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SOME INEQUALITIES FOR GENERAL SUM–CONNECTIVITY INDEX

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ABSTRACT. Let G be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$ sequences of vertex and edge degrees, respectively. If vertices v_i and v_j are adjacent, we write $i \sim j$. The general sum–connectivity index is defined as $\chi_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha$, where α is an arbitrary real number. Firstly, we determine a relation between $\chi_\alpha(G)$ and $\chi_{\alpha-1}(G)$. Then we use it to obtain some new bounds for $\chi_\alpha(G)$.

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1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with n vertices, $m \geq 1$ edges, and a sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$. Let $e = \{v_i, v_j\}$ be an arbitrary edge in G connecting vertices v_i and v_j. The degree of an edge e is defined as $d(e) = d_i + d_j - 2$. Denote by $\Delta_e = d(e_1) + 2 \geq d(e_2) + 2 \geq \cdots \geq d(e_m) + 2 = \delta_e$ be a sequence of modified edge degrees in G. If vertices v_i and v_j are adjacent we write $i \sim j$.

In graph theory, an invariant is a a numerical quantity associated with graphs that depends only on their abstract structure, not on the labeling of vertices or edges. In mathematical chemistry, such quantities are also referred to as topological indices. Topological indices are an important class of molecular structure descriptors used for quantifying information on molecules. Hundreds of topological indices have been introduced in order to describe physical and chemical properties of molecules. Various mathematical properties of topological indices have been investigated, as well. As topological indices have been defined for

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quantifying information of graphs, this area could be classified into the so-called quantitative graph theory [4].

One of the most popular and extensively studied graph based molecular structure descriptors is the first Zagreb index introduced by Gutman and Trinajstić in [10]. It is defined as

$$M_1(G)=\sum_{i=1}^{n} d_i^2 .$$

Recently [8], a graph invariant similar to M_1 came into the focus of attention, defined as

$$F(G)=\sum_{i=1}^{n} d_i^3=\sum_{i\sim j}(d_i^2+d_j^2)\,,$$

which for historcal reasons [9] was named *forgotten* topological index.

Zhou and Trinajstić [32] introduced general sum–connectivity index, conceived as

$$\chi_\alpha(G)=\sum_{i\sim j}(d_i+d_j)^\alpha,$$

where α is an arbitrary real number. It can be easily seen that

$$\chi_\alpha(G)=\sum_{i=1}^{m}(d(e_i)+2)^\alpha\,.$$

For specific values of α, specific notations (and hence specific names) are being used. Here we list some particular indices of this kind for which we are interested in.

- Sum–connectivity index [30], obtained for $\alpha=-1/2$,

$$SC(G)=\chi_{-1/2}(G)=\sum_{i\sim j}\frac{1}{\sqrt{d_i+d_j}}=\sum_{i=1}^{m}\frac{1}{\sqrt{d(e_i)+2}}\,.$$

- Harmonic index [6], obtained for $\alpha=-1$,

$$H(G)=2\chi_{-1}(G)=\sum_{i\sim j}\frac{2}{d_i+d_j}=\sum_{i=1}^{m}\frac{2}{d(e_i)+2}\,.$$

- First Zagreb index, obtained for $\alpha=1$,

$$M_1(G)=\chi_1(G)=\sum_{i\sim j}(d_i+d_j)=\sum_{i=1}^{m}(d(e_i)+2)\,.$$

- Hyper–Zagreb index [26] obtained for $\alpha=2$,

$$HM(G)=\chi_2(G)=\sum_{i\sim j}(d_i+d_j)^2=\sum_{i=1}^{m}(d(e_i)+2)^2\,.$$

It is not difficult to observe that

$$HM(G)=F(G)+2M_2(G)\,,$$

where

$$M_2(G) = \sum_{i\sim j} d_i d_j$$

is the second Zagreb index defined in [11].

