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SOME IDENTITIES INVOLVING THE GENERALIZED POLYNOMIALS OF DERANGEMENTS ARISING FROM DIFFERENTIAL EQUATION†

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Abstract. In this paper we define a new generalized polynomials of derangements. It also derives the differential equations that occur in the generating function of the generalized polynomials of derangements. We establish some new identities for the generalized polynomials of derangements. Finally, we perform a survey of the distribution of zeros of the generalized polynomials of derangements.

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1. Introduction

Many mathematicians have studied in the area of the special numbers and polynomials. Many generalizations of these polynomials have been studied(see [1, 2, 3, 4, 5, 6, 8, 9]). The numbers of derangements D_n are defined by the generating function(see [1, 2]):

$$\frac{e^{-t}}{1-t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.$$

Clark and Sved [1] obtained an interesting relationship between the number of derangements D_n and the Bell numbers, as follows,

$$\sum_{k=0}^{n} \binom{n}{k} k^s D_k = n! \sum_{k=0}^{s} \binom{s}{k} (-1)^k n^{s-k} B_k, \quad 0 \le s \le n,$$

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where B_n is the familiar Bell numbers satisfying the generating function:

$$e^{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

The following interesting properties of D_n can be obtained easily,

$$\begin{aligned}
&\frac{D_n}{n!}=1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!},\\
&\lim_{n\to\infty}\frac{D(n)}{n!}=\frac{1}{e},\\
&\sum_{k=0}^{n}\binom{n}{k}D_k=n!,\\
&D_n=\frac{\Gamma(n+1,-1)}{e},
\end{aligned}$$

where $\Gamma(z,a)$ is the incomplete gamma function(see [1, 2]). The polynomials of derangements $D_n(x)$ are defined by the generating function:

$$\left(\frac{e^{-t}}{1-t}\right)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}. \tag{1.1}$$

Note that, by taking $x=1$, (1.1) gives $D_n(1)=D_n$.

The generalized polynomials of derangements $D_n^{(\alpha)}(x)$ are defined by the generating function:

$$\left(\frac{e^{-t}}{1-t}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}D_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.2}$$

In the special case $x=0$ and $\alpha=1$, we have $D_n^{(1)}(0)=D_n$. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [9])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k \text{ and } x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k, \tag{1.3}$$

respectively. Here $(x)_n=x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. If x is a variable, we use the following notation:

$$<x>_k=x(x+1)\cdots(x+k-1),\quad \binom{x}{k}=\frac{(x)_k}{k!},\quad (1+t)^x=\sum_{k=0}^{\infty}\binom{x}{k}t^k. \tag{1.4}$$

Recently, in order to give explicit identities for special polynomials, differential equations arising from the generating functions of special polynomials are studied by many authors(see [10, 11, 12, 13]). Inspired by their work, we construct a differential equations by generating function of generalized polynomials of derangements as follow. Let D denote differentiation with respect to t, D^2 denote differentiation twice with respect to t, and so on; that is, for positive integer N,

$$D^N F = \left(\frac{\partial}{\partial t}\right)^N F(t,x,\alpha).$$

We derive a differential equations with coefficients $a_i(N,x,\alpha)$, which is satisfied by

$$\left(\frac{\partial}{\partial t}\right)^N F(t,x,\alpha) - a_0(N,x,\alpha)(1-t)^{-N}F(t,x,\alpha) - \cdots - a_N(N,x,\alpha)F(t,x,\alpha) = 0.$$

For $0 \le i \le N$, by using the coefficients $a_i(N,x,\alpha)$ of this differential equation, we have explicit identities for the generalized polynomials of derangements. This paper is organized as follows. In Sect.2, we construct differential equations arising from the generating functions of generalized polynomials of derangements. We establish some new identities for the generalized polynomials of derangements. In Sect.3, using numerical methods, we investigate the structure of zeros of the generalized polynomials of derangements.

