlinear expectations(see [19],[22]). It is to state the consequence of Theorem 3.4 corresponding $g_n(x)=x^2$ and $a_n=n^{1/p},\ n=1,2,\cdots$.

Corollary 3.6. *Suppose* $\{X_n, n \geq 1\}$ *is a sequence of independent and identical random variable in the sub-linear expectation space with* $\widehat{\mathbb{E}}[X_1] = 0 = \widehat{\mathcal{E}}[X_1]$ *and* $\mathbb{V}$ *is countably sub-additive. If the series*

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[X_n^2]}{n^{2/p}} < \infty$$

is satisfied.
(I) *For* $0 < p < 1$*, then*

$$\frac{S_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad \mathbb{V}$$

(II) *For* $1 \leq p < 2$*, suppose* $\lim_{a\to\infty} \widehat{\mathbb{E}}[(|X_1| - a)^+] = 0$*, then*

$$\frac{S_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad \mathbb{V}$$

References

[1] Z. Chen, *Strong laws of large numbers for sub-linear expectation*, Sci. China Math. **59** (2016), no. 5, 945–954.
[2] Z. Chen, C. Hu and G. Zong, *Strong laws of large numbers for sub-linear expectation without independence*, Comm. Stati.-Theory and Mathods **46** (2017), no. 15, 7529–7545.
[3] Z. Chen. Z. & F. Hu, *A law of the iterated logarithm under sublinear expectations*, J. Financial Engineering **1** (2014), no. 2, 1450015. Available at https://doi.org/10.1142/S2345768614500159
[4] Z. Chen, P. Wu & B. Li, *A strong laws of large numbers for non-additive probabilities*, Interna. J. Approx. Reason **54** (2013), no. 3, 365–377.
[5] H. Cheng, *A strong law of large numbers for sub-linear under a general moment condition*, Stat. & Probab. Lett. **116** (2016), 248–258.
[6] K. L. Chung, *Note on some strong laws of large numbers*, Amer. J. Math. **69** (1947), no. 1, 189–192.
[7] C. Denis & C. Martini, *A theoretical framework for the pricing of contingent claims in the presence of model uncertainty*, Ann. Appl. Probab. **16** (2006), 827–852.
[8] I. Gilboa, *Expected utility theory with purely subjective non-additive probabilities*, J. Math. Econom. **16** (1987),65–68.
[9] C. Hu, *Strong laws of large numbers for sublinear expectation under controlled 1st moment condition*, Chines Ann Math. Ser. B **39** (2019), no. 5, 791–804.
[10] C. Jardas, J. Pecaric & N. Sarapa, *A note on Chung's strong law of large numbers*, JMAA **217** (1998), 328–334.
[11] M. Marinacci, *Limit laws for non-additive probabilities and their frequentist interpretation*, J. Econom, Theory **84** (1999), 145–195.
[12] S. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Ito type.* In: Benth, F. E., et al., Proceedings of the 2005 Abel Symposium. Springer, Berlin-Heidelberg, (2006), 541–567.
[13] S. Peng, *Law of large numbers and central limit theorem under nonlinear expectations*, Probability, Uncertainty and Quantitative risk **4** (2019), no. 4, 1–8.
[14] S. Peng, *A new central limit theorem under sublinear expectations*, J. Math. **53** (2008), no. 8, 1989–1994.
[15] S. Peng, *Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations*, Sci. China Ser. A: Math. **52** (2009), no. 7, 1391–1411.

[16] S. Peng, *Nonlinear expectations and stochastic calculus under uncertainty-with robust central limit theorem and G-Brownian motion*, (2010), arXiv:1002.4546v1 [math.PR].

[17] V. V. Petrov, *On the strong law of large numbers*, Theory Probab. Appli. **14** (1969), no. 2, 183–192.

[18] V. V. Petrov, *On the order of growth and sums of dependent variables*, Theory Probab. Appli. **18** (1974), no. 2, 348–350.

[19] J. P. Xu & L. X. Zhang, *Three series theorem for independent random variables under sub-linear expectations with applications*, Acta Math. Sin., English Ser. **35** (2019), no. 2, 172–184.

[20] Q. Y. Wu & Y. Y. Jiang, *Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectation*, JMAA **460** (2018), 252–270.

[21] L. X. Zhang, *Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications*, Sci. China Math. **59** (2016), no. 4, 751–768.

[22] L. X. Zhang & J. H. Lin, *Marcinkiewicz's strong law of large numbers for nonlinear expectations*, Stat. & Probab. Lett. **137** (2018), 269-276.

[23] L. X. Zhang, *The convergencee of the sums of independent random variables under the sub-linear expectation*, (2019). Available at https://arxiv.org/abs/1902.10872.

Kyo-Shin Hwang

Graduate School of Education, Yeungnam University, Gyeongsan 38541, Korea

E-mail address: kshwang@yu.ac.kr