- For $\alpha = 3$, one could obtain

$$\chi_3(G) = \sum_{i\sim j} (d_i + d_j)^3 = \sum_{i=1}^{m} (d(e_i) + 2)^3 \,.$$

It can be verified that the following identity is valid

$$\chi_3(G) = EF(G) + 6HM(G) - 12M_1(G) + 8m,$$

where

$$EF(G) = \sum_{i=1}^{m} d(e_i)^3$$

is reformulated forgotten topological index [14].

More on these and some other topological indices one can find, for example, in [1, 2, 12, 15–18, 20, 21].

In this paper we determine a relationship between $\chi_\alpha(G)$ and $\chi_{\alpha-1}(G)$ and based on it obtain some new bounds for $\chi_\alpha(G)$.

2. Preliminaries

In this section we recall some results from the literature that are of interest for our work.

Lemma 2.1. [22] *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequence. Then for any real r such that $r \geq 1$ or $r \leq 0$ holds*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r . \tag{1}$$

If $0 \leq r \leq 1$, then (1) reverses. Equality holds if and only if either $r = 0$, or $r = 1$, or for some t, $1 \leq t \leq n-1$, holds $p_1 = p_2 = \cdots = p_t = 0$ and $a_{t+1} = a_{t+2} = \cdots = a_n$.

The following relation between $\chi_\alpha(G)$ and $M_1(G)$ was determined in [32].

Lemma 2.2. [32] *Let G be a simple graph of size m. The for any α, $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$\chi_\alpha(G) \geq \frac{M_1(G)^\alpha}{m^{\alpha-1}}. \tag{2}$$

If $0 \leq \alpha \leq 1$, then the opposite inequality holds. Equality is attained if and only if G is regular or semiregular bipartite graph.

The inequality (2) for $\alpha = 2$ was proven in [8] (see also [7])

Lemma 2.3. [8] *Let G be a simple graph of size m. Then*

$$HM(G) \geq \frac{M_1(G)^2}{m}, \tag{3}$$

with equality holding if and only if G is regular or semiregular bipartite graph.

The following relationship between $\chi_\alpha(G)$ and $HM(G)$ and $M_1(G)$ was established in [20].

Lemma 2.4. [20]. *Let G be a simple graph of size m. Then for any α, $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$\chi_\alpha(G) \geq \frac{HM(G)^{\alpha-1}}{M_1(G)^{\alpha-2}}. \tag{4}$$

If $1 \leq \alpha \leq 2$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or G is regular or semiregular bipartite graph.

The inequality (4) for $\alpha = 3$ was proven in [25].

Lemma 2.5. [25] *Let G be a simple connected graph. Then*

$$\chi_3(G) \geq \frac{HM(G)^2}{M_1(G)}. \tag{5}$$

Equality is attained if and only if G is regular or semiregular bipartite graph.

3. Main results

In the next theorem we determine a relationship between $\chi_\alpha(G)$ and $\chi_{\alpha-1}(G)$, $M_1(G)$ and $H(G)$.

Theorem 3.1. *Let G be a simple connected graph with $m \geq 1$ edges. Then for any real α such that $\alpha \leq 0$ or $\alpha \geq 1$ holds*

$$\begin{aligned}\chi_\alpha(G) \geq \max\Bigg\{ & \frac{(M_1(G)-m)^\alpha}{\left(m - \frac{1}{2}H(G)\right)^{\alpha-1}} + \chi_{\alpha-1}(G), \\ & \frac{(M_1(G)+m)^\alpha}{\left(m + \frac{1}{2}H(G)\right)^{\alpha-1}} - \chi_{\alpha-1}(G)\Bigg\}.\end{aligned} \tag{6}$$

If $0 \leq \alpha \leq 1$, then

$$\chi_\alpha(G) \leq \min\left\{ \frac{(M_1(G)-m)^\alpha}{\left(m - \frac{1}{2}H(G)\right)^{\alpha-1}} + \chi_{\alpha-1}(G), \frac{(M_1(G)+m)^\alpha}{\left(m + \frac{1}{2}H(G)\right)^{\alpha-1}} - \chi_{\alpha-1}(G)\right\}.$$

Equalities hold if and only if either $\alpha = 0$, or $\alpha = 1$, or G is regular or semiregular bipartite graph.