2. Differential equations associated with the generalized polynomials of derangements

In this section, we consider differential equations arising from the generating functions of the generalized polynomials of derangements. Let

$$F = F(t,x,\alpha) = \left(\frac{e^{-t}}{1-t}\right)^{\alpha} e^{xt}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}
F^{(1)} &= \frac{d}{dt}F(t,x,\alpha) = \frac{d}{dt}\left(\frac{e^{-t}}{1-t}\right)^{\alpha} e^{xt} \\
&= \alpha\left(\frac{e^{-t}}{1-t}\right)^{\alpha-1}\left\{\frac{-e^{-t}}{1-t} + \frac{-e^{-t}}{(1-t)^2}\right\} e^{xt} + x\left(\frac{e^{-t}}{1-t}\right)^{\alpha} e^{xt} \\
&= (x-\alpha)F(t,x,\alpha) + \alpha(1-t)^{-1}F(t,x,\alpha), \\
&= ((x-\alpha) + \alpha(1-t)^{-1})F(t,x,\alpha), \qquad (2.2)\\
F^{(2)} &= \frac{d}{dt}F^{(1)} \\
&= \alpha(1-t)^{-2}F(t,x,\alpha) + ((x-\alpha)+\alpha(1-t)^{-1})F^{(1)}(t,x,\alpha) \\
&= \alpha(1-t)^{-2}F(t,x,\alpha) + ((x-\alpha)+\alpha(1-t)^{-1})^2 F(t,x,\alpha), \\
&= \left\{(x-\alpha) + 2\alpha(x-\alpha)(1-t)^{-1} + (\alpha^2+\alpha)(1-t)^{-2}\right\} F(t,x,\alpha),
\end{aligned}$$

and

$$\begin{aligned}
F^{(3)} &= \frac{d}{dt}F^{(2)} \\
&= \left\{2\alpha(x-\alpha)(1-t)^{-2} + 2(\alpha^2+\alpha)(1-t)^{-3}\right\} F(t,x,\alpha), \\
&\quad + \left\{(x-\alpha)^2 + 2\alpha(x-\alpha)(1-t)^{-1} + (\alpha^2+\alpha)(1-t)^{-2}\right\} F^{(1)}(t,x,\alpha) \\
&= (x+\alpha)^3 F(t,x,\alpha) \\
&\quad + \left\{\alpha(x-\alpha)^2 + 2\alpha(x-\alpha)^2\right\}(1-t)^{-1}F(t,x,\alpha) \\
&\quad + \left\{2\alpha(x-\alpha) + (\alpha^2+\alpha)(x-\alpha) + 2\alpha^2(x-\alpha)\right\}(1-t)^{-2}F(t,x,\alpha) \\
&\quad + \left\{2\alpha^3 + 2\alpha^2\right\}(1-t)^{-3}F(t,x,\alpha).
\end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned}
F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x,\alpha) \\
&= \sum_{i=0}^{N} a_i(N,x,\alpha)(1-t)^{-N+i}F(t,x,\alpha), \quad (N=1,2,\ldots).
\end{aligned} \tag{2.4}$$

Differentiating (2.4) with respect to t, we have

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \sum_{i=0}^{N} a_i(N,x,\alpha)(N-i)(1-t)^{-N+i-1}F(t,x,\alpha) \\
&\quad + \sum_{i=0}^{N} a_i(N,x,\alpha)(1-t)^{-N+i}F^{(1)}(t,x,\alpha) \\
&= \sum_{i=0}^{N} a_i(N,x,\alpha)(\alpha+N-i)(1-t)^{-N+i-1}F(t,x,\alpha) \\
&\quad + \sum_{i=1}^{N+1} (x-\alpha)a_{i-1}(N,x,\alpha)(1-t)^{-N+i-1}F(t,x,\alpha).
\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,\alpha)(1-t)^{-N+i-1}F(t,x,\alpha). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}&\sum_{i=0}^{N}(-1)^{-i}a_i(N,x,\alpha)(\alpha+N-i)(1-t)^{-N+i-1}F(t,x,\alpha)\\&\qquad+\sum_{i=1}^{N+1}(x-\alpha)a_{i-1}(N,x,\alpha)(1-t)^{-N+i-1}F(t,x,\alpha)\\&=\sum_{i=0}^{N+1}a_i(N+1,x,\alpha)(1-t)^{-N+i-1}F(t,x,\alpha).\end{aligned}\tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned}&a_0(N+1,x,\alpha)=(\alpha+N)a_0(N,x,\alpha),\\&a_{N+1}(N+1,x,\alpha)=(x-\alpha)a_N(N,x,\alpha),\end{aligned}\tag{2.8}$$

and

$$a_i(N+1,x,\alpha)=(\alpha+N-i)a_i(N,x,\alpha)+(x-\alpha)a_{i-1}(N,x,\alpha),(1\le i\le N).\tag{2.9}$$