Proof. For $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $n = m$, $p_i = 1 - \frac{1}{d(e_i)+2}$, $a_i = d(e_i) + 2$, $i = 1, 2, \dots, m$, the inequality (1) becomes

$$\left(\sum_{i=1}^{m}\left(1 - \frac{1}{d(e_i)+2}\right)\right)^{\alpha-1} \sum_{i=1}^{m}\left(1 - \frac{1}{d(e_i)+2}\right)(d(e_i)+2)^\alpha$$

$$\geq \left(\sum_{i=1}^{m} \left(1 - \frac{1}{d(e_i)+2}\right)(d(e_i)+2) \right)^{\alpha},$$

that is

$$\sum_{i=1}^{m} (d(e_i)+1)(d(e_i)+2)^{\alpha-1} \geq \frac{(M_1(G)-m)^{\alpha}}{\left(m - \frac{1}{2}H(G)\right)^{\alpha-1}}. \tag{7}$$

On the other hand, for any real α we have that

$$\chi_{\alpha}(G) - \chi_{\alpha-1}(G) = \sum_{i=1}^{m} (d(e_i)+1)(d(e_i)+2)^{\alpha-1}. \tag{8}$$

From (7) and (8) follows

$$\chi_{\alpha}(G) \geq \chi_{\alpha-1}(G) + \frac{(M_1(G)-m)^{\alpha}}{\left(m - \frac{1}{2}H(G)\right)^{\alpha-1}}. \tag{9}$$

Now, for $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $n = m$, $p_i = 1 + \frac{1}{d(e_i)+2}$, $a_i = d(e_i)+2$, $i = 1, 2, \ldots, m$, the inequality (1) transforms into

$$\left(\sum_{i=1}^{m} \left(1 + \frac{1}{d(e_i)+2}\right) \right)^{\alpha-1} \sum_{i=1}^{m} (d(e_i)+3)(d(e_i)+2)^{\alpha-1} \geq \left(\sum_{i=1}^{m} (d(e_i)+3) \right)^{\alpha},$$

i.e.

$$\sum_{i=1}^{m} (d(e_i)+3)(d(e_i)+2)^{\alpha-1} \geq \frac{(M_1(G)+m)^{\alpha}}{\left(m + \frac{1}{2}H(G)\right)^{\alpha-1}}. \tag{10}$$

For any real α we have that

$$\chi_{\alpha}(G) + \chi_{\alpha-1}(G) = \sum_{i=1}^{m} (d(e_i)+3)(d(e_i)+2)^{\alpha-1}.$$

From the above and (10) we get

$$\chi_{\alpha}(G) + \chi_{\alpha-1}(G) \geq \frac{(M_1(G)+m)^{\alpha}}{\left(m + \frac{1}{2}H(G)\right)^{\alpha-1}}. \tag{11}$$

The inequality (6) is obtained according to (9) and (11).

Equalities in (9) and (11) hold if and only if either $\alpha = 0$, or $\alpha = 1$, or $d(e_1)+2 = d(e_2)+2 = \cdots = d(e_m)+2$, which implies that equality in (6) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or G is regular or semiregular bipartite graph.

By a similar procedure we prove inequality when $0 \leq \alpha \leq 1$. □

Corollary 3.2. *Let G be a simple connected graph with $m \geq 1$ edges. Then for every real $\alpha \geq 1$ holds*

$$\chi_{\alpha}(G) \geq \chi_{\alpha-1}(G) + (M_1(G)-m)\left(\frac{M_1(G)}{m}\right)^{\alpha-1}. \tag{12}$$

Equality holds if and only if $\alpha = 1$ or G is regular or semiregular bipartite graph.