By (2.1) and (2.2), we have

$$a_0(0,x,\alpha)=1.$$

In addition, by (2.2) and (2.4), we get

$$\begin{aligned}F^{(1)}&=a_0(1,x,\alpha)(1-t)^{-1}F(t,x,\alpha)+a_1(1,x,\alpha)F(t,x,\alpha)\\&=\alpha(1-t)^{-1}F(t,x,\alpha)+(x-\alpha)F(t,x,\alpha).\end{aligned}\tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(1,x,\alpha)=\alpha,\quad a_1(1,x,\alpha)=x-\alpha.\tag{2.11}$$

It is not difficult to show that

$$\begin{aligned}&\left\{(x-\alpha)^2+2\alpha(x-\alpha)(1-t)^{-1}+(\alpha^2+\alpha)(1-t)^{-2}\right\}F(t,x,\alpha)\\&=\sum_{i=0}^{2}a_i(2,x,\alpha)(1-t)^{-2+i}F(t,x,\alpha)\\&=\left\{a_0(2,x,\alpha)(1-t)^{-2}+a_1(2,x,\alpha)(1-t)^{-1}+a_2(2,x,\alpha)\right\}F(t,x,\alpha).\end{aligned}\tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(2,x,\alpha)=\alpha(\alpha+1),\quad a_1(2,x,\alpha)=2\alpha(x-\alpha),\quad a_2(2,x,\alpha)=(x-\alpha)^2.\tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x,\alpha)=(\alpha+N)a_0(N,x,\alpha)=\cdots=<\alpha>_{N+1},$$

and

$$a_N(N+1,x,\alpha)=\alpha a_{N-1}(N,x,\alpha)=\cdots=(x-\alpha)^{N+1}.\tag{2.14}$$

For $i = 1, 2, 3$ in (2.9), we get

$$a_1(N+1, x, \alpha) = (x-\alpha)\sum_{k=0}^{N}(\alpha+N-1)_k a_0(N-k, x, \alpha),$$

$$a_2(N+1, x, \alpha) = (x-\alpha)\sum_{k=0}^{N-1}(\alpha+N-2)_k a_1(N-k, x, \alpha),$$

and

$$a_3(N+1, x, \alpha) = (x-\alpha)\sum_{k=0}^{N-2}(\alpha+N-2)_k a_2(N-k, x, \alpha).$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1, x, \alpha) = (x-\alpha)\sum_{k=0}^{N+1-i}(\alpha+N-i)_k a_{i-1}(N-k, x, \alpha). \tag{2.15}$$

Note that, here the matrix $a_i(j, x, \alpha)_{0 \le i,j \le N}$ is given by

$$\begin{pmatrix} 1 & <\alpha>_1 & <\alpha>_2 & <\alpha>_3 & \cdots & <\alpha>_N \\ 0 & x-\alpha & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & (x-\alpha)^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & (x-\alpha)^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (x-\alpha)^N \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1, x, \alpha)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_1(N+1, x, \alpha) &= (x-\alpha)\sum_{k_1=0}^{N}(\alpha+N-1)_{k_1} a_0(N-k_1, x, \alpha) \\ &= (x-\alpha)\sum_{k_1=0}^{N}(\alpha+N-1)_{k_1} <\alpha>_{N-k_1} \end{aligned}$$

$$\begin{aligned} a_2(N+1, x, \alpha) &= (x-\alpha)\sum_{k_2=0}^{N-1}(\alpha+N-2)_{k_2} a_1(N-k_2, x, \alpha) \\ &= (x-\alpha)^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_1-1}(\alpha+N-2)_{k_2}(\alpha+N-k_2-2)_{k_1} \\ &\quad \times <\alpha>_{N-k_2-k_1-1}, \end{aligned}$$

and

$$
\begin{aligned}
&a_3(N+1,x,\alpha)\\
&= x\sum_{k_3=0}^{N-2}(\alpha+N-3)_{k_3}a_2(N-k_3,x,\alpha)\\
&= (x-\alpha)^3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}(\alpha+N-3)_{k_3}\\
&\times(\alpha+N-k_3-3)_{k_2}(\alpha+N-k_3-k_2-3)_{k_1} <\alpha>_{N-k_3-k_2-k_1-2}\,.
\end{aligned}
$$