Proof. According to the arithmetic–harmonic mean inequality for real numbers [23], we have that

$$\frac{1}{2}H(G)M_1(G) \geq m^2. \tag{13}$$

From the above inequality follows

$$\frac{M_1(G)-m}{m-\frac{1}{2}H(G)} \geq \frac{M_1(G)}{m},$$

wherefrom we obtain that for any real $\alpha \geq 1$ holds

$$\left(\frac{M_1(G)-m}{m-\frac{1}{2}H(G)}\right)^{\alpha-1} \geq \left(\frac{M_1(G)}{m}\right)^{\alpha-1}.$$

From the above and (6) we arrive at (12). □

The inequality (13) was proven in [13] (see also [27, 28]).

For $\alpha = 2$ and $\alpha = 3$ from (6) and (12) the following corollary of Theorem 3.1 is obtained.

Corollary 3.3. *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$F(G) \geq M_1(G) - 2M_2(G) + \frac{(M_1(G)-m)^2}{m-\frac{1}{2}H(G)}, \tag{14}$$

$$\chi_3(G) \geq HM(G) + \frac{(M_1(G)-m)^3}{\left(m-\frac{1}{2}H(G)\right)^2}$$

and

$$\chi_3(G) \geq HM(G) + (M_1(G)-m)\left(\frac{M_1(G)}{m}\right)^2.$$

Equalities hold if and only if G is regular or semiregular bipartite graph.

Corollary 3.4. *Let G be a simple connected graph with $m \geq 1$ edges. Then for any real α such that $\alpha \leq 0$ or $\alpha \geq 1$ holds*

$$\chi_\alpha(G) \geq \frac{1}{2}\left(\frac{(M_1(G)-m)^\alpha}{\left(m-\frac{1}{2}H(G)\right)^{\alpha-1}} + \frac{(M_1(G)+m)^\alpha}{\left(m+\frac{1}{2}H(G)\right)^{\alpha-1}}\right).$$

If $0 \leq \alpha \leq 1$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or G is regular or semiregular bipartite graph.

Corollary 3.5. *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$SC(G) \geq \frac{1}{2}\left(\frac{\left(m-\frac{1}{2}H(G)\right)^{3/2}}{(M_1(G)-m)^{1/2}} + \frac{\left(m+\frac{1}{2}H(G)\right)^{3/2}}{(M_1(G)+m)^{1/2}}\right),$$

$$HM(G) \geq \frac{1}{2}\left(\frac{(M_1(G)-m)^2}{m-\frac{1}{2}H(G)} + \frac{(M_1(G)+m)^2}{m+\frac{1}{2}H(G)}\right),$$

$$\chi_3(G) \geq \frac{1}{2}\left(\frac{(M_1(G)-m)^3}{\left(m-\frac{1}{2}H(G)\right)^2}+\frac{(M_1(G)+m)^3}{\left(m+\frac{1}{2}H(G)\right)^2}\right).$$

Equalities hold if and only if G is regular or semiregular bipartite graph.

By a similar procedure as in case of Theorem 3.1, the following results are proved.

Theorem 3.6. *Let G be a simple connected graph with $m \geq 1$ edges. Then for any real α such that $\alpha \leq 1$ or $\alpha \geq 2$ holds*

$$\chi_\alpha(G) \geq \max\left\{\frac{(HM(G)-M_1(G))^{\alpha-1}}{(M_1(G)-m)^{\alpha-2}}+\chi_{\alpha-1}(G),\right. \left.\frac{(HM(G)+M_1(G))^{\alpha-1}}{(M_1(G)+m)^{\alpha-2}}-\chi_{\alpha-1}(G)\right\}. \tag{15}$$

If $1 \leq \alpha \leq 2$, then

$$\chi_\alpha(G) \leq \min\left\{\frac{(HM(G)-M_1(G))^{\alpha-1}}{(M_1(G)-m)^{\alpha-2}}+\chi_{\alpha-1}(G),\right. \left.\frac{(HM(G)+M_1(G))^{\alpha-1}}{(M_1(G)+m)^{\alpha-2}}-\chi_{\alpha-1}(G)\right\}.$$

Equalities hold if and only if either $\alpha = 1$, or $\alpha = 2$, or G is regular or semiregular bipartite graph.