Continuing this process, we have

$$
\begin{aligned}
&a_i(N+1,x,\alpha) = (x-\alpha)^i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1}\\
&\times(\alpha+N-i)_{k_i}(\alpha+N-k_i-i)_{k_{i-1}}\cdots(\alpha+N-k_i-\cdots-k_2-i)_{k_1}\\
&\times <\alpha>_{N-k_i-\cdots-k_2-k_1-i+1}\,.
\end{aligned}
\tag{2.16}
$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.1. *For $N=0,1,2,\ldots,$ the functional equation*

$$F^{(N)} = \sum_{i=0}^{N}a_i(N,x,\alpha)(1-t)^{-N+i}F(t,x,\alpha)$$

has a solution

$$F = F(t,x,\alpha) = \left(\frac{e^{-t}}{1-t}\right)^{\alpha}e^{xt},$$

where

$$
\begin{aligned}
&a_0(N,x,\alpha) = <\alpha>_N,\\
&a_N(N,x,\alpha) = (x-\alpha)^N,\\
&a_i(N,x,\alpha) = (x-\alpha)^i\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(\alpha+N-i-1)_{k_i}\\
&\qquad\times(\alpha+N-k_i-i-1)_{k_{i-1}}\cdots(\alpha+N-k_i-\cdots-k_2-i-1)_{k_1}\\
&\qquad\times <\alpha>_{N-k_i-\cdots-k_2-k_1-i}, (1\le i\le N-1).
\end{aligned}
$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x,\alpha) = \sum_{k=0}^{\infty}D_{k+N}^{(\alpha)}(x)\frac{t^k}{k!}. \tag{2.17}$$

From Theorem 2.1 and (2.17), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{\infty} D_{k+N}^{(\alpha)}(x)\frac{t^k}{k!} &= F^{(N)} \\ &= \sum_{i=0}^{N} a_i(N,x,\alpha)(1-t)^{-N+i}F(t,x,\alpha) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N} a_i(N,x,\alpha)D_k^{(\alpha+N-i)}(x+N-i)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2.2. *For $k=0,1,\ldots,$ and $N=0,1,2,\ldots,$ we have*

$$\sum_{k=0}^{\infty} D_{k+N}^{(\alpha)}(x) = \sum_{i=0}^{N} a_i(N,x,\alpha)D_k^{(\alpha+N-i)}(x+N-i), \tag{2.19}$$

where

$$\begin{aligned}&a_0(N,x,\alpha) = <\alpha>_N, \\ &a_N(N,x,\alpha) = (x-\alpha)^N, \\ &a_i(N,x,\alpha) = (x-\alpha)^i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(\alpha+N-i-1)_{k_i} \\ &\quad\times(\alpha+N-k_i-i-1)_{k_{i-1}}\cdots(\alpha+N-k_i-\cdots-k_2-i-1)_{k_1} \\ &\quad\times <\alpha>_{N-k_i-\cdots-k_2-k_1-i}, (1\le i\le N-1).\end{aligned}$$

By (1.1) and (1.5), we have

$$\begin{aligned}F^{(N)} &= \sum_{k=0}^{\infty} D_{k+N}^{(\alpha)}(x)\frac{t^k}{k!} \\ &= \sum_{i=0}^{N} a_i(N,x,\alpha)\frac{1}{(1-t)^{N-i}}F(t,x,\alpha) \\ &= \sum_{i=0}^{N} a_i(N,x,\alpha)\left(\sum_{l=0}^{\infty}(N-i+l-1)_l\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}D_k^{(\alpha)}(x)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{l=0}^{k}\binom{k}{l}a_i(N,x,\alpha)(N-i+l-1)_l D_{k-l}^{(\alpha)}(x)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.20}$$

By (2.18) and (2.20), we obtain the following theorem.

Theorem 2.3. *For $k = 0, 1, \ldots,$ and $N = 0, 1, 2, \ldots,$ we have*

$$\sum_{k=0}^{\infty} D_{k+N}^{(\alpha)}(x) = \sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} a_i(N, x, \alpha)(N - i + l - 1)_l D_{k-l}^{(\alpha)}(x).$$

If we take $k = 0$ in Theorem 2.3, then we have the following corollary.

Corollary 2.4. *For $N = 0, 1, 2, \ldots,$ we have*

$$D_N^{(\alpha)}(x) = \sum_{i=0}^{N} a_i(N, x, \alpha).$$

By (2.17), we get

$$\begin{aligned} e^{-nt} \left(\frac{d}{dt}\right)^N F(t, x, \alpha) &= \left(\sum_{m=0}^{\infty} (-n)^m \frac{t^m}{m!}\right) \left(\sum_{m=0}^{\infty} D_{m+N}^{(\alpha)}(x) \frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} D_{N+k}^{(\alpha)}(x)\right) \frac{t^m}{m!}. \end{aligned} \tag{2.21}$$

By the Leibniz rule and the inverse relation, we have

$$\begin{aligned} e^{-nt} \left(\frac{d}{dt}\right)^N F(t, x, \alpha) &= \sum_{k=0}^{N} \binom{N}{k} n^{N-k} \left(\frac{d}{dt}\right)^k F(t, x - n, \alpha) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{N} \binom{N}{k} n^{N-k} D_{m+k}^{(\alpha)}(x - n)\right) \frac{t^m}{m!}. \end{aligned} \tag{2.22}$$

Hence, by (2.21) and (2.22), and comparing the coefficients of $\dfrac{t^m}{m!}$ gives the following theorem.

Theorem 2.5. *Let m, n, N be nonnegative integers. Then*

$$\sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} D_{N+k}^{(\alpha)}(x) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} D_{m+k}^{(\alpha)}(x - n). \tag{2.23}$$

Let us take $m = 0$ in (2.23). Then, we have the following corollary.

Corollary 2.6. *For $N = 0, 1, 2, \ldots,$ we have*

$$D_N^{(\alpha)}(x) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} D_k^{(\alpha)}(x - n).$$

For $N = 0, 1, 2, \ldots,$ the functional equation

$$F^{(N)} = \sum_{i=0}^{N} a_i(N, x, \alpha)(1 - t)^{-N+i} F(t, x, \alpha)$$

has a solution

$$F = F(t, x, \alpha) = \left(\frac{e^{-t}}{1-t}\right)^{\alpha} e^{xt}.$$

Here is a plot of the surface for this solution. In Figure 1(left), we plot of the

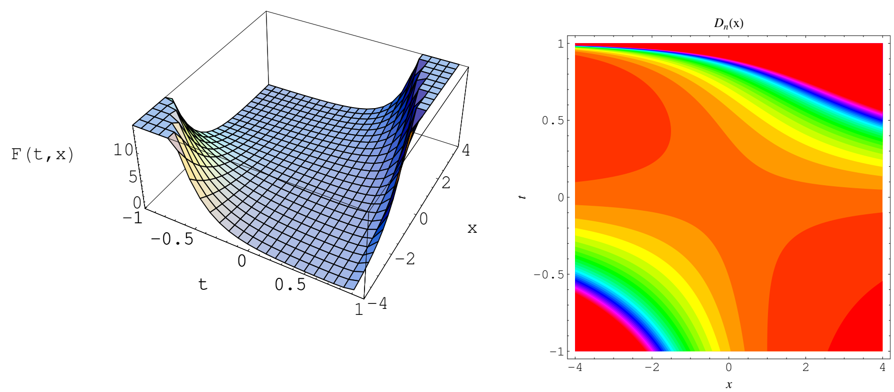

FIGURE 1. The surface for the solution $F(t, x, \alpha)$

surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution. In Figure 1, we choose $\alpha = 2$.