Corollary 3.7. *Let G be a simple connected graph with $m \geq 1$ edges. Then for any real α such that $\alpha \leq 1$ or $\alpha \geq 2$ holds*

$$\chi_\alpha(G) \geq \frac{1}{2}\left(\frac{(HM(G)-M_1(G))^{\alpha-1}}{(M_1(G)-m)^{\alpha-2}}+\frac{(HM(G)+M_1(G))^{\alpha-1}}{(M_1(G)+m)^{\alpha-2}}\right).$$

For $1 \leq \alpha \leq 2$, the sense of the above inequality reverses. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or G is regular or semiregular bipartite graph.

Corollary 3.8. *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$SC(G) \geq \frac{1}{2}\left(\frac{(M_1(G)-m)^{5/2}}{(HM(G)-M_1(G))^{3/2}}+\frac{(M_1(G)+m)^{5/2}}{(HM(G)+M_1(G))^{3/2}}\right),$$

$$\chi_3(G) \geq \frac{1}{2}\left(\frac{(HM(G)-M_1(G))^2}{M_1(G)-m}+\frac{(HM(G)+M_1(G))^2}{M_1(G)+m}\right),$$

$$\chi_3(G) \geq HM(G)+\frac{(HM(G)-M_1(G))^2}{M_1(G)-m}. \tag{16}$$

Equalities hold if and only if G is regular or semiregular bipartite graph.

In the following theorems we establish lower bounds for $\chi_\alpha(G)$ in terms of $M_1(G)$ and parameters m, Δ_e and δ_e.

Theorem 3.9. *Let G be a simple connected graph with $m \geq 3$ edges. Then for any real α, $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$\chi_\alpha(G) \geq \Delta_e^\alpha + \delta_e^\alpha + \frac{(M_1(G) - \Delta_e - \delta_e)^\alpha}{(m-2)^{\alpha-1}}. \tag{17}$$

For $0 \leq \alpha \leq 1$, the sense of inequality reverses. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $d(e_2) + 2 = d(e_3) + 2 = \cdots = d(e_{m-1}) + 2$.

Proof. The inequality (1) can be considered as

$$\left(\sum_{i=2}^{m-1} p_i\right)^{r-1} \sum_{i=2}^{m-1} p_i a_i^r \geq \left(\sum_{i=2}^{m-1} p_i a_i\right)^r. \tag{18}$$

For $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $p_i = 1$, $a_i = d(e_i) + 2$, $i = 2, 3, \ldots, m-1$, the above inequality becomes

$$(m-2)^{\alpha-1} \sum_{i=2}^{m-1} (d(e_i) + 2)^\alpha \geq \left(\sum_{i=2}^{m-1} (d(e_i) + 2)\right)^\alpha, \tag{19}$$

that is

$$(m-2)^{\alpha-1} (\chi_\alpha(G) - \Delta_e^\alpha - \delta_e^\alpha) \geq (M_1(G) - \Delta_e - \delta_e)^\alpha,$$

wherefrom (17) is obtained.

In a similar way we obtain that in (17) the opposite inequality holds for $0 \leq \alpha \leq 1$.

Equality in (19), i.e. in (17), holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $d(e_2) + 2 = d(e_3) + 2 = \cdots = d(e_{m-1}) + 2$. □

By the similar arguments as in case of Theorem 3.9 we prove the following results.

Theorem 3.10. *Let G be a simple connected graph with $m \geq 2$ edges. Then for any real α, such that $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$\chi_\alpha(G) \geq \max\left\{\Delta_e^\alpha + \frac{(M_1(G) - \Delta_e)^\alpha}{(m-1)^{\alpha-1}}, \delta_e^\alpha + \frac{(M_1(G) - \delta_e)^\alpha}{(m-1)^{\alpha-1}}\right\}.$$

If $0 \leq \alpha \leq 1$, then

$$\chi_\alpha(G) \leq \min\left\{\Delta_e^\alpha + \frac{(M_1(G) - \Delta_e)^\alpha}{(m-1)^{\alpha-1}}, \delta_e^\alpha + \frac{(M_1(G) - \delta_e)^\alpha}{(m-1)^{\alpha-1}}\right\}.$$

Equalities hold if and only if either $\alpha = 0$, or $\alpha = 1$, or $d(e_2) = d(e_3) = \cdots = d(e_m)$, or $d(e_1) = d(e_2) = \cdots = d(e_{m-1})$.

Corollary 3.11. *Let G be a simple connected graph. Then*

$$SC(G) \geq \frac{\sqrt{\Delta_e} + \sqrt{\delta_e}}{\sqrt{\Delta_e \delta_e}} + \frac{(m-2)^{3/2}}{(M_1(G) - \Delta_e - \delta_e)^{1/2}} \quad (m \geq 2), \tag{20}$$

$$
\begin{aligned}
SC(G) \geq \max \Bigg\{ & \frac{1}{\sqrt{\Delta_e}} + \frac{(m-1)^{\frac{3}{2}}}{(M_1(G)-\Delta_e)^{\frac{1}{2}}}, \\
& \frac{1}{\sqrt{\delta_e}} + \frac{(m-1)^{\frac{3}{2}}}{(M_1(G)-\delta_e)^{\frac{1}{2}}} \Bigg\} \quad (m \geq 2),
\end{aligned} \tag{21}
$$

$$
HM(G) \geq \Delta_e^2 + \delta_e^2 + \frac{(M_1(G)-\Delta_e-\delta_e)^2}{m-2} \quad (m \geq 3), \tag{22}
$$

$$
\begin{aligned}
HM(G) \geq \max \Bigg\{ & \Delta_e^2 + \frac{(M_1(G)-\Delta_e)^2}{m-1}, \\
& \delta_e^2 + \frac{(M_1(G)-\delta_e)^2}{m-1} \Bigg\} \quad (m \geq 2),
\end{aligned} \tag{23}
$$

$$
\chi_3(G) \geq \Delta_e^3 + \delta_e^3 + \frac{(M_1(G)-\Delta_e-\delta_e)^3}{(m-2)^2} \quad (m \geq 3), \tag{24}
$$

$$
\chi_3(G) \geq \max \left\{ \Delta_e^3 + \frac{(M_1(G)-\Delta_e)^3}{(m-1)^2}, \delta_e^3 + \frac{(M_1(G)-\delta_e)^3}{(m-1)^2} \right\} \quad (m \geq 2). \tag{25}
$$

Equalities in (20), (22) *and* (24) *hold if and only if* $d(e_2)+2 = d(e_3)+2 = \cdots = d(e_{m-1})+2$, *while in* (21), (23) *and* (25) *if and only if* $d(e_2)+2 = d(e_3)+2 = \cdots = d(e_m)+2$, *or* $d(e_1)+2 = d(e_2)+2 = \cdots = d(e_{m-1})+2$.

Remark 3.1. It can be easily observed that for $r = \alpha$, $n = m$, $p_i = 1$, $a_i = d(e_i)+2$, $i = 1, 2, \dots, m$, from (1) the inequality (2) is obtained. The inequalities proven in Theorems 3.9 and 3.10 are stronger than (2). Similarly, the inequalities (22) and (23) are stronger than (3).

In the next theorem we determine lower bound for $\chi_\alpha(G)$ in terms of $M_1(G)$, $HM(G)$, Δ_e and δ_e.

Theorem 3.12. *Let G be a simple connected graph with $m \geq 3$ edges. Then for any real number α, $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$
\chi_\alpha(G) \geq \Delta_e^\alpha + \delta_e^\alpha + \frac{(HM(G)-\Delta_e^2-\delta_e^2)^{\alpha-1}}{(M_1(G)-\Delta_e-\delta_e)^{\alpha-2}}. \tag{26}
$$

For $1 \leq \alpha \leq 2$, the sense of the above inequality reverses. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $d(e_2)+2 = d(e_3)+2 = \cdots = d(e_{m-1})+2$.