3. Distribution of the zeros of the generalized polynomials of derangements

In this section, we investigate various properties of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$ such as the distribution of the roots and the symmetry of the roots. The first few examples of generalized polynomials of derangements $D_n^{(\alpha)}(x)$ are

$$D_0^{(\alpha)}(x) = 1,$$

$$D_1^{(\alpha)}(x) = x,$$

$$D_2^{(\alpha)}(x) = \alpha + x^2,$$

$$D_3^{(\alpha)}(x) = 2\alpha + 3\alpha x + x^3,$$

$$D_4^{(\alpha)}(x) = 6\alpha + 3\alpha^2 + 8\alpha x + 6\alpha x^2 + x^4,$$

$$D_5^{(\alpha)}(x) = 24\alpha + 20\alpha^2 + 30\alpha x + 15\alpha^2 x + 20\alpha x^2 + 10\alpha x^3 + x^5,$$

$$\begin{aligned} D_6^{(\alpha)}(x) = {} & 120\alpha + 130\alpha^2 + 15\alpha^3 + 144\alpha x + 120\alpha^2 x + 90\alpha x^2 + 45\alpha^2 x^2 + 40\alpha x^3 \\ & + 15\alpha x^4 + x^6. \end{aligned}$$

We display the shapes of the generalized polynomials of derangements $D_n^{(\alpha)}(x)$ and investigate its zeros. We plot the graph of the generalized polynomials of derangements $D_n^{(\alpha)}(x)$ for $n = 1, \cdots, 10$ combined together. The shape of generalized polynomials of derangements $D_n^{(\alpha)}(x)$ for $-20 \le x \le 20$ are displayed in Figure 2.

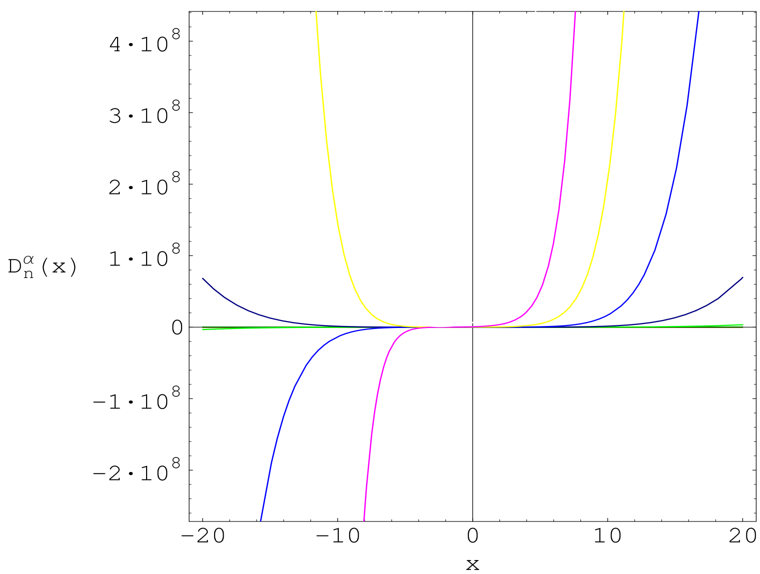

FIGURE 2. Curve of the generalized polynomials of derangements $D_n^{(\alpha)}(x)$

We investigate the zeros of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$ by using a computer. We plot the zeros of the $D_n^{(\alpha)}(x) = 0$ for $n = 20, 40, \alpha = -2, 2$ and $x \in \mathbb{C}$(Figure 3). In Figure 2(top-left), we choose $n = 20$ and $\alpha = 2$. In Figure 2(top-right), we choose $n = 40$ and $\alpha = 2$. In Figure 2(bottom-left), we choose $n = 20$ and $\alpha = -2$. In Figure 2(bottom-right), we choose $n = 40$ and $\alpha = -2$. Prove that $D_n^{(\alpha)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions(see Figure 3).

Stacks of zeros of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$ for $1 \le n \le 30$ from a 3-D structure are presented(Figure 4). In Figure 4(left), we choose $\alpha = 2$. In Figure 4(right), we choose $\alpha = -2$. Our numerical results for approximate solutions of real zeros of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$ are displayed(Tables 1, 2).

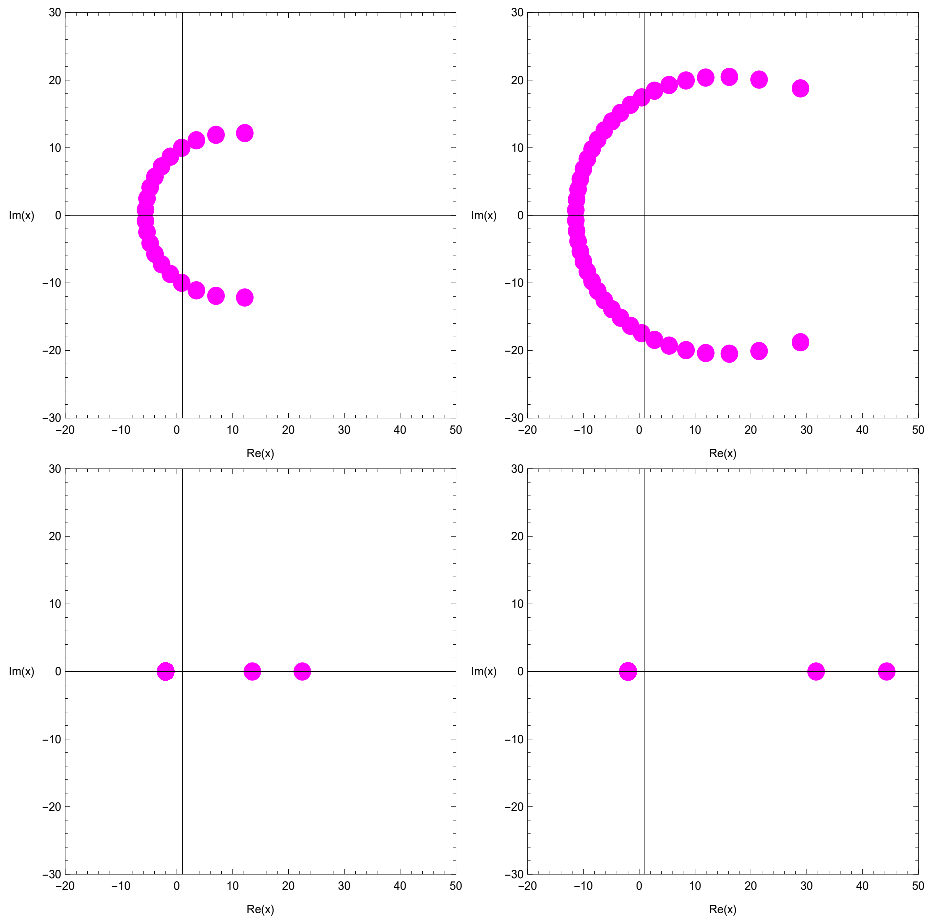

FIGURE 3. Zeros of $D_n^{(\alpha)}(x)$

Table 1. Numbers of real and complex zeros of $D_n^{(\alpha)}(x)$

degree n	$\alpha = 2$		$\alpha = -2$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	0	2	2	0
3	1	2	3	0
4	0	4	4	0
5	1	4	5	0
6	0	6	6	0
7	1	6	7	0
8	0	8	8	0
9	1	8	9	0
10	0	10	10	0

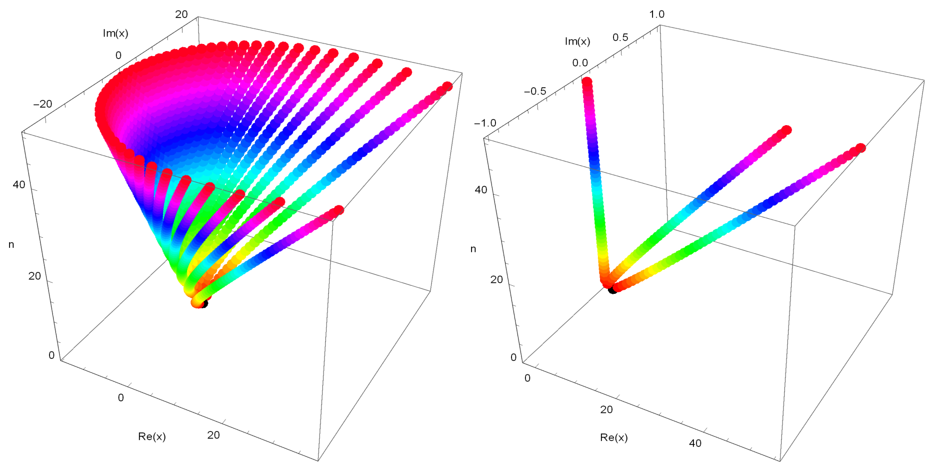

FIGURE 4. Stacks of zeros of $D_n^{(\alpha)}(x), 1 \leq n \leq 30$

Plot of real zeros of $D_n^{(\alpha)}(x) = 0$ for $1 \leq n \leq 50$ structure are presented(Figure 5). In Figure 5(left), we choose $\alpha = 2$. In Figure 5(right), we choose $\alpha = -2$.