Proof. For $r = \alpha - 1$, $\alpha \leq 1$ or $\alpha \geq 2$, $p_i = d(e_i)+2$, $a_i = d(e_i)+2$, $i = 2, 3, \dots, m-1$, the inequality (18) becomes

$$
\left(\sum_{i=2}^{m-1} (d(e_i)+2) \right)^{\alpha-2} \sum_{i=2}^{m-1} (d(e_i)+2)^\alpha \geq \left(\sum_{i=2}^{m-1} (d(e_i)+2)^2 \right)^{\alpha-1}, \tag{27}
$$

that is

$$(M_1(G) - \Delta_e - \delta_e)^{\alpha-2} (\chi_\alpha(G) - \Delta_e^\alpha - \delta_e^\alpha) \geq \left(HM(G) - \Delta_e^2 - \delta_e^2\right)^{\alpha-1},$$

wherefrom (26) follows.

Similarly, we conclude that in (26) the opposite inequality holds for $1 \leq \alpha \leq 2$.

Equality in (27), i.e. in (26), holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $d(e_2) + 2 = d(e_3) + 2 = \cdots = d(e_{m-1}) + 2$. □

The proof of the next theorem is fully analogous to that of Theorem 3.12, hence omitted.

Theorem 3.13. *Let G be a simple connected graph with $m \geq 2$ edges. Then for any real α, such that $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$\chi_\alpha(G) \geq \max\left\{\Delta_e^\alpha + \frac{\left(HM(G) - \Delta_e^2\right)^{\alpha-1}}{(M_1(G) - \Delta_e)^{\alpha-2}}, \delta_e^\alpha + \frac{\left(HM(G) - \delta_e^2\right)^{\alpha-1}}{(M_1(G) - \delta_e)^{\alpha-2}}\right\}.$$

If $1 \leq \alpha \leq 2$, then

$$\chi_\alpha(G) \leq \min\left\{\Delta_e^\alpha + \frac{\left(HM(G) - \Delta_e^2\right)^{\alpha-1}}{(M_1(G) - \Delta_e)^{\alpha-2}}, \delta_e^\alpha + \frac{\left(HM(G) - \delta_e^2\right)^{\alpha-1}}{(M_1(G) - \delta_e)^{\alpha-2}}\right\}.$$

Equalities hold if and only if either $\alpha = 1$, or $\alpha = 2$, or $d(e_2) = d(e_3) = \cdots = d(e_m)$, or $d(e_1) = d(e_2) = \cdots = d(e_{m-1})$.

Remark 3.2. For $r = \alpha - 1$, $n = m$, $\alpha \leq 1$ or $\alpha \geq 2$, $p_i = a_i = d(e_i) + 2$, $i = 1, 2, \ldots, m$, from (1) we get (4). Therefore, the inequalities proven in Theorems 3.12 and 3.13 are stronger than inequality (4).

Corollary 3.14. *Let G be a simple connected graph with m edges. Then*

$$\chi_3(G) \geq \Delta_e^3 + \delta_e^3 + \frac{\left(HM(G) - \Delta_e^2 - \delta_e^2\right)^2}{M_1(G) - \Delta_e - \delta_e} \quad (m \geq 3), \tag{28}$$

with equality holding if and only if $d(e_2) = d(e_3) = \cdots = d(e_{m-1})$, and

$$\chi_3(G) \geq \max\left\{\Delta_e^3 + \frac{\left(HM(G) - \Delta_e^2\right)^2}{M_1(G) - \Delta_e}, \delta_e^3 + \frac{\left(HM(G) - \delta_e^2\right)^2}{M_1(G) - \delta_e}\right\} \quad (m \geq 2), \tag{29}$$

with equality holding if and only if $d(e_2) = d(e_3) = \cdots = d(e_m)$, or $d(e_1) = d(e_2) = \cdots = d(e_{m-1})$.

The inequalities (28) and (29) are stronger than (5).

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