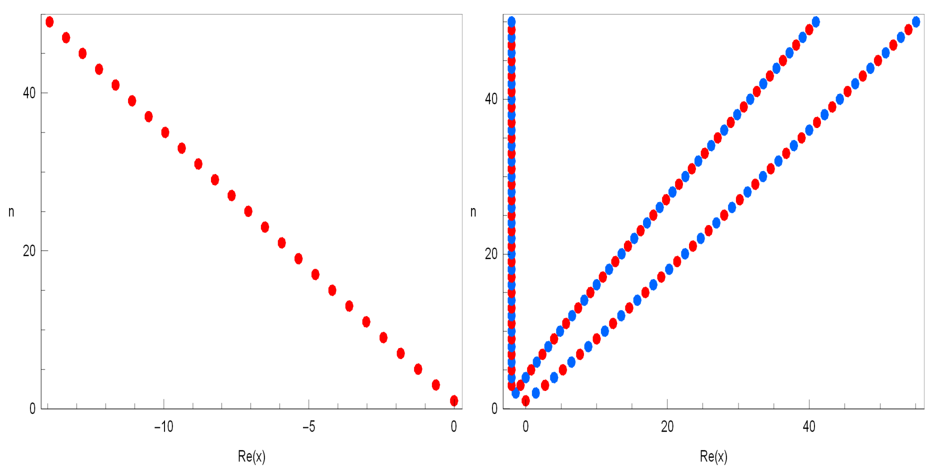

FIGURE 5. Stacks of zeros of $D_n^{(\alpha)}(x), 1 \leq n \leq 50$

We observed a remarkable regular structure of zeros of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$ and also to verify same kind of remarkable regular structure of zeros of the of the generalized polynomials of derangements $D_n^{(\alpha)}(x) = 0$(Table 1). Next, an approximate solution satisfying $D_n^{(\alpha)}(x) = 0$ for $x \in \mathbb{R}$ are listed in Tables 2.

Table 2. Approximate solutions of $D_n^{(\alpha)}(x) = 0, \alpha = 2, x \in \mathbb{R}$

degree n	x
1	0
2	–
3	-0.625816819
4	–
5	-1.237112761
6	–
7	-1.83920261
8	–

Table 3. Solutions of $D_n^{(\alpha)}(x) = 0, \alpha = -2, x \in \mathbb{R}$

degree n	x
1	0
2	$-\sqrt{2}, \quad \sqrt{2}$
3	$-2 \quad 1-\sqrt{3}, \quad 1+\sqrt{3}$
4	$-2, \quad -2, \quad 0, \quad 4$
5	$-2, \quad -2, \quad -2, \quad 3-\sqrt{5}, \quad 3+\sqrt{5}$
6	$-2, \quad -2, \quad -2, \quad -2, \quad 4-\sqrt{6}, \quad 4+\sqrt{6}$
7	$-2, \quad -2, \quad -2, \quad -2, \quad -2, \quad 5-\sqrt{7}, \quad 5+\sqrt{7}$
8	$-2, \quad -2, \quad -2, \quad -2, \quad -2, \quad -2, \quad 2(3-\sqrt{2}), \quad 2(3+\sqrt{2})$

From all the numerical computations done in this research work, we give the following problems: How many zeros do $D_n^{(\alpha)}(x) = 0$ have? We are not able to decide if $D_n^{(\alpha)}(x) = 0$ has n distinct solutions(see Tables 1, 2, and 3). We like to know the number of real zeros $D_n^{(\alpha)}(x) = 0$. Finally, we prove that $D_n^{(\alpha)}(x)$ has not refection symmetry for $a \in \mathbb{R}$(see Figures 3, 4, and 5).